



Newton method for estimation of the Robin coefficient

Yan-Bo Ma

Department of Mathematics and Statist, Hanshan Normal University, Chaozhou, Guangdong, 521041, P. R. China.

Communicated by P. Kumam

Abstract

This paper considers estimation of Robin parameter by using measurements on partial boundary and solving a Robin inverse problem associated with the Laplace equation. Typically, such problems are solved utilizing a Gauss-Newton method in which the forward model constraints are implicitly incorporated. Variants of Newton's method which use second derivative information are rarely employed because their perceived disadvantage in computational cost per step offsets their potential benefits of fast convergence. In this paper, we show that by formulating the inversion as a constrained or unconstrained optimization problem, we can carry out the sequential quadratic programming and the full Newton iteration with only a modest additional cost. Our numerical results illustrate that Newton's method can produce a solution in fewer iterations and, in some cases where the data contain significant noise, requires fewer floating point operations than Gauss-Newton methods.

Keywords: Robin inverse problem, ill-posedness, boundary integral equations, Newton method, Gauss-Newton method.

2010 MSC: 45Q05, 65N21.

1. Introduction

Let Ω be a bounded domain in R^2 with smooth boundary $\partial\Omega = \Gamma$. Consider the Robin boundary value problem for the Laplace equation

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + pu = g, & \text{on } \partial\Omega = \Gamma, \end{cases} \quad (1.1)$$

Email address: yanboma@hstc.edu.cn (Yan-Bo Ma)

where ν is the unit outward normal direction on Γ , $p = p(x)$ is the Robin coefficient with support contained in $\Gamma_1 \subset \Gamma$, and $g = g(x)$ is a given input function. Many results concerned with important properties of the forward map from p to u , such as uniqueness, continuity with respect to proper norms, and differentiability and stability in various forms, have been developed; see [1, 7, 8, 9, 14, 15].

In this paper, we consider the Robin inverse problem: to recover the Robin coefficient p from a partial boundary measurement of function u . That is, recover p by using $u = u_0$ on a part Γ_0 of the boundary with $\Gamma_0 \cap \Gamma_1 = \emptyset$. It is well-known that the above inverse problem is severely ill-conditioned. Applications of the problem include various nondestructive evaluation methods.

Some numerical studies for this inverse problem have been studied; (see [4, 5, 6, 12, 13, 19, 20, 21, 22, 23, 25, 26, 27, 30]).

We first introduce the boundary integral equation formulation for the boundary value problem (1.1). Let $\Phi = \Phi(x, y)$ be the fundamental solution for the Laplace equation in R^2 :

$$\Phi(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} \quad \text{for } x \neq y.$$

It is known that the boundary value of u satisfies the following boundary integral equation [3, 24, 28]:

$$\frac{1}{2}u(x) + \int_{\Gamma} \left(\frac{\partial \Phi(x, y)}{\partial \nu_y} + p(y)\Phi(x, y) \right) u(y)ds_y = \int_{\Gamma} \Phi(x, y)g(y)ds_y, \quad x \in \Gamma. \tag{1.2}$$

Let the operators \mathcal{D} and \mathcal{S} be defined by

$$\begin{aligned} (\mathcal{D}u)(x) &= \int_{\Gamma} \frac{\partial \Phi(x, y)}{\partial \nu_y} u(y)ds_y, \\ (\mathcal{S}u)(x) &= \int_{\Gamma} \Phi(x, y)u(y)ds_y, \end{aligned} \quad \text{for } x \in \Gamma.$$

Then (1.2) can be written as

$$\mathcal{A}(p)(u) = f, \tag{1.3}$$

where $\mathcal{A}(p)(u) = (\frac{1}{2}\mathcal{I} + \mathcal{D})u + \mathcal{S}(pu)$, $f = \mathcal{S}g$.

Assume that the discretize form of the operator equation is given by

$$A(\mathbf{p})\mathbf{u} = \mathbf{f}. \tag{1.4}$$

(see Appendix A for more details) and denote the location of the observations by R_0 , then the Robin inverse problem is to find \mathbf{p} such that (1.4) holds and

$$\|R_0\mathbf{u} - \mathbf{u}_0\| \approx Tol, \tag{1.5}$$

where Tol depends on the noise level.

