Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 600–609
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

# Generalized monotone iterative method for integral boundary value problems with causal operators

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**Abstract**

This paper investigates the existence of solutions for a class of integral boundary value problems with causal operators. The arguments are based upon the developed monotone iterative method. As applications, two examples are worked out to demonstrate the main results. 

*Keywords:* Generalized monotone iterative method, integral boundary value problems, causal operators, upper and lower solutions.
*2010 MSC:* 39A10, 34A34.

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## 1. Introduction and Preliminaries

The general theory of integral boundary value problems arises naturally from a wide variety of applications such as heat conduction, chemical engineering, underground water flow. Recently, the existence of solutions for such problems have received a great deal of attentions; for details see for example [1, 2, 10, 11, 12, 19, 20, 21]. It is well known that monotone iterative technique is quite useful, see [3, 13, 14, 15, 16, 17] and references therein. In [5, 6, 7, 9, 22], this method, combining upper and lower solutions, has been successfully applied to obtain the existence of extremal solutions for boundary value problems with integral boundary conditions. Bhaskar [4] and West [18] developed monotone iterative method, considered the generalized monotone iterative method for initial value problems, obtained the existence of extremal solutions for differential equations where the forcing function is the sum of two monotone functions, one of which is monotone non-decreasing and the other is non-increasing. In [8], Lakshmikantham also discussed initial value problems for causal differential equations by using the generalized monotone iterative method. The theory of causal differential equations has the powerful quality of unifying ordinary differential

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*Received 2014-12-29*

equations, integro differential equations, differential equations with finite or infinite delay, Volterra integral equations and neutral equations. For more information about the general theory of causal differential equations, we refer to the book of Lakshmikantham [8].

However, integral boundary value problems for causal differential equations where the right-hand side is the sum of two monotone functions, are not investigated till now. In this paper, we deal with the following integral boundary value problems with causal operators:

$$\begin{cases} y'(t) = (Py)(t) + (Qy)(t), \quad t \in J = [0, T], \\ y(0) = \lambda_1 y(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t, y(t))dt + c, \end{cases} \tag{1.1}$$

where $P, Q : E \to E = C(J, \mathbb{R})$ are causal operators. $\tau \in (0, T]$ is a given point and $\varphi \in C(J \times \mathbb{R}, \mathbb{R})$, $\lambda_1, \lambda_2, c \in \mathbb{R}$.

Note that the integral boundary value problem (1.1) reduces to initial value problems for $\lambda_1 = \lambda_2 = 0$ which has been studied in [8], periodic boundary value problems when $\lambda_1 = 1, \lambda_2 = c = 0$ and $\tau = T$ and anti-periodic boundary value problems if $\lambda_1 = -1, \lambda_2 = c = 0$ and $\tau = T$. Thus problem (1.1) can be regarded as a generalization of the boundary value problems mentioned above.

The rest of this paper is organized as follows. In Sections 2, we develop the monotone technique and establish the existence of coupled extremal solutions for (1.1). In Section 3, two examples are added to verify the assumption and theoretical results.

## 2. Main results

In order to verify our main results, we first give the following definitions.

**Definition 2.1.** For the causal differential equation (1.1):
(1) a function $y \in C^1(J, \mathbb{R})$ is said to be a natural solution if it satisfies (1.1).
(2) $y, z \in C^1(J, \mathbb{R})$ are said to be coupled solutions of type I, if

$$\begin{cases} y'(t) = (Py)(t) + (Qz)(t), \quad y(0) = \lambda_1 y(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t, y(t))dt + c, \\ z'(t) = (Pz)(t) + (Qy)(t), \quad z(0) = \lambda_1 z(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t, z(t))dt + c; \end{cases}$$

(3) $y, z \in C^1(J, \mathbb{R})$ are said to be coupled solutions of type II, if

$$\begin{cases} y'(t) = (Pz)(t) + (Qy)(t), \quad y(0) = \lambda_1 y(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t, y(t))dt + c, \\ z'(t) = (Py)(t) + (Qz)(t), \quad z(0) = \lambda_1 z(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t, z(t))dt + c; \end{cases}$$

(4) $y, z \in C^1(J, \mathbb{R})$ are said to be coupled solutions of type III, if

$$\begin{cases} y'(t) = (Pz)(t) + (Qz)(t), \quad y(0) = \lambda_1 y(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t, y(t))dt + c, \\ z'(t) = (Py)(t) + (Qy)(t), \quad z(0) = \lambda_1 z(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t, z(t))dt + c. \end{cases}$$

