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Evolutes of fronts on Euclidean 2-sphere

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Abstract

We define framed curves (or frontals) on Euclidean 2-sphere, give a moving frame of the framed curve and define a pair of smooth functions as the geodesic curvature of a regular curve. It is quite useful for analysing curves with singular points. In general, we can not define evolutes at singular points of curves on Euclidean 2-sphere, but we can define evolutes of fronts under some conditions. Moreover, some properties of such evolutes at singular points are given.

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1. Introduction

Singularity theory is a developing area which is related to nonlinear sciences. In particular, it has been extensively applied in studying classifications of singularities associated with some objects in Euclidean and semi-Euclidean spaces [14, 15]. In this paper, we focus on the evolutes of curves at singular points on 2-sphere.

The evolute of a plane curve is defined to be the locus of the center of its osculating circles. The evolute of a spherical curve is defined to be the locus of the center of its osculating spheres [4, 11, 12]. In particular, the evolute of a regular curve is a classical object from the view point of differential geometry. The evolute of a regular curve without inflection points is given by not only the locus of all its centres of curvature, but also the envelope of its normal lines [6, 7, 10]. The properties of evolutes can be discussed by Frenet-Serret formulas, distance squared functions and the theories of Lagrangian and Legendre singularities [1, 2, 3, 13].

In general, there exist singular points along the evolute of a regular curve. And the singular points correspond to vertices of the regular curve. There are at least four vertices for a simple closed curve. One

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can not define the evolutes of curves at singular points. However we can define evolutes of fronts under conditions. In [8, 9], T. Fukunaga and M. Takahashi defined Legendre curves in Euclidean plane and studied evolutes of Legendre curves. We also do some work associated with the evolutes of fronts in hyperbolic plane [5]. In this paper, we define framed curves on Euclidean 2-sphere and research the evolutes of fronts.

In section 2 we define a framed curve, and give a moving frame of the framed curve (or, frontal) on the Euclidean 2-sphere. Moreover, we define a pair of smooth functions of framed curves as a geodesic curvature for a regular curve. It is quite useful for analysing framed curves. We define evolutes of fronts on Euclidean 2-sphere. The evolute of a front is a generalisation of the notion of a evolute of a regular curve. In section 3, we discuss some properties of evolutes without inflection points. By the representation, we give properties for an evolute of the front. For example, the evolute of a front is also a front, see Theorem 3.3. Moreover, we extend the notion of the vertex for a front and give a kind of four vertices theorem for a front, see Proposition 3.10. It follows that we can consider the repeated evolute, namely, the evolute of an evolute of a front, see Theorem 4.1.

We shall assume throughout the whole paper that all manifolds and maps are C^∞ unless the contrary is explicitly stated.

2. Definitions and basis concepts

Let $\mathbb{R}^3$ be a 3-dimensional vector space. For any $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, their scalar product is defined by $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1+x_2y_2+x_3y_3$. Euclidean 2-sphere is denoted by $\mathrm{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 | \langle \mathbf{x}, \mathbf{x}\rangle = 1\}$.

Let $\boldsymbol{\gamma} : I \to \mathrm{S}^2$ be a regular curve on S^2 (i.e. $\dot{\boldsymbol{\gamma}}(t) = d\boldsymbol{\gamma}/dt \neq 0$), where I is an open interval. The norm of the vector $\mathbf{x} \in \mathbb{R}^3$ is defined by $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}$. We can take s as the arc-length parameter of $\boldsymbol{\gamma}$ satisfying $\|\boldsymbol{\gamma}'(s)\| = 1$, and take t as the general parameter of $\boldsymbol{\gamma}$. Then we have the tangent vector $\mathbf{t}(s) = \boldsymbol{\gamma}'(s)$, obviously $\|\mathbf{t}(s)\| = 1$, $\mathbf{e}(s) = \boldsymbol{\gamma}(s) \wedge \mathbf{t}(s)$. Then we have an orthonormal frame $\{\boldsymbol{\gamma}(s), \mathbf{t}(s), \mathbf{e}(s)\}$ of S^2 along $\boldsymbol{\gamma}$. By directly calculating, the following Frenet-Serret formula is displayed

$$\begin{cases} \boldsymbol{\gamma}'(s) = \mathbf{t}(s) \\ \mathbf{t}'(s) = -\boldsymbol{\gamma}(s) + \kappa_g(s)\mathbf{e}(s) \\ \mathbf{e}'(s) = -\kappa_g(s)\mathbf{t}(s). \end{cases} \tag{2.1}$$