Since the data are noisy, and the inverse problem of recovering \mathbf{p} that satisfies (1.4) and (1.5) is ill-posed, a process of regularization is required to recover stably a relatively smooth solution to a nearby problem. A model which is often utilized in practice is to minimize a least squares residual vector where a regularization term is added:

$$\begin{aligned} \min_{\mathbf{p}, \mathbf{u}} \frac{1}{2} \|R_0\mathbf{u} - \mathbf{u}_0\|^2 + \frac{\alpha}{2} \|E(\mathbf{p} - \mathbf{p}_{ref})\|^2 \\ \text{s.t. } A(\mathbf{p})\mathbf{u} = \mathbf{f}, \end{aligned} \tag{1.6}$$

where E is a weighting matrix involving discretized derivatives which does not depend on \mathbf{p} , \mathbf{p}_{ref} is a reference model, and $\alpha > 0$ is a regularization parameter.

The problem (1.6) is a nonlinear constrained optimization problem. Since the forward model is linear in \mathbf{u} and allows an explicit elimination

$$\mathbf{u} = A(\mathbf{p})^{-1}\mathbf{f},$$

the constrained optimization problem (1.6) can then be written as an unconstrained, nonlinear least squares problems:

$$\min_{\mathbf{p}} \frac{1}{2} \|R_0 A(\mathbf{p})^{-1}\mathbf{f} - \mathbf{u}_0\|_2^2 + \frac{\alpha}{2} \|E(\mathbf{p} - \mathbf{p}_{ref})\|^2. \tag{1.7}$$

A common approach in the literature hitherto is to solve the unconstrained minimization (1.7) by the Gauss-Newton method in which the term $\|R_0 A(\mathbf{p})^{-1}\mathbf{f} - \mathbf{u}_0\|_2^2$ is approximated by a quadratic function. However, the Gauss-Newton method may be costly since a lot of iterations are often required to achieve a given tolerance.

In this paper, we propose a full Newton method for solving the Robin inverse problem and show that the proposed Newton steps do not require too much additional computations compared to the Gauss-Newton approximation. We notice that the Newton method converges quadratically while the Gauss-Newton method converges linearly, and it follows that the Newton method can significantly reduce the number of iterations required for convergence. Numerical results verify the above conclusion.

The rest of the paper is arranged as follows. In Section 2, we introduce the Lagrangian for the constrained minimization and obtain the Newton and Gauss-Newton approximation for solving the constrained and unconstrained minimization problem. We give the equivalent techniques for the Newton and Gauss-Newton approximation, respectively. In Section 3, we present numerical results.

2. Newton’s method and Gauss-Newton approximation

Consider solving constrained minimization problem (1.6) by using the Lagrangian

$$\mathcal{L}(\mathbf{u}, \mathbf{p}, \lambda) = \frac{1}{2} \|R_0 \mathbf{u} - \mathbf{u}_0\|^2 + \frac{\alpha}{2} \|E(\mathbf{p} - \mathbf{p}_{ref})\|^2 + \lambda^T [A(\mathbf{p})\mathbf{u} - \mathbf{f}], \tag{2.1}$$

where λ is the Lagrange multiplier. A necessary condition for $(\mathbf{u}^*, \mathbf{p}^*)$ to be an optimal solution of problem (1.6) is that there exists a multiplier λ^* such that $(\mathbf{u}^*, \mathbf{p}^*, \lambda^*)$ satisfies the Karush-Kuhn-Tucker (KKT) condition, i.e. the first derivatives $\mathcal{L}_{\mathbf{u}}$, $\mathcal{L}_{\mathbf{p}}$, and \mathcal{L}_{λ} of the Lagrangian vanish:

$$\mathcal{L}_{\mathbf{u}} = R_0^T (R_0 \mathbf{u} - \mathbf{u}_0) + A(\mathbf{p})^T \lambda = 0, \tag{2.2a}$$

$$\mathcal{L}_{\mathbf{p}} = \alpha E^T E (\mathbf{p} - \mathbf{p}_{ref}) + G^T \lambda = 0, \tag{2.2b}$$

$$\mathcal{L}_{\lambda} = A(\mathbf{p})\mathbf{u} - \mathbf{f} = 0, \tag{2.2c}$$

where

$$G = G(\mathbf{u}, \mathbf{p}) = \frac{\partial(A(\mathbf{p})\mathbf{u})}{\partial \mathbf{p}}.$$

To solve the nonlinear equations (2.2), Newton’s method can be used. Given the approximation $\mathbf{u}, \mathbf{p}, \lambda$, the Newton correction direction is the solution of the linear system

$$\begin{pmatrix} R_0^T R_0 & K & A(\mathbf{p})^T \\ K^T & \alpha E^T E + T & G^T \\ A(\mathbf{p}) & G & 0 \end{pmatrix} \begin{pmatrix} \delta \mathbf{u} \\ \delta \mathbf{p} \\ \delta \lambda \end{pmatrix} = - \begin{pmatrix} \mathcal{L}_{\mathbf{u}} \\ \mathcal{L}_{\mathbf{p}} \\ \mathcal{L}_{\lambda} \end{pmatrix}, \tag{2.3}$$

where

$$K = K(\mathbf{p}, \lambda) = \frac{\partial(A(\mathbf{p})^T \lambda)}{\partial \mathbf{p}}, \quad T = T(\mathbf{u}, \mathbf{p}, \lambda) = \frac{\partial(G^T \lambda)}{\partial \mathbf{p}}$$

are two matrices introduced as part of the second derivative information. The coefficient matrix in Eq. (2.3) is known as a saddle point (or KKT) matrix. The term of saddle point comes from the fact that a solution to Eq. (2.3) is a saddle point for the Lagrangian.