**Definition 2.2.** Related to equation (1.1), the functions $\alpha, \beta \in C^1(J, \mathbb{R})$ are said to be
(1) natural lower and upper solutions if

$$\begin{cases} \alpha'(t) \le (P\alpha)(t) + (Q\alpha)(t), \ \ \alpha(0) \le \lambda_1\alpha(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t,\alpha(t))dt + c, \\ \beta'(t) \ge (P\beta)(t) + (Q\beta)(t), \ \ \beta(0) \ge \lambda_1\beta(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t,\beta(t))dt + c; \end{cases}$$

(2) coupled lower and upper solutions of type I, if

$$\begin{cases} \alpha'(t) \le (P\alpha)(t) + (Q\beta)(t), \ \ \alpha(0) \le \lambda_1\alpha(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t,\alpha(t))dt + c, \\ \beta'(t) \ge (P\beta)(t) + (Q\alpha)(t), \ \ \beta(0) \ge \lambda_1\beta(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t,\beta(t))dt + c; \end{cases}$$

(3) coupled lower and upper solutions of type II, if

$$\begin{cases} \alpha'(t) \le (P\beta)(t) + (Q\alpha)(t), \ \ \alpha(0) \le \lambda_1\alpha(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t,\alpha(t))dt + c, \\ \beta'(t) \ge (P\alpha)(t) + (Q\beta)(t), \ \ \beta(0) \ge \lambda_1\beta(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t,\beta(t))dt + c; \end{cases}$$

(4) coupled lower and upper solutions of type III, if

$$\begin{cases} \alpha'(t) \le (P\beta)(t) + (Q\beta)(t), \ \ \alpha(0) \le \lambda_1\alpha(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t,\alpha(t))dt + c, \\ \beta'(t) \ge (P\alpha)(t) + (Q\alpha)(t), \ \ \beta(0) \ge \lambda_1\beta(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(t,\beta(t))dt + c. \end{cases}$$

In this paper, we set $\alpha(t) \le \beta(t)$, $t \in J$, then $(Py)$ is non-decreasing and $(Qy)$ is non-increasing. Therefore, the natural lower and upper solutions satisfy type II and the coupled lower and upper solutions of type III satisfy type II. Thus, we only need to consider the cases of the coupled lower and upper solutions of type I and II for (1.1).

**Definition 2.3.** Coupled solutions $\rho, r \in C^1(J,\mathbb{R})$ are said to be coupled minimal and maximal solutions of (1.1), respectively, if for any coupled solutions $y, z \in C^1(J,\mathbb{R})$, we have
$\rho(t) \le y(t) \le r(t)$ and $\rho(t) \le z(t) \le r(t)$, $t \in J$.

**Theorem 2.4.** *Assume that the following conditions hold,*
$(A_1)$ *the functions* $\alpha_0, \beta_0 \in C^1(J,\mathbb{R})$ *are coupled lower and upper solutions of type I for* (1.1) *with* $\alpha_0(t) \le \beta_0(t)$ *on* $J$;
$(A_2)$ *the operators* $P, Q$ *in* (1.1) *are such that* $P, Q : E \to E$, $(Py)$ *is non-decreasing in* $y$ *and* $(Qy)$ *is non-increasing in* $y$;
$(A_3)$ $\lambda_1(u(\tau) - v(\tau)) + \lambda_2 \int_0^T \big(\varphi(s,u(s)) - \varphi(s,v(s))\big)ds \le 0$ *where* $\alpha_0(t) \le u(t) \le v(t) \le \beta_0(t)$, $\lambda_1, \lambda_2 \in \mathbb{R}$.

*Then there exist monotone sequences* $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *which converge uniformly to the coupled minimal and maximal solutions of type I for* (1.1).

*Proof.* First, we consider the following linear problems:

$$\begin{cases} \alpha'_{n+1}(t) = (P\alpha_n)(t) + (Q\beta_n)(t), \\ \alpha_{n+1}(0) = \lambda_1\alpha_n(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(s,\alpha_n(s))ds + c. \end{cases} \tag{2.1}$$

$$\begin{cases} \beta'_{n+1}(t) = (P\beta_n)(t) + (Q\alpha_n)(t), \\ \beta_{n+1}(0) = \lambda_1\beta_n(\tau) + \lambda_2 \int_0^T \varphi(s, \beta_n(s))ds + c. \end{cases} \tag{2.2}$$

Obviously, by general results on the initial value problems of causal differential equations [8], (2.1) and (2.2) have solutions, hence the above definitions are adequate.

Due to (2.1) and (2.2), we get two monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$. Next, we shall show that the sequences satisfy

$$\alpha_0(t) \leq \alpha_1(t) \leq \cdots \leq \alpha_n(t) \leq \beta_n(t) \leq \cdots \leq \beta_1(t) \leq \beta_0(t) \tag{2.3}$$

for $t \in J$, $n = 1, 2, \cdots$ through two steps.