Here $\kappa_g(s)$ is a geodesic curvature of $\boldsymbol{\gamma}$ on S^2, which is given by $\kappa_g(s) = \det\,(\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{t}'(s))$.

Even if t is not the arc-length parameter, we have the unit tangent vector $\boldsymbol{t}(t) = \dot{\boldsymbol{\gamma}}(t)/\|\dot{\boldsymbol{\gamma}}(t)\|$, the unit normal vector $\boldsymbol{e}(t) = \boldsymbol{\gamma}(t) \wedge \boldsymbol{t}(t)$ and the Frenet formula

$$\begin{cases} \dot{\boldsymbol{\gamma}}(t) = \|\dot{\boldsymbol{\gamma}}(t)\|\mathbf{t}(t) \\ \dot{\mathbf{t}}(t) = \|\dot{\boldsymbol{\gamma}}(t)\|(-\boldsymbol{\gamma}(t) + \kappa_g(t)\mathbf{e}(t)) \\ \dot{\mathbf{e}}(t) = -\|\dot{\boldsymbol{\gamma}}(t)\|\kappa_g(t)\mathbf{t}(t). \end{cases} \tag{2.2}$$

Here $\dot{\boldsymbol{\gamma}}(t) = d\boldsymbol{\gamma}/dt(t)$, $\|\dot{\boldsymbol{\gamma}}(t)\| = \sqrt{\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle}$ and $\kappa_g(t) = \det(\boldsymbol{\gamma}(t), \boldsymbol{t}(t), \dot{\boldsymbol{t}}(t))/\|\dot{\boldsymbol{\gamma}}(t)\|^3$. Note that $\kappa_g(t)$ is independent on the choice of a parametrisation.

Definition 2.1. We say that $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : I \to \mathrm{S}^2\times\mathrm{S}^2$ is a framed curve on S^2, if $\langle \dot{\boldsymbol{\gamma}}(t), \boldsymbol{\nu}(t)\rangle = 0$ and $\langle \boldsymbol{\gamma}(t), \boldsymbol{\nu}(t)\rangle = 0$ for all $t \in I$. Moreover, if $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ is an immersion, namely, $(\dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\nu}}(t)) \neq (0, 0)$, we call $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ a framed immersion.

Definition 2.2. We say that $\boldsymbol{\gamma} : I \to \mathrm{S}^2$ is a frontal if there exists a smooth mapping $\boldsymbol{\nu} : I \to \mathrm{S}^2$ such that $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ ia a framed curve. We also say that $\boldsymbol{\gamma} : I \to \mathrm{S}^2$ is a front or a wave front if there exists a smooth mapping $\boldsymbol{\nu} : I \to \mathrm{S}^2$ such that $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ is a framed immersion.

We put on $\boldsymbol{\mu}(t) = \boldsymbol{\nu}(t) \wedge \boldsymbol{\gamma}(t)$. We call the pair $\{\boldsymbol{\gamma}(t), \boldsymbol{\mu}(t), \boldsymbol{\nu}(t)\}$ is a moving frame of a frontal $\boldsymbol{\gamma}$. Then we have the Frenet formula of a frontal $\boldsymbol{\gamma}$ as follows

$$\begin{cases} \dot{\boldsymbol{\gamma}}(t) = \beta(t)\boldsymbol{\mu}(t) \\ \dot{\boldsymbol{\mu}}(t) = -\beta(t)\boldsymbol{\gamma}(t) - l(t)\boldsymbol{\nu}(\boldsymbol{t}) \\ \dot{\boldsymbol{\nu}}(t) = l(t)\boldsymbol{\mu}(t), \end{cases} \tag{2.3}$$

where $l(t) = \langle \dot{\boldsymbol{\nu}}(t), \boldsymbol{\mu}(t)\rangle$. If $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ is a framed immersion, we have $l(t), \beta(t) \neq (0,0)$ for each $t \in I$. The pair $(l(t), \beta(t))$ is an important pair of functions of framed curves like the geodesic curvature of a regular curve. We call the pair $(l(t), \beta(t))$ the geodesic curvature of the framed curve.