For the linear KKT system (2.3), we use the reduced Hessian methods to get the solution of it. That is, we apply a block elimination for $\delta \mathbf{u}$ and $\delta \lambda$. Firstly from the last block of rows of (2.3) we get

$$\delta \mathbf{u} = -A(\mathbf{p})^{-1}[\mathcal{L}_\lambda + G\delta \mathbf{p}]. \tag{2.4}$$

Next, we substitute $\delta \mathbf{u}$ in the first block rows to get

$$\delta \lambda = A(\mathbf{p})^{-T}[R_0^T R_0 A(\mathbf{p})^{-1}G - K]\delta \mathbf{p} - A(\mathbf{p})^{-T}[\mathcal{L}_\mathbf{u} - R_0^T R_0 A(\mathbf{p})^{-1}\mathcal{L}_\lambda]. \tag{2.5}$$

Finally, from the second block rows we obtain a linear system for $\delta \mathbf{p}$ alone:

$$H\delta \mathbf{p} = -\mathbf{q}, \tag{2.6}$$

where

$$H = H(\mathbf{u}, \mathbf{p}, \lambda) = M^T M + \alpha E^T E + T - S - S^T \tag{2.7}$$

with

$$M = M(\mathbf{u}, \mathbf{p}) = -R_0 A(\mathbf{p})^{-1}G, \tag{2.8a}$$

$$S = S(\mathbf{u}, \mathbf{p}, \lambda) = K^T A(\mathbf{p})^{-1}G, \tag{2.8b}$$

$$\mathbf{q} = \mathbf{q}(\mathbf{u}, \mathbf{p}, \lambda) = \alpha E^T E(\mathbf{p} - \mathbf{p}_{ref}) + M^T (R_0 A(\mathbf{p})^{-1}\mathbf{f} - \mathbf{u}_0) - K^T (\mathbf{u} - A(\mathbf{p})^{-1}\mathbf{f}). \tag{2.8c}$$

From (2.6) we can obtain $\delta \mathbf{p}$ and use it to update \mathbf{p} . Then we update \mathbf{u} by

$$\mathbf{u} = A(\mathbf{p})^{-1}\mathbf{f} \tag{2.9}$$

and solve λ from (2.2a)

$$\lambda = A(\mathbf{p})^{-T}R_0^T(\mathbf{u}_0 - R_0\mathbf{u}). \tag{2.10}$$

These formulas satisfy (2.2c) and (2.2a), respectively. Alternatively, the basic Newton step calculates $\delta \mathbf{p}$, $\delta \mathbf{u}$, $\delta \lambda$ from (2.6), (2.4) and (2.5) respectively, and updates \mathbf{p} , \mathbf{u} , λ simultaneously. These Newton method variants was proposed in [11, 18].

It is well-known that the Newton method for solving the nonlinear equation (2.2) is equivalent to an sequential quadratic programming (SQP) method (see [29]). The SQP method which converges globally and typically requires only a few iterations and function evaluations to locate a solution point, is one of the leading methods for solving constrained optimization problems. In fact, an SQP algorithm defines an appropriate direction $(\delta \mathbf{u}, \delta \mathbf{p})$ from the the current approximate point as the minimizer of a quadratic model of the objective subject to a linearization of the constraints:

$$\min_{\delta \mathbf{u}, \delta \mathbf{p}} \begin{pmatrix} \delta \mathbf{u} & \delta \mathbf{p} \end{pmatrix} \begin{pmatrix} R_0^T R_0 & K \\ K^T & \alpha E^T E + T \end{pmatrix} \begin{pmatrix} \delta \mathbf{u} \\ \delta \mathbf{p} \end{pmatrix} + \begin{pmatrix} \mathcal{L}_\mathbf{u} & \mathcal{L}_\mathbf{p} \end{pmatrix} \begin{pmatrix} \delta \mathbf{u} \\ \delta \mathbf{p} \end{pmatrix}, \tag{2.11a}$$

$$\text{s.t.} \quad \begin{pmatrix} A(\mathbf{p}) & G \end{pmatrix} \begin{pmatrix} \delta \mathbf{u} \\ \delta \mathbf{p} \end{pmatrix} = \mathbf{f} - A(\mathbf{p})\mathbf{u}. \tag{2.11b}$$

See for instance [29].