**Step 1**. We prove that (2.3) is true for $n = 1$, i.e., $\alpha_0(t) \leq \alpha_1(t) \leq \beta_1(t) \leq \beta_0(t)$, $t \in J$. For this purpose, let $p(t) = \alpha_0(t) - \alpha_1(t)$, we acquire

$$p'(t) = \alpha'_0(t) - \alpha'_1(t) \leq (P\alpha_0)(t) + (Q\beta_0)(t) - (P\alpha_0)(t) - (Q\beta_0)(t) = 0,$$

note that $p(0) \leq 0$. We get $p(t) \leq 0$ on $J$, which implies $\alpha_0(t) \leq \alpha_1(t)$, $t \in J$. Similarly, we can obtain $\beta_1(t) \leq \beta_0(t)$, $t \in J$.

If we set $p(t) = \alpha_1(t) - \beta_1(t), t \in J$, then from $(A_2)$, $(A_3)$ and the fact $\alpha_0(t) \leq \beta_0(t)$, $t \in J$, we have

$$p'(t) = \alpha'_1(t) - \beta'_1(t) = (P\alpha_0)(t) + (Q\beta_0)(t) - (P\beta_0)(t) - (Q\alpha_0)(t) \leq 0$$

and

$$\begin{aligned} p(0) &= \alpha_1(0) - \beta_1(0) \\ &= \lambda_1(\alpha_0(\tau) - \beta_0(\tau)) + \lambda_1 \int_0^T \big(\varphi(s, \alpha_0(s)) - \varphi(s, \beta_0(s))\big)ds \\ &\leq 0. \end{aligned}$$

Therefore, we get $p(t) \leq 0$ on $J$, which gives $\alpha_1(t) \leq \beta_1(t)$, $t \in J$.

**Step 2**. Assume that for some integer $n > 1$,

$$\alpha_{n-1}(t) \leq \alpha_n(t) \leq \beta_n(t) \leq \beta_{n-1}(t), \quad t \in J.$$

We need to prove that

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad t \in J.$$

For this purpose, setting $p(t) = \alpha_n(t) - \alpha_{n+1}(t)$ and use $(A_2), (A_3)$, then we get

$$p'(t) = \alpha'_n(t) - \alpha'_{n+1}(t) = (P\alpha_{n-1})(t) + (Q\beta_{n-1})(t) - (P\alpha_n)(t) - (Q\beta_n)(t) \leq 0,$$

$$\begin{aligned} p(0) &= \alpha_n(0) - \alpha_{n+1}(0) \\ &= \lambda_1(\alpha_{n-1}(\tau) - \alpha_n(\tau)) + \lambda_2 \int_0^T \big(\varphi(s, \alpha_{n-1}(s)) - \varphi(s, \alpha_n(s))\big)ds \leq 0. \end{aligned}$$

It follows that $\alpha_n(t) \leq \alpha_{n+1}(t)$, $t \in J$. In the same way, we can get $\beta_{n+1}(t) \leq \beta_n(t)$, $t \in J$. To prove $\alpha_{n+1}(t) \leq \beta_{n+1}(t)$ on $J$, taking $p(t) = \alpha_{n+1}(t) - \beta_{n+1}(t)$, employing $(A_2)$, $(A_3)$ and the fact that $\alpha_n(t) \leq \beta_n(t)$, $t \in J$, we obtain

$$p'(t) = \alpha'_{n+1}(t) - \beta'_{n+1}(t) = (P\alpha_n)(t) + (Q\beta_n)(t) - (P\beta_n)(t) - (Q\alpha_n)(t) \leq 0$$

and

$$\begin{aligned} p(0) &= \alpha_{n+1}(0) - \beta_{n+1}(0) \\ &= \lambda_1(\alpha_n(\tau) - \beta_n(\tau)) + \lambda_2 \int_0^T \big(\varphi(s, \alpha_n(s)) - \varphi(s, \beta_n(s))\big)ds \leq 0. \end{aligned}$$

Then $p(t) \le 0$, which gives $\alpha_{n+1}(t) \le \beta_{n+1}(t)$, $t \in J$.

From the above discussion, applying the mathematical induction, we have

$$\alpha_0(t) \le \alpha_1(t) \le \cdots \le \alpha_n(t) \le \beta_n(t) \le \cdots \le \beta_1(t) \le \beta_0(t), \quad t \in J.$$

Consequently, there exist $\rho, r \in C^1(J,\mathbb{R})$ such that $\lim\limits_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim\limits_{n\to\infty} \beta_n(t) = r(t)$ on $J$. Apparently, $\rho, r$ are coupled solutions of type I for (1.1).