Example 2.3. Let $\boldsymbol{\gamma} : I \to \mathrm{S}^2$ be a regular curve,

$$\boldsymbol{\gamma}(t) = (\cos(t^2)\cos(t^3),\ \sin(t^2)\cos(t^3),\ \sin(t^3)). \tag{2.4}$$

We get

$$\dot{\boldsymbol{\gamma}}(t) = (-2\sin(t^2)\cos(t^3)t - 3\cos(t^2)\sin(t^3)t^2,\ 2\cos(t^2)\cos(t^3)t - 3\sin(t^2)\sin(t^3)t^2,\ 3\cos(t^3)t^2). \tag{2.5}$$

So $\boldsymbol{\gamma}$ is singular at $t = 0$. Take $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$, where

$$\begin{aligned} \nu_1(t) &= P(2t\sin(t^3)\cos(t^2)\cos(t^3) - 3t^2\sin(t^2)),\\ \nu_2(t) &= P(3t^2\cos(t^2) + 2t\sin(t^2)\sin(t^3)\cos(t^3)),\\ \nu_3(t) &= P(-2\cos^2(t^3)). \end{aligned} \tag{2.6}$$

Here, $P = 1/\sqrt{(2t\sin(t^3)\cos(t^2)\cos(t^3) - 3t^2\sin(t^2))^2 + (3t^2\cos(t^2) + 2t\sin(t^2)\sin(t^3)\cos(t^3))^2 + 4\cos^4(t^3)}$.
It satisfies $\langle \boldsymbol{\gamma}(t), \boldsymbol{\nu}(t)\rangle = \langle \dot{\boldsymbol{\gamma}}(t), \boldsymbol{\nu}(t)\rangle = 0$ and $\langle \boldsymbol{\nu}(t), \boldsymbol{\nu}(t)\rangle = 1$. Hence, $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ is a framed curve. See Fig.1.

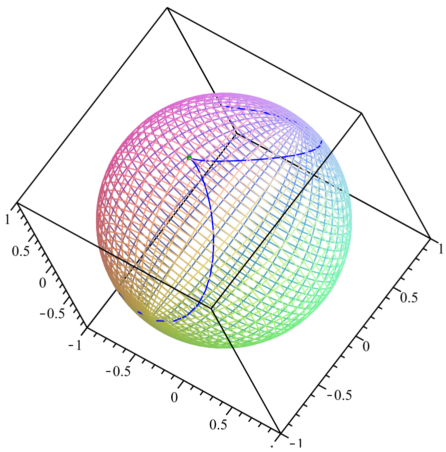

Fig. 1: framed curve

In this paper, we consider evolutes of curves on S^2. Let $E_{\boldsymbol{\gamma}}(t) : I \to \mathrm{S}^2$ be given by

$$E_{\boldsymbol{\gamma}}(t) = \frac{1}{\sqrt{1 + \kappa_g^2(t)}}(\kappa_g(t)\boldsymbol{\gamma}(t) + \boldsymbol{e}(t)). \tag{2.7}$$