However, a step may move away from a minimizer, or may be trapped near a stationary point of the Lagrangian. Thus, merit functions more appropriate to constrained optimization should be considered. In this paper, we make use of the commonly used l_1 penalty function

$$\phi(\mathbf{u}, \mathbf{p}; \mu) = \frac{1}{2}\|R_0\mathbf{u} - \mathbf{u}_0\|^2 + \frac{\alpha}{2}\|E(\mathbf{p} - \mathbf{p}_{ref})\|^2 + \mu\|A(\mathbf{p})\mathbf{u} - \mathbf{f}\|_1,$$

where $\mu > 0$ known as the penalty parameter.

Upon the calculation and acceptance of the search direction $(\delta \mathbf{u}, \delta \mathbf{p})$ for a particular value μ_k of the penalty parameter, we perform a backtracking line search to compute a step length τ satisfying the Armijo condition, i.e.

$$\phi(\mathbf{u} + \tau \delta \mathbf{u}, \mathbf{p} + \tau \delta \mathbf{p}; \mu_k) \leq \phi(\mathbf{u}, \mathbf{p}; \mu_k) + \gamma \tau D(\phi(\mathbf{u}, \mathbf{p}; \mu_k))$$

for some $0 < \gamma < 1$, where $D(\phi(\mathbf{u}, \mathbf{p}; \mu_k))$ denotes the directional derivative of $\phi(\mathbf{u}, \mathbf{p}; \mu)$ in the direction $(\delta \mathbf{u}, \delta \mathbf{p})$.

In summary, our approach which solving the optimal problem (1.7) with fixed regular parameter α follows a standard line search SQP framework. In each iteration, a step is computed as a solution to the system (2.11) satisfying appropriate conditions that deem the step acceptable. The penalty parameter is then set based on properties of the computed step, after which a backtracking line search is performed to compute a step length τ satisfying the Armijo condition. Finally, the iteration is updated.

Because the matrix of the second derivative of the Lagrangian in the SQP formulation (2.11) may not be positive definite on the constraint null-space when the approximate solution is not close to the solution, the reduced Hessian matrix H (cf. (2.7)) is not positive definite. Thus computational difficulty can occur and special care is required, e.g. applying the trust region method to ensure the positive definiteness of the reduced Hessian.

Positive definiteness is immediately obtained by dropping the second derivative information which get the Gauss-Newton approximation. Thus, setting $K = 0$ and $T = 0$, we obtain also $S = 0$ in (2.8b) and

$$H_{GN}(\mathbf{u}, \mathbf{p}, \lambda) = M^T M + \alpha E^T E \tag{2.12}$$

is positive definite. The direction vector $\delta \mathbf{p} = \delta \mathbf{p}_{GN}$ is now the solution of the linear system

$$H_{GN} \delta \mathbf{p}_{GN} = -\mathbf{q}_{GN}, \tag{2.13}$$

where

$$\mathbf{q}_{GN} = \alpha E^T E(\mathbf{p} - \mathbf{p}_{ref}) + M^T (R_0 A(\mathbf{p})^{-1} \mathbf{f} - \mathbf{u}_0).$$

It is often the case in inverse problems that the cost of the formation of $M^T M$ is prohibitively large. Alternatively, we solve the least squares problem

$$\min_{\delta \mathbf{p}} \left\| \begin{pmatrix} M \\ \sqrt{\alpha} E \end{pmatrix} \delta \mathbf{p} + \begin{pmatrix} R_0 A(\mathbf{p})^{-1} \mathbf{f} - \mathbf{u}_0 \\ \sqrt{\alpha} E(\mathbf{p} - \mathbf{p}_{ref}) \end{pmatrix} \right\|_2^2 \tag{2.14}$$

by using the LSQR technique to obtain $\delta \mathbf{p}$; See for instance [29]. The explicit calculation of λ is now unnecessary as well, because it appears only in the discarded K and T .

2.1. Determination of regularization parameter

The methods developed in the previous section are for solving the nonlinear problem with a specific regularization parameter α . Different regularization parameters will result in different numerical solutions. Thus, in order to obtain a reasonable numerical solution, the optimization problem (1.7) or (1.6) must be solved several times for different regularization parameter values α . This contributes in a major way to the total cost of the solution process. A solution is accepted when some stopping criterion is satisfied. The simplest criterion is the discrepancy principle. That is, one seeks α such that

$$\|R_0 A(\mathbf{p})^{-1} \mathbf{f} - \mathbf{u}_0\|^2 \approx Tol^2, \tag{2.15}$$

for some specified tolerance level Tol which depends on the noise level and is assumed given.