Finally, we show that $\rho(t), r(t)$ are coupled minimal and maximal solutions of type I for (1.1). To do it, put $[\alpha_0, \beta_0] = \{y \in C(J,\mathbb{R}) : \alpha_0(t) \le y(t) \le \beta_0(t), t \in J\}$. Let $y_1, y_2 \in [\alpha_0, \beta_0]$ be any coupled solutions of type I for (1.1) and assume that there exists a positive integer $n$ such that $\alpha_n(t) \le y_i(t) \le \beta_n(t)$ $(i = 1, 2)$ on $J$. Then setting $p(t) = \alpha_{n+1}(t) - y_1(t)$ on $J$ and using $(A_2)$ and $(A_3)$, we have

$$p'(t) = \alpha'_{n+1}(t) - y'_1(t) = (P\alpha_n)(t) + (P\beta_n)(t) - (Py_1)(t) - (Qy_2)(t) \le 0$$

and

$$p(0) = \alpha_{n+1}(0) - y_1(0) \le 0.$$

Hence $p(t) \le 0$, $t \in J$, which implies $\alpha_{n+1}(t) \le y_1(t)$, $t \in J$. Similarly, we can conclude $y_1(t) \le \beta_{n+1}(t)$ and $\alpha_{n+1}(t) \le y_2(t) \le \beta_{n+1}(t)$, $t \in J$. Since $\alpha_0(t) \le y_i(t) \le \beta_0(t)$ $(i = 1, 2)$ on $J$, by induction we see that $\alpha_n(t) \le y_i(t) \le \beta_n(t)$ $(i = 1, 2)$ holds for all $n$. Therefore $\rho(t) \le y_i(t) \le r(t)$ $(i = 1, 2)$ on $J$ as $n \to \infty$. $\rho(t)$ and $r(t)$ are coupled minimal and maximal solutions of type I for (1.1). Since that any natural solution $y(t)$ of (1.1) can be considered as $(y, y)$ coupled solutions of type I, we also have $\rho(t) \le y(t) \le r(t)$, $t \in J$. Then the proof is finished. □

*Remark* 2.5. When $\lambda_1 = \lambda_2 = 0$, the problem (1.1) is reduced to initial value problems with causal operators, see causal differential equations Theorem 3.4.1. in [8].

**Corollary 2.6.** *Assume that $\lambda_1 = 1$, $\lambda_2 = c = 0$, $\tau = T$ and conditions of Theorem 2.4 hold. The problem* (1.1) *is reduced to periodic boundary value problems with causal operators. Then there exist two monotone sequences which converge uniformly to the coupled minimal and maximal solutions of type I for* (1.1).

**Corollary 2.7.** *If $\lambda_1 = -1$, $\lambda_2 = c = 0$ and conditions of Theorem 2.4 hold. The problem* (1.1) *is reduced to anti-periodic boundary value problems with causal operators. We can get two monotone sequences converging to the coupled minimal and maximal solutions of type I for* (1.1).

**Theorem 2.8.** *Assume conditions of Theorem 2.4 hold. Then for any natural solution $y$ of* (1.1)*with $\alpha_0(t) \le y(t) \le \beta_0(t)$, $t \in J$, there exist alternating sequences*

$$\{\alpha_{2n}(t), \beta_{2n+1}(t)\} \to \rho(t), \{\beta_{2n(t)}, \alpha_{2n+1}(t)\} \to r(t)$$

*on $J$ with $\alpha_0(t) \le \beta_1(t) \le \cdots \le \alpha_{2n}(t) \le \beta_{2n+1}(t) \le y(t) \le \alpha_{2n+1}(t) \le \beta_{2n}(t) \le \cdots \le \alpha_1(t) \le \beta_0(t)$ on $J$ and $\rho, r \in C^1(J,\mathbb{R})$ are the coupled minimal and maximal solutions of type I for* (1.1).

*Proof.* Consider the equations

$$\begin{cases} \alpha'_{n+1}(t) = (P\beta_n)(t) + (Q\alpha_n)(t), \;\; \alpha_{n+1}(0) = \lambda_1\beta_n(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(s, \beta_n(s))ds + c, \\ \beta'_{n+1}(t) = (P\alpha_n)(t) + (Q\beta_n)(t), \;\; \beta_{n+1}(0) = \lambda_1\alpha_n(\tau) + \lambda_2 \displaystyle\int_0^T \varphi(s, \alpha_n(s))ds + c. \end{cases} \tag{2.4}$$

Notice that the initial value problemd of causal differential equations (2.4) have unique solution, hence the above definitions are adequate.

We complete the proof by five steps.