Example 2.4. Let $\boldsymbol{\gamma} : I \to \mathrm{S}^2$ be a regular curve, $\boldsymbol{\gamma}(t) = (\cos^2(t), \cos(t)\sin(t), \sin(t))$, since $\dot{\boldsymbol{\gamma}}(t) \neq 0$.
We get $\kappa_g(t) = \sin(t)(\cos^2(t) + 2)/\sqrt{(1 + \cos^2(t))^3}$. Take $E_{\boldsymbol{\gamma}} = (E_1,\ E_2,\ E_3)$, we have

$$\begin{aligned} E_1(t) &= \frac{1}{\sqrt{(T(\cos^2(t) + 2)/P^2) + 1}}\left(\frac{T}{P}\cos^2(t) + \frac{1}{P^{1/3}}\sin^3(t)\right),\\ E_2(t) &= \frac{1}{\sqrt{(T(\cos^2(t) + 2)/P^2) + 1}}\left(\frac{T}{P}\cos(t)\sin(t) - \frac{1}{P^{1/3}}\cos(t)(1 + \sin^2(t))\right),\\ E_3(t) &= \frac{1}{\sqrt{(T(\cos^2(t) + 2)/P^2) + 1}}\left(\frac{T}{P}\sin(t) + \frac{1}{P^{1/3}}\cos^2(t)\right). \end{aligned} \tag{2.8}$$

Here $T = \sin(t)(\cos^2(t) + 2)$, $P = \sqrt{(1 + \cos^2(t))^3}$. See Fig.2. The blue part is $\boldsymbol{\gamma}$, and the red part is $E_{\boldsymbol{\gamma}}$.

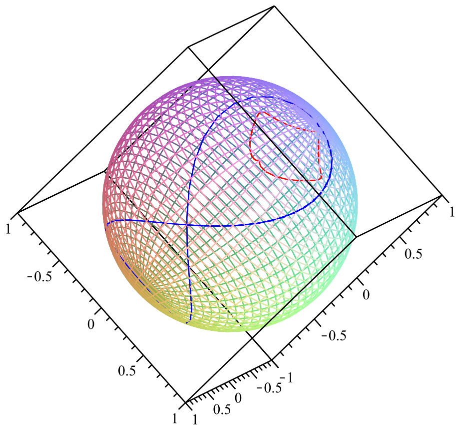

Fig. 2: regular curve and its evolute

If $\boldsymbol{\gamma}$ is not a regular curve, then we can not define the evolute as above, since the geodesic curvature may be divergence at a singular point. However, we can define an evolute of a front on S^2, see Definition 2.8 and Theorem 3.3. It is a generalisation of the evolute of a regular curve on S^2.

Now, we give the definition of the evolute of a front. First, we introduce the notion of parallel curve of $\boldsymbol{\gamma}$.

Let $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : I \to \mathrm{S}^2 \times \mathrm{S}^2$ be a framed immersion. We define a parallel curve $\boldsymbol{\gamma}_\lambda : I \to \mathrm{S}^2$ of $\boldsymbol{\gamma}$ by

$$\boldsymbol{\gamma}_\lambda(t) = \frac{\boldsymbol{\gamma}(t) + \lambda \boldsymbol{\nu}(t)}{\sqrt{1+\lambda^2}} \tag{2.9}$$

for each $\lambda \in \mathbb{R}$. Then we have following results.

Proposition 2.5. *For a framed immersion $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : I \to \mathrm{S}^2 \times \mathrm{S}^2$, the parallel curve $\boldsymbol{\gamma}_\lambda : I \to \mathrm{S}^2$ is a front for each $\lambda \in \mathbb{R}$.*

Proof. We take $\boldsymbol{\nu}_\lambda : I \to \mathrm{S}^2$ by

$$\boldsymbol{\nu}_\lambda(t) = \frac{-\lambda \boldsymbol{\gamma}(t) + \boldsymbol{\nu}(t)}{\sqrt{1+\lambda^2}}. \tag{2.10}$$

Since

$$\boldsymbol{\gamma}_\lambda(t) = \frac{\boldsymbol{\gamma}(t) + \lambda \boldsymbol{\nu}(t)}{\sqrt{1+\lambda^2}}, \quad \dot{\boldsymbol{\gamma}}_\lambda(t) = \frac{\dot{\boldsymbol{\gamma}}(t) + \lambda \dot{\boldsymbol{\nu}}(t)}{\sqrt{1+\lambda^2}}. \tag{2.11}$$