In this paper, we apply simple continuation, or cooling [2, 16, 17], in order to find the regularization parameter. We start with a relative large value of α , for which we solve an almost quadratic problem, and we then gradually reduce α and solve each new problem with the solution for the previous α as the initial guess. The minimization process with a specific α is referred to as an outer iteration and the Newton or Gauss-Newton iteration within each outer iteration is referred to an inner iteration. The algorithm is terminated when (2.15) is reached.

In this work α was reduced to be 0.1 of its previous value upon convergence of the inner iteration. If that value was deemed unacceptable, α was increased by the formula $\alpha = \alpha + 0.5(\alpha_{pre} - \alpha)$.

3. Numerical results

In this section, we compare the methods presented in the previous section for solving the Robin inverse problem (1.7) on the ellipse $x_1^2/a^2 + x_2^2/b^2 \leq 1$ with $a = 1, b = 0.2$; see Appendix A. The parametrization for Γ is

$$(x_1, x_2) = (\phi(t), \psi(t)) = (a \cos(2\pi t), b \sin(2\pi t)) \quad \text{for } 0 \leq t \leq 1.$$

The segments Γ_0 and Γ_1 are chosen as

$$\Gamma_0 = \{x(t) : t \in [0.55, 0.85]\} \quad \text{and} \quad \Gamma_1 = \{x(t) : t \in [0.15, 0.45]\}.$$

As for the input function g , we choose it as

$$g(a \cos(2\pi t), b \sin(2\pi t)) = \begin{cases} 1 & \text{if } t \in [0.4, 0.6], \\ 0 & \text{elsewhere on } \Gamma. \end{cases}$$

We use two different profiles for the Robin coefficient $p(t)$ that have been tested in the literature (e.g. [25]). In our tests below, we first use the true $p(t)$ to obtain approximate values of the solution $u(t)$ at the grid points by solving a linear system corresponding to (2.9), i.e.,

$$A(\mathbf{p})\mathbf{u} = \mathbf{f}.$$

by using the Gaussian Elimination method. Then we generate the synthetic data \mathbf{u}_0 from $\mathbf{u}|_{\Gamma_0}$ with certain level δ of uniformly distributed random noise.

We choose $\mathbf{p}_{ref} = 0, E$ to be the discretized matrix of the first derivative, and the initial iteration guess \mathbf{p}_0 is to be the vector with all elements being 0.1. In order to evaluate an initial guess for the regularization parameter α , we estimate the largest generalized singular value of the sensitivity matrix $M(p_0)$ and the matrix E . We then set α such that the initial value, the leading term which corresponds to the misfit function in the Hessian, $M(p_0)^T M(p_0)$, is very small compared with the regularization term $\alpha E^T E$, i.e., set α ensure the condition:

$$\alpha E^T E + M(p_0)^T M(p_0) \approx \alpha E^T E.$$

We choose a regularization parameter $\alpha_0 = 3 \max(\text{svd}(M(p_0)))$.

In the tests, we compare the following factors: The total number of inner iterations which were needed in order to converge. This number is given by the sum of all iterations for different α . The number of α values (outer iterations) which were needed to achieve convergence within the tolerance. The stopping criterion for all algorithms is that the norm of the gradient is not more than 10^{-6} times that of the initial step.

The recovered profiles are shown in Figure 1. We can see that one can obtain a satisfiable profile by using the Newton method or the Newton-Gauss method.

To compare the number of iterations required by each method, we use noisy data with two different noisy levels, namely, 2% (which implies a low residual problem) and for 20% (which implies large residuals). The results of the two-hump profile are presented in Table 1 and those of the smooth single-hump case are presented in Table 2. The results in Tables 1–2 indicate that when the noise is low, both the Gauss-Newton method and the Newton method perform well. In the other hand, if the noise is large, the Newton method converges much faster than the Gauss-Newton method. This is expected because for large residual problems the second order terms are important to achieve rapid convergence.

Noise=2%	Outer	Total	Noise=20%	Outer	Total
N	7	46	N	7	44
GN	8	41	GN	7	125

Table 1: The number of iterations required by the Newton method and the Gauss-Newton method (two-hump case).