**Step 1**. We prove that $\alpha_0(t) \le \alpha_1(t)$ and $\beta_1(t) \le \beta_0(t)$, $t \in J$.

Let $p(t) = \alpha_0(t) - \alpha_1(t)$. From (2.4), we have

$$p'(t) = \alpha_0'(t) - \alpha_1'(t) = (P\alpha_0)(t) + (Q\beta_0)(t) - (P\beta_0)(t) - (Q\alpha_0)(t) \le 0,$$

and

$$p(0) = \alpha_0(0) - \alpha_1(0) \le 0.$$

Then we get $p(t) \le 0$, $t \in J$, which gives $\alpha_0(t) \le \alpha_1(t)$, $t \in J$. Similarly, we can prove $\beta_1(t) \le \beta_0(t)$, $t \in J$.

**Step 2**. We shall show that

$$\alpha_0(t) \le \beta_1(t) \le \alpha_2(t) \le \beta_3(t) \le y(t) \le \alpha_3(t) \le \beta_2(t) \le \alpha_1(t) \le \beta_0(t), \ \ t \in J. \tag{2.5}$$

First taking $p(t) = y(t) - \alpha_1(t)$, $t \in J$, using the fact $\alpha_0(t) \le y(t) \le \beta_0(t)$ on $J$, we get

$$p'(t) = y'(t) - \alpha_1'(t) = (Py)(t) + (Qy)(t) - (P\beta_0)(t) - (Q\alpha_0)(t) \le 0$$

and

$$\begin{aligned} p(0) &= y(0) - \alpha_1(0) \\ &= \lambda_1(y(\tau) - \beta_0(\tau)) + \lambda_2 \int_0^T \big(\varphi(s, y(s)) - \varphi(s, \beta_0(s))\big) ds \le 0. \end{aligned}$$

Hence $p(t) \le 0$ on $J$, i.e., $y(t) \le \alpha_1(t)$, $t \in J$. Similarly, we can conclude $\beta_1(t) \le y(t)$, $t \in J$. In order to avoid repetition, we can prove $\alpha_2(t) \le y(t)$, $y(t) \le \beta_2(t)$, $y(t) \le \alpha_3(t)$ and $\beta_3(t) \le y(t)$, $t \in J$ by using similar arguments.

Next we shall show that $\alpha_0(t) \le \beta_1(t) \le \alpha_2(t) \le \beta_3(t)$ and $\alpha_3(t) \le \beta_2(t) \le \alpha_1(t) \le \beta_0(t)$, $t \in J$. Set $p(t) = \alpha_0(t) - \beta_1(t)$, then we have $p(0) \le 0$ and

$$p'(t) = \alpha_0'(t) - \beta_1'(t) \le (P\alpha_0)(t) + (Q\beta_0)(t) - (P\alpha_0)(t) - (Q\beta_0)(t) = 0.$$

Then $p(t) \le 0$ on $J$, thus $\alpha_0(t) \le \beta_1(t)$, $t \in J$. Similarly, we have $\alpha_1(t) \le \beta_0(t)$, $t \in J$.

Taking $p(t) = \beta_1(t) - \alpha_2(t)$, from $(A_2)$ and $(A_3)$, we derive

$$p'(t) = \beta_1'(t) - \alpha_2'(t) = (P\alpha_0)(t) + (Q\beta_0)(t) - (P\beta_1)(t) - (Q\alpha_1)(t) \le 0,$$

and

$$p(0) = \lambda_1(\alpha_0(\tau) - \beta_1(\tau)) + \lambda_2 \int_0^T \big(\varphi(s, \alpha_0(s)) - \varphi(s, \beta_1(s))\big) ds \le 0.$$

This implies that $p(t) \le 0$ on $J$, which gives $\beta_1(t) \le \alpha_2(t)$, $t \in J$. In the same way, we can obtain $\alpha_2(t) \le \beta_3(t)$, $\alpha_3(t) \le \beta_2(t)$, $\beta_2(t) \le \alpha_1(t)$, $t \in J$. Combining all these arguments, we get (2.5).

**Step 3**. Assume there exists an integer $k \ge 2$ such that $\beta_{2k-1}(t) \le \alpha_{2k}(t) \le \beta_{2k+1}(t) \le y(t) \le \alpha_{2k+1}(t) \le \beta_{2k}(t) \le \alpha_{2k-1}(t)$ on $J$. Now, we need to prove

$$\beta_{2k+1}(t) \le \alpha_{2k+2}(t) \le \beta_{2k+3}(t) \le y(t) \le \alpha_{2k+3}(t) \le \beta_{2k+2}(t) \le \alpha_{2k+1}(t), \quad t \in J.$$