If $\dot{\boldsymbol{\gamma}}_\lambda(t_0) = 0$ at a point $t_0 \in I$, then we have

$$\dot{\boldsymbol{\gamma}}(t_0) + \lambda \dot{\boldsymbol{\nu}}(t_0) = 0. \tag{2.12}$$

If $\dot{\boldsymbol{\nu}}(t_0) = 0$, then $\dot{\boldsymbol{\gamma}}(t_0) = 0$. It is contradict from the fact that $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ is a framed immersion and hence $(\boldsymbol{\gamma}_\lambda, \boldsymbol{\nu}_\lambda)$ is a framed immersion. By $\|\boldsymbol{\nu}(t)\| = 1$, we have $\langle \boldsymbol{\nu}(t), \dot{\boldsymbol{\nu}}(t) \rangle = 0$. Then

$$\langle \dot{\boldsymbol{\gamma}}_\lambda(t), \boldsymbol{\nu}_\lambda(t) \rangle = \frac{1}{1+\lambda^2} \langle \dot{\boldsymbol{\gamma}}(t) + \lambda \dot{\boldsymbol{\nu}}(t), -\lambda \boldsymbol{\gamma}(t) + \boldsymbol{\nu}(t) \rangle = 0. \tag{2.13}$$

It follows that $(\boldsymbol{\gamma}_\lambda, \boldsymbol{\nu}_\lambda)$ is a framed immersion and hence $\boldsymbol{\gamma}_\lambda$ is a front. □

We denote the geodesic curvature of the parallel curve $\boldsymbol{\gamma}_\lambda$ by $\kappa_{g\lambda}$, when $\boldsymbol{\gamma}_\lambda$ is a regular curve.

Proposition 2.6. *Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be a framed immersion. If $\boldsymbol{\gamma}$ is a regular curve and $\lambda \neq 1/\kappa_g(t)$, then the parallel curve $\boldsymbol{\gamma}_\lambda$ is a also regular curve and $E_{\gamma_\lambda}(t)$ is consistent with $E_\gamma(t)$.*

Proof. Since

$$\boldsymbol{\gamma}_\lambda(t) = \frac{\boldsymbol{\gamma}(t) + \lambda \boldsymbol{e}(t)}{\sqrt{1+\lambda^2}}, \quad \dot{\boldsymbol{\gamma}}_\lambda(t) = \frac{|\dot{\boldsymbol{\gamma}}(t)|(1-\lambda\kappa_g(t))\boldsymbol{t}(t)}{\sqrt{1+\lambda^2}}. \tag{2.14}$$

By the assumption $\lambda \neq 1/\kappa_g(t)$, $\boldsymbol{\gamma}_\lambda$ is a regular curve. By a direct calculation, we have

$$\kappa_{g\lambda}(t) = \frac{\lambda + \kappa_g(t)}{|1-\lambda\kappa_g(t)|}, \quad \boldsymbol{e}_\lambda(t) = \frac{1-\lambda\kappa_g(t)}{|1-\lambda\kappa_g(t)|}\frac{\boldsymbol{e}(t) - \lambda\boldsymbol{\gamma}(t)}{\sqrt{1+\lambda^2}}. \tag{2.15}$$

Hence

$$\begin{aligned} E_{\boldsymbol{\gamma}_\lambda}(t) &= \frac{\kappa_{g\lambda}(t)\boldsymbol{\gamma}_\lambda(t) + \boldsymbol{e}_\lambda(t)}{\sqrt{1+\kappa_{g\lambda}(t)^2}} \\ &= \frac{1}{\sqrt{1+(\frac{\lambda+\kappa_g}{|1-\lambda\kappa_g|})^2}}\Big(\frac{\lambda+\kappa_g(t)}{|1-\lambda\kappa_g(t)|}\frac{\boldsymbol{\gamma}(t)+\lambda\boldsymbol{e}(t)}{\sqrt{1+\lambda^2}} + \frac{1-\lambda\kappa_g(t)}{|1-\lambda\kappa_g(t)|}\frac{\boldsymbol{e}(t)-\lambda\boldsymbol{\gamma}(t)}{\sqrt{1+\lambda^2}}\Big) \\ &= \frac{1}{\sqrt{(1+\kappa_g(t)^2)(1+\lambda^2)}}\Big(\frac{1+\lambda^2}{\sqrt{1+\lambda^2}}\kappa_g(t)\boldsymbol{\gamma}(t) + \frac{1+\lambda^2}{\sqrt{1+\lambda^2}}\boldsymbol{e}(t)\Big) \\ &= \frac{\kappa_g\boldsymbol{\gamma}(t)+\boldsymbol{e}(t)}{\sqrt{1+\kappa_g(t)^2}} \\ &= E_{\boldsymbol{\gamma}}(t). \end{aligned} \tag{2.16}$$