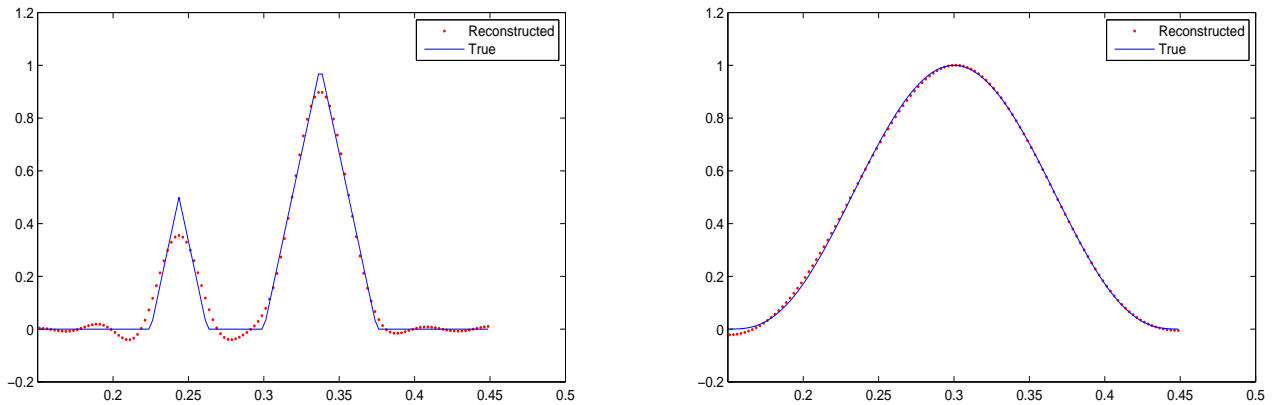


Figure 1: The reconstructed Robin coefficient with noise level $\delta = 10\%$.

Noise=2%	Outer	Total	Noise=20%	Outer	Total
N	6	31	N	6	30
GN	6	34	GN	6	73

Table 2: The number of iterations required by the Newton method and the Gauss-Newton method (single-hump case).

Finally, we plot the convergence process of the last outer iteration for the case of the two-hump profile with 20% of noise is added to the input data; see Figure 2. We observe that while the Newton method converges in 3 iterations and displays a quadratic rate of convergence, the Gauss-Newton method requires 6 iterations and displays linear convergence, which is typical to the whole process.

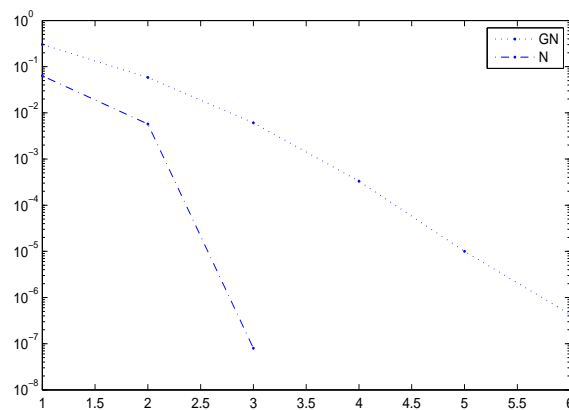


Figure 2: The norm of the gradient for the Full Newton and Gauss-Newton method for the last α

Appendix A. The discretization of the integral operators

For the sake of completeness, we introduce the parametrization of Γ and the discretization of relevant operators, which were presented in [25]. We use the mid-point quadrature rule to discretize the integral operators and central difference quotients to approximate derivatives.

Suppose the boundary Γ of a planar domain Ω has a regular parametrization given by

$$(x_1, x_2) = (\phi(t), \psi(t)) \text{ for } 0 \leq t \leq L,$$

with $(\phi(0), \psi(0)) = (\phi(L), \psi(L))$. Assume also that $\phi(\cdot)$ and $\psi(\cdot)$ are both \mathcal{C}^2 functions.

Define

$$K_d(t, s) = \frac{1}{2\pi} \frac{\psi'(s)(\phi(t) - \phi(s)) - \phi'(s)(\psi(t) - \psi(s))}{(\phi(s) - \phi(t))^2 + (\psi(s) - \psi(t))^2}$$

and

$$K_s(t, s) = -\frac{1}{2\pi} \left(\ln \sqrt{(\phi(s) - \phi(t))^2 + (\psi(s) - \psi(t))^2} \right) \sqrt{(\phi'(s))^2 + (\psi'(s))^2}$$

for $x = (\phi(t), \psi(t))$ and $y = (\phi(s), \psi(s))$ on Γ with $x \neq y$ ($t \neq s$). We can write the operators \mathcal{D} and \mathcal{S} as

$$\begin{aligned} (\mathcal{D}u)(t) &= \int_0^L K_d(t, s)u(s)ds, \\ (\mathcal{S}u)(t) &= \int_0^L K_s(t, s)u(s)ds, \end{aligned}$$

where $u(s) = u(\phi(s), \psi(s))$.