Take $p(t) = \beta_{2k+1}(t) - \alpha_{2k+2}(t)$, then for $t \in J$, we get

$$p'(t) = \beta_{2k+1}'(t) - \alpha_{2k+2}'(t) = (P\alpha_{2k})(t) + (Q\beta_{2k})(t) - (P\beta_{2k+1})(t) - (Q\alpha_{2k+1})(t) \le 0$$

and

$$p(0) = \lambda_1(\alpha_{2k}(\tau) - \beta_{2k+1}(\tau)) + \lambda_2 \int_0^T \big(\varphi(s, \alpha_{2k}(s)) - \varphi(s, \beta_{2k+1}(s))\big) ds \le 0,$$

by using the hypotheses $\alpha_{2k}(t) \le \beta_{2k+1}(t)$, $\alpha_{2k+1}(t) \le \beta_{2k}(t)$, $t \in J$ and $(A_2)$, $(A_3)$. This implies that $p(t) \le 0$ on $J$ and $\beta_{2k+1}(t) \le \alpha_{2k+2}(t)$, $t \in J$. Using similar argument we obtain $\beta_{2k+2}(t) \le \alpha_{2k+1}(t)$, $\alpha_{2k+2}(t) \le \beta_{2k+3}(t)$, and $\alpha_{2k+3}(t) \le \beta_{2k+2}(t)$, $t \in J$.

To prove $\alpha_{2k+2}(t) \leq y(t)$, let $p(t) = \alpha_{2k+2}(t) - y(t)$, then we have

$$p'(t) = \alpha'_{2k+2}(t) - y'(t) = (P\beta_{2k+1})(t) + (Q\alpha_{2k+1})(t) - (Py)(t) - (Qy)(t) \leq 0$$

and

$$\begin{aligned} p(0) &= \alpha_{2k+2}(0) - y(0) \\ &= \lambda_1(\beta_{2k+1}(\tau) - y(\tau)) + \lambda_2 \int_0^T \big(\varphi(s, \beta_{2k+1}(s)) - \varphi(s, y(s))\big) ds \leq 0. \end{aligned}$$

Then we get $p(t) \leq 0$ on $J$, which gives $\alpha_{2k+2}(t) \leq y(t)$, $t \in J$. Similarly, we can conclude that $y(t) \leq \beta_{2k+2}(t)$, $\beta_{2k+3}(t) \leq y(t)$ and $y(t) \leq \alpha_{2k+3}(t)$, $t \in J$.

**Step 4**. Following the first three steps, we obtain $\alpha_0(t) \leq \beta_1(t) \leq \cdots \leq \alpha_{2n}(t) \leq \beta_{2n+1}(t) \leq y(t) \leq \alpha_{2n+1}(t) \leq \beta_{2n}(t) \leq \cdots \leq \alpha_1(t) \leq \beta_0(t)$, $t \in J$.

Obviously, each $\alpha_n(t), \beta_n(t)$ $(n = 1, 2, \cdots)$ satisfy (2.4). Therefore there exist $\rho(t)$ and $r(t)$ such that the sequences $\{\alpha_{2n}(t), \beta_{2n+1}(t)\}$ converge uniformly to $\rho(t)$ and $\{\beta_{2n}(t), \alpha_{2n+1}(t)\}$ converge uniformly to $r(t)$ on $J$ respectively. Clearly $\rho, r$ are coupled solutions of type I for (1.1).

**Step 5**. We prove that $\rho(t)$ and $r(t)$ are coupled minimal and maximal solutions of equation (1.1).

Put $[\alpha_0, \beta_0] = \{y \in C(J, \mathbb{R}) : \alpha_0(t) \leq y(t) \leq \beta_0(t), t \in J\}$. Let $y_1, y_2 \in [\alpha_0, \beta_0]$ be any coupled solutions of type I for (1.1). Suppose that there exists an integer $k$ such that $\beta_{2k-1}(t) \leq \alpha_{2k}(t) \leq \beta_{2k+1}(t) \leq y_i(t) \leq \alpha_{2k+1}(t) \leq \beta_{2k}(t) \leq \alpha_{2k-1}(t)$ $(i = 1, 2)$ for $t \in J$. Using similar arguments above, we can see that $\beta_{2k+1}(t) \leq \alpha_{2k+2}(t) \leq \beta_{2k+3}(t) \leq y_i(t) \leq \alpha_{2k+3}(t) \leq \beta_{2k+2}(t) \leq \alpha_{2k+1}(t)$ $(i = 1, 2)$ on $J$. By the induction, we have $\alpha_{2n}(t) \leq \beta_{2n+1}(t) \leq y_i(t) \leq \alpha_{2n+1}(t) \leq \beta_{2n}(t)$ $(i = 1, 2)$ hold on $J$ for all $n$. Taking the limit as $n \to \infty$, we have $\rho(t) \leq y_i(t) \leq r(t)$ $(i = 1, 2)$ on $J$ proving $\rho, r$ are coupled minimal and maximal solutions of type I for (1.1). Since that we have shown $\alpha_{2n}(t) \leq \beta_{2n+1}(t) \leq y(t) \leq \alpha_{2n+1}(t) \leq \beta_{2n}(t)$, if $n \to \infty$, then $\rho(t) \leq y(t) \leq r(t)$ on $J$. This ends the proof. □