Remark 2.7. Let$(\boldsymbol{\gamma},\boldsymbol{\nu})$ be a framed immersion. If t_0 is a singular point of the front $\boldsymbol{\gamma}$, then $\lim\limits_{t\to t_0}|\kappa_g(t)| = \infty$. By the equality $\kappa_{g\lambda}(t) = \lambda - \kappa_g(t)/|1-\lambda\kappa_g(t)|$, we have $\lim_{t\to t_0}\kappa_{g\lambda}(t) \neq 0$.

We now define an evolute of a front on the S^2.

Definition 2.8. Let $(\boldsymbol{\gamma},\boldsymbol{\nu}) : I \to \mathrm{S}^2 \times \mathrm{S}^2$ be a framed immersion. We define an evolute $\mathcal{E}_{\boldsymbol{\gamma}}(t) : I \to \mathrm{S}^2$ of the front $\boldsymbol{\gamma}$ as follows. If t is a regular point,

$$\mathcal{E}_{\boldsymbol{\gamma}}(t) = \frac{\kappa_g\boldsymbol{\gamma}(t)+\boldsymbol{e}(t)}{\sqrt{1+\kappa_g(t)^2}}, \tag{2.17}$$

if $t \in (t_0-\delta, t_0+\delta)$, t_0 is a singular point of $\boldsymbol{\gamma}$,

$$\mathcal{E}_{\boldsymbol{\gamma}}(t) = \frac{\kappa_{g\lambda}(t)\boldsymbol{\gamma}_\lambda(t)+\boldsymbol{e}_\lambda(t)}{\sqrt{1+\kappa_{g\lambda}(t)^2}}, \tag{2.18}$$

where δ is a sufficiently small positive real number and $\lambda \neq 1/\kappa_g(t)$.

By the assumption of the finiteness of singularities of a front, there exists $\lambda \in \mathbb{R}$ with the condition $\lambda \neq 1/\kappa_g(t)$. Moreover, by Proposition 2.6, we can glue on the regular interval of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_\lambda$. Then the evolute of a front is well-defined. Furthermore, by definition, the evolute of a front $\mathcal{E}_{\boldsymbol{\gamma}}$ is a $\mathcal{C}^\infty$ mapping.

3. Properties of evolutes of fronts

In this section, we consider properties of the evolute of fronts. Let $(\boldsymbol{\gamma},\boldsymbol{\nu}) : I \to \mathrm{S}^2 \times \mathrm{S}^2$ be a framed immersion with the geodesic curvature (l,β).

First we give a relationship between the geodesic curvature $(l(t),\beta(t))$ of the framed immersion and the geodesic curvature $\kappa_g(t)$, if $\boldsymbol{\gamma}$ is a regular curve.

Lemma 3.1. 1. *If $\boldsymbol{\gamma}$ is a regular curve, then $l(t) = |\beta(t)|\kappa_g(t)$.*

2. *If $\boldsymbol{\gamma}_\lambda$ is a regular curve, then $\lambda\beta(t) - l(t) = |\beta(t)+\lambda l(t)|\kappa_{g\lambda}(t)$.*