In our numerical tests, we choose the ellipse domain $x_1^2/a^2 + x_2^2/b^2 \leq 1$ with $a = 1, b = 0.2$. The usual parametrization for Γ is

$$(x_1, x_2) = (\phi(t), \psi(t)) = (a \cos(2\pi t), b \sin(2\pi t)) \quad \text{for } 0 \leq t \leq 1.$$

In this case, the kernel in the integral operator \mathcal{D} is

$$K_d(t, s) = -\frac{ab}{2(a^2 \sin^2(\pi(t+s)) + b^2 \cos^2(\pi(t+s)))}.$$

The kernel in the integral operator \mathcal{S} is weakly singular at $s = t$ and $(t, s) = (0, 1)$ and $(1, 0)$, special case should be taken in discretizing $\int_0^1 K_s(t, s)u(s)ds$ to avoid large errors. By decomposing $K_s(t, s)$ as

$$K_s(t, s) = (K_{s1}(t, s) + K_{s2}(t, s))K_{s3}(t, s),$$

where

$$\begin{aligned} K_{s1}(t, s) &= \ln(2|\sin(\pi(t-s))|), \\ K_{s2}(t, s) &= \ln \sqrt{a^2 \sin^2(\pi(t+s)) + b^2 \cos^2(\pi(t+s))}, \\ K_{s3}(s) &= -\sqrt{a^2 \sin^2(2\pi(s)) + b^2 \cos^2(2\pi(s))}, \end{aligned}$$

and using the singularity subtraction technique (see for instance [10]), the integrals involving singularity can be evaluated exactly.

Let the interval $[0, 1]$ be partitioned into n uniform subintervals: $[(i-1)h, ih], i = 1, 2, \dots, n$, where $h = 1/n$. Then the quadrature points are $t_i = (i-1/2)h, i = 1, 2, \dots, n$. Suppose

$$\{t_i\}_{i=1}^n \cap \{t : (\phi(t), \psi(t)) \in \Gamma_0\} = \{t_{m_1+1}, \dots, t_{m_2}\}$$

and

$$\{t_i\}_{i=1}^n \cap \{t : (\phi(t), \psi(t)) \in \Gamma_1\} = \{t_{m_3+1}, \dots, t_{m_4}\}.$$

Denote by \mathbf{p} and \mathbf{u} discretized functions of $p(t)$ on Γ_1 , and $u(t)$ on Γ , respectively:

$$\mathbf{p} = [p(t_{m_3+1}), \dots, p(t_{m_4})]^T, \quad \mathbf{u} = [p(t_1), \dots, p(t_n)]^T,$$

and the discrete data

$$\mathbf{u}_0 = [u_0(t_{m_1+1}), \dots, u_0(t_{m_2})]^T, \quad \mathbf{g} = [g(t_1), \dots, g(t_n)]^T.$$

Let $D = h[K_d(t_i, t_j)]_{i,j=1}^n$ and $S = h[K_s(t_i, t_j)]_{i,j=1}^n$, then the matrix in the discretize form of the operator equation (1.3) is given by

$$A(\mathbf{p}) = \frac{1}{2}I + D + S_1P$$

where $S_1 = S(:, m_3 + 1 : m_4)$ and $P = \text{diag}(\mathbf{p}(m_3 + 1 : m_4))$.

The matrix $G = \frac{\partial A(\mathbf{p})u}{\partial \mathbf{p}}$ is easily obtained by the differentiating the product $S(\mathbf{p})\mathbf{u}$ with respect to \mathbf{p} . It is easy to show that G is given by

$$G = S_1U,$$

where $S_1 = S(:, m_3 + 1 : m_4)$ and $U = \text{diag}(u(m_3 + 1 : m_4))$. Similarly, we can obtain that the matrix $K = \frac{\partial A^T \lambda}{\partial \mathbf{p}}$ is a sparsity matrix which the elements of the $m_3 + 1$ diagonal are that of $S_1^T \lambda$ and the others vanish. the matrix $T = \frac{\partial G^T \lambda}{\partial \mathbf{p}}$ is a matrix which the elements are all zeros because G is independent of \mathbf{p} .

Acknowledgements

This research was supported by NSF of China No. 11271238.

References

- [1] G. Alessandrini, L. Del Piero, L. Rondi, *Stable determination of corrosion by single electrostatic boundary measurement*, Inverse Problems, **19** (2003), 973–984.1
- [2] U. Ascher, R. Mattheij, R. Russell, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, SIAM, Philadelphia, (1995).2.1
- [3] K. Atkinson, *The Numerical Solution of Integral Equation of Second Kind*, Cambridge University Press, Cambridge, (1997).1
- [4] S. Busenberg, W. Fang, *Identification of semiconductor contact resistivity*, Quart. Appl. Math., **49** (1991), 639–649.1
- [5] S. Chaabane, C. Elhechmi, M. Jaoua, *A stable recovery method for the Robin inverse problem*, Math.Comput. Simulation, **66** (2004), 367–383.1
- [6] S. Chaabane, I. Feki, N. Mars, *Numerical reconstruction of a piecewise constant Robin parameter in the two- or three-dimensional case*, Inverse Problems, **28** (2012), 19 pages.1
- [7] S. Chaabane, I. Fellah, M. Jaoua, J. Leblond, *Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems*, Inverse Problems, **20** (2004), 47–59.1
- [8] S. Chaabane, J. Ferchichi, K. Kunisch, *Differentiability properties of L^1 -tracking functional and application to the Robin inverse problem*, Inverse Problems, **20** (2004), 1083–1097.1
- [9] S. Chaabane, M. Jaoua, *Identification of Robin coefficients by means of boundary measurements*, Inverse Problems, **15** (1999), 1425–1438.1
- [10] L. M. Delves, J. L. Mohamed, *Computational Methods for Integral Equations*, Cambridge University Press, Cambridge, (1985).Appendix A
- [11] J. E. Dennis, M. Heinkenschloss, L. N. Vicente, *Trust-region interior-point SQP algorithms for a class of nonlinear programming problems*, SIAM J. Control Optim., **36** (1998), 1750–1794.2
- [12] W. Fang, E. Cumberbatch, *Inverse problems for metal oxide semiconductor field-effect transistor contact resistivity*, SIAM J. Appl. Math., **52** (1992), 699–709.1
- [13] W. Fang, X. Zeng, *A direct solution of the Robin inverse problem*, J. Integral Equations Appl., **21** (2009), 545–557.1
- [14] D. Fasino, G. Inglese, *An inverse Robin problem for Laplace's equation: theoretical results and numerical methods*, Inverse Problems, **15** (1999), 41–48.1
- [15] D. Fasino, G. Inglese, *Discrete methods in the study of an inverse problem for Laplace's equation*, SIAM J. Numer. Anal., **19** (1999), 105–118.1
- [16] E. Haber, *Numerical strategies for the solution of inverse problems*, PhD thesis, University of British Columbia, (1998).2.1
- [17] E. Haber, U. M. Ascher, D. Oldenburg, *On optimization techniques for solving nonlinear inverse problems*, Inverse problems, **16** (2000), 1263–1280.2.1

-
- [18] M. Heinkenschloss, *Mesh independence for nonlinear least squares problems with norm constraints*, SIAM J. Optim., **3** (1993), 81–117.2
 - [19] G. Inglese, *An inverse problem in corrosion detection*, Inverse Problems, **13** (1997), 977–994. 1
 - [20] B. Jin, *Conjugate gradient method for the Robin inverse problem associated with the Laplace equation*, Internat. J. Numer. Methods Engrg., **71** (2007), 433–453.1
 - [21] B. Jin, J. Zou, *Inversion of Robin coefficient by a spectral stochastic finite element approach*, J. Comput. Phys., **227** (2008), 3282–3306.1
 - [22] B. Jin, J. Zou, *Numerical estimation of piecewise constant Robin coefficient*, SIAM J. Control Optim., **48** (2009), 1977–2002.1
 - [23] P. G. Kaup, F. Santosa, *Nondestructive evaluation of corrosion damage using electrostatic measurements*, J. Nondestruct. Eval., **14** (1995), 127–136.1
 - [24] R. Kress, *Linear Integral Equations*, Springer, New York (1999).1
 - [25] F. Lin, W. Fang, *A linear integral equation approach to the Robin inverse problem*, Inverse Problems, **21** (2005), 1757–1772.1, 3, Appendix A
 - [26] W. H. Loh, K. Saraswat, R. W. Dutton, *Analysis and scaling of Kelvin resistors for extraction of specific contact resistivity*, IEEE Electron. Device Lett., **6** (1985), 105–108.1
 - [27] W. H. Loh, S. E. Swirhun, T. A. Schreyer, R. M. Swanson, K. C. Saraswat, *Modeling and measurement of contact resistances*, IEEE Transac. Elect. Dev., **34** (1987), 512–524.1
 - [28] V. G. Maz'ya, *Boundary integral equations analysis IV*, Springer, New York, (1991).1
 - [29] J. Nocedal, S. Wright, *Numerical Optimization*, Springer, New York, (2006).2, 2, 2
 - [30] F. Santosa, M. Vogelius, J. M. Xu, *An effective nonlinear boundary condition for corroding surface identification of damage based on steady state electric data*, Z. Angew. Math. Phys., **49** (1998), 656–679. 1