*Remark* 2.9. When $\lambda_1 = \lambda_2 = 0$, the problem (1.1) is reduced to initial value problem with causal operators, see causal differential equations Theorem 3.4.2 in [8].

**Corollary 2.10.** *Assume that $\lambda_1 = 1$, $\lambda_2 = c = 0$, $\tau = T$ and conditions of Theorem 2.8 hold. The problem* (1.1) *is reduced to periodic boundary value problems with causal operators. Then there exist alternating sequences which converge uniformly to the coupled minimal and maximal solutions of type I for* (1.1).

**Corollary 2.11.** *If $\lambda_1 = -1$, $\lambda_2 = c = 0$, and conditions of Theorem 2.8 hold. The problem* (1.1) *is reduced to anti-periodic boundary value problems with causal operators. We can get alternating sequences converging to the coupled minimal and maximal solutions of type I for* (1.1).

**Theorem 2.12.** *Assume conditions $(A_2)$ and $(A_3)$ of Theorem 2.4 hold, let $\alpha_0, \beta_0 \in C^1(J, \mathbb{R})$ be the coupled lower and upper solutions of type II with $\alpha_0(t) \leq \beta_0(t)$ on $J$.*

*Then there exist two monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ such that*

$$\alpha_0(t) \leq \alpha_1(t) \leq \cdots \leq \alpha_n(t) \leq \beta_n(t) \leq \cdots \leq \beta_1(t) \leq \beta_0(t), \quad t \in J,$$

*provided $\alpha_0(t) \leq \alpha_1(t)$ and $\beta_1(t) \leq \beta_0(t)$, $t \in J$, where the sequences are given by*

$$\alpha'_{n+1}(t) = (P\alpha_n)(t) + (Q\beta_n)(t), \;\; \alpha_{n+1}(0) = \lambda_1 \alpha_n(\tau) + \lambda_2 \int_0^T \varphi(s, \alpha_n(s)) ds + c$$

*and*

$$\beta'_{n+1}(t) = (P\beta_n)(t) + (Q\alpha_n)(t), \;\; \beta_{n+1}(0) = \lambda_1 \beta_n(\tau) + \lambda_2 \int_0^T \varphi(s, \beta_n(s)) ds + c.$$

*Moreover, the monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ converge uniformly to the coupled minimal and maximal solutions of type I for* (1.1).

It is analogous to the proof of Theorem 2.4, so we omit it.

**Theorem 2.13.** *Assume the hypotheses of Theorem 2.12 hold. Then for any natural solution $y$ of (1.1) with $\alpha_0(t) \le y(t) \le \beta_0(t)$, $t \in J$, there exist the alternating sequences $\{\alpha_{2n}(t), \beta_{2n+1}(t)\}$, $\{\beta_{2n}(t), \alpha_{2n+1}(t)\}$ satisfying*

$$\alpha_0(t) \le \beta_1(t) \le \cdots \le \alpha_{2n}(t) \le \beta_{2n+1}(t) \le y(t) \le \alpha_{2n+1}(t) \le \beta_{2n}(t) \le \cdots \le \alpha_1(t) \le \beta_0(t),$$

*provided $\alpha_0(t) \le \alpha_1(t)$ and $\beta_1(t) \le \beta_0(t)$ on $J$, for every $n \ge 1$, where the iterative schemes are developed by*

$$\alpha'_{n+1}(t) = (P\beta_n)(t) + (Q\alpha_n)(t), \ \ \alpha_{n+1}(0) = \lambda_1\beta_n(\tau) + \lambda_2 \int_0^T \varphi(s, \beta_n(s))ds + c$$

*and*

$$\beta'_{n+1}(t) = (P\alpha_n)(t) + (Q\beta_n)(t), \ \ \beta_{n+1}(0) = \lambda_1\alpha_n(\tau) + \lambda_2 \int_0^T \varphi(s, \alpha_n(s))ds + c.$$

*Moreover, the monotone sequences $\{\alpha_{2n}(t), \beta_{2n+1}(t)\}$ converge to $\rho(t)$ and $\{\beta_{2n}(t), \alpha_{2n+1}(t)\}$ converge to $r(t)$ on $J$, where $\rho, r$ are coupled minimal and maximal solutions of type I for (1.1), respectively.*

One can imitate the proof of Theorem 2.4, so we omit it.