Proof. (1) By a direct calculation, $\dot{\boldsymbol{\gamma}}(t) = \beta(t)\boldsymbol{\mu}(t)$, $\ddot{\boldsymbol{\gamma}}(t) = \dot{\beta}(t)\boldsymbol{\mu}(t) - \beta(t)^2\boldsymbol{\gamma}(t) - \beta(t)l(t)\boldsymbol{\nu}(t)$, and

$$\kappa_g(t) = \frac{1}{\|\dot{\boldsymbol{\gamma}}(t)\|^3}(\boldsymbol{\gamma}(t), \dot{\boldsymbol{\gamma}}(t), \ddot{\boldsymbol{\gamma}}(t)) = \frac{l(t)}{|\beta(t)|}. \tag{3.1}$$

(2) We can also prove by the same calculation of (1). □

Remark 3.2. Since $(l(t), \beta(t)) \neq (0, 0)$, if t_0 is a singular point of $\boldsymbol{\gamma}$, then $\boldsymbol{\gamma}_\lambda$ is a regular curve. By Lemma 3.1, $-l(t_0) = |\lambda l(t_0)|\kappa_{g\lambda}(t_0)$. It follows from $\lambda l(t_0) \neq 0$ that $\kappa_{g\lambda}(t_0) \neq 0$.

We give another representation of the evolute of a front by using the moving frame $(\boldsymbol{\gamma}(t), \boldsymbol{\mu}(t), \boldsymbol{\nu}(t))$ and the geodesic curvature $(l(t), \beta(t))$.

Theorem 3.3. *Under the above notations, the evolute of a front $\mathcal{E}_{\boldsymbol{\gamma}}(t)$ is represented by*

$$\mathcal{E}_{\boldsymbol{\gamma}}(t) = \frac{1}{\sqrt{l(t)^2 + \beta(t)^2}}(-l(t)\boldsymbol{\gamma}(t) + \beta(t)\boldsymbol{\nu}(t)) \tag{3.2}$$

and $\mathcal{E}_{\boldsymbol{\gamma}}(t)$ is a front.

Proof. First suppose that $\boldsymbol{\gamma}$ is a regular curve. Since $\dot{\boldsymbol{\gamma}}(t) = \beta(t)\boldsymbol{\mu}(t)$, we have $|\beta(t)| \neq 0$ and

$$\boldsymbol{t}(t) = \frac{\beta(t)}{|\beta(t)|}\boldsymbol{\mu}(t), \quad \boldsymbol{e}(t) = -\frac{\beta(t)}{|\beta(t)|}\boldsymbol{\nu}(t). \tag{3.3}$$

By Lemma 3.1 (1), $\kappa_g(t) = l(t)/|\beta(t)|$. Then

$$\mathcal{E}_{\boldsymbol{\gamma}}(t) = \frac{\kappa_g(t)\boldsymbol{\gamma}(t) + \boldsymbol{e}(t)}{\sqrt{1 + \kappa_g(t)^2}} = \frac{1}{\sqrt{l(t)^2 + \beta(t)^2}}(-l(t)\boldsymbol{\gamma}(t) + \beta(t)\boldsymbol{\nu}(t)). \tag{3.4}$$

Second suppose that t_0 is a singular point of $\boldsymbol{\gamma}$, and $\boldsymbol{\gamma}_\lambda$ is a regular curve with $\lambda \neq 1/\kappa_g(t)$. Since $\dot{\boldsymbol{\gamma}}_\lambda(t) = (\beta(t) + \lambda l(t))\boldsymbol{\mu}(t)/\sqrt{1 + \lambda^2}$, we have $|\beta(t) + \lambda l(t)| \neq 0$ and

$$\begin{aligned} \boldsymbol{t}_\lambda &= \frac{\beta(t) + \lambda l(t)}{|\beta(t) + \lambda l(t)|}\boldsymbol{\mu}(t), \\ \boldsymbol{e}_\lambda &= \boldsymbol{\gamma}_\lambda(t) \wedge \boldsymbol{t}_\lambda(t) = \frac{\beta(t) + \lambda l(t)}{|\beta(t) + \lambda l(t)|}\frac{\boldsymbol{\nu}(t) - \lambda\boldsymbol{\gamma}(t)}{\sqrt{1 + \lambda^2}}. \end{aligned} \tag{3.5}$$

By Lemma 3.1 (2), $\kappa_{g\lambda}(t) = \lambda\beta(