*Remark* 2.14. Theorem 2.12 and Theorem 2.13 hold if $\lambda_1 = \lambda_2 = 0$ and the problem (1.1) is reduced to initial value problems with causal operators.

*Remark* 2.15. Theorem 2.12 and Theorem 2.13 hold when $\lambda_1 = 1$, $\lambda_2 = c = 0$, and $\tau = T$. Problem (1.1) is reduced to periodic boundary value problems with causal operators.

*Remark* 2.16. Theorem 2.12 and Theorem 2.13 hold when $\lambda_1 = -1$, $\lambda_2 = c = 0$. Problem (1.1) is reduced to anti-periodic boundary value problems with causal operators.

## 3. Examples

In this section, we give two simple but illustrative examples, thereby validating the proposed theorems.

**Example 3.1.** Consider the problem:

$$\begin{cases} y'(t) = (\sin t)y(t) + \dfrac{1}{15}e^{-y(t)}, \ \ t \in J = [0,1], \\ y(0) = -y(1) + 2\displaystyle\int_0^1 y^2(s)ds + \dfrac{1}{100}. \end{cases} \tag{3.1}$$

Set

$$\alpha_0(t) = 0, \quad \beta_0(t) = t + \frac{1}{10}, \ \ t \in J.$$

We can easily verify that $\alpha_0(t) \le \beta_0(t)$ and

$$\begin{cases} \alpha'_0(t) = 0 \le (\sin t)\alpha_0(t) + \dfrac{1}{15}e^{-\beta_0(t)} = \dfrac{1}{15}e^{-t-\frac{1}{10}}, \quad t \in [0,1], \\ \alpha_0(0) = 0 < \dfrac{1}{100}, \end{cases}$$

$$\begin{cases} \beta'_0(t) = 1 \ge (\sin t)\beta_0(t) + \dfrac{1}{15}e^{-\alpha_0(t)} = (\sin t)(t + \dfrac{1}{10}) + \dfrac{1}{15}, \quad t \in [0,1], \\ \beta_0(0) = \dfrac{1}{10} > -(1 + \dfrac{1}{10}) + 2\displaystyle\int_0^1 (s + \dfrac{1}{10})^2 ds + \dfrac{1}{100} = -\dfrac{1}{300}. \end{cases}$$

This implies that $\alpha_0(t)$, $\beta_0(t)$ are coupled lower and upper solutions of type I for problem (3.1). Then the conditions of Theorem 2.4 are all satisfied. So problem (3.1) has coupled minimal and maximal solutions of type I in the sector $[0, t + \frac{1}{10}]$. Moreover, from Theorem 2.8, we obtain the existence of alternating sequences that also converge to the coupled minimal and maximal solutions.

**Example 3.2.** Consider the following problem:

$$\begin{cases} y'(t) = y^3(t) - t^2 y(t), \quad t \in J = [0,1], \\ y(0) = y\big(\frac{1}{3}\big) - \int_0^1 (s + y(s))ds + \frac{1}{2} = 0. \end{cases} \tag{3.2}$$

Setting $\alpha_0(t) = -1, \ \beta_0(t) = 1, \ t \in J$, we can easily verify that $\alpha_0(t) \le \beta_0(t)$, and

$$\begin{cases} \alpha_0'(t) = 0 \le \beta_0^3(t) - t^2\alpha_0(t) = 1 + t^2, \\ \alpha_0(0) = -1 < -1 + 1/2 + 1/2 = 0, \end{cases}$$

$$\begin{cases} \beta_0'(t) = 0 \ge \alpha_0^3(t) - t^2\beta_0(t) = -1 - t^2, \\ \beta_0(0) = 1 > 1 - 3/2 + 1/2 = 0. \end{cases}$$

Then functions $\alpha_0(t), \ \beta_0(t)$ are coupled lower and upper solutions of type II for problem (3.2), then the conditions of Theorem 2.12 are all satisfied. Applying Theorem 2.12 and using (2.1) and (2.2), problem (3.2) has coupled minimal and maximal solutions of type I for (3.2) in the segment $[-1, 1]$. From Theorem 2.13, we obtain the existence of alternating sequences that also converge to the coupled minimal and maximal solutions.

## Acknowledgements

The authors would like to thank the reviewers and the editors for their valuable suggestions and comments.

This work was supported by the Fundamental Research Funds for the Central Universities (No. 2015ZD29), the Science and Technology Research Projects of Higher Education Institutions of Hebei Province of China (No. Z2013038), and the Research Project of China University of Geosciences Great Wall College (ZD-CYK01402).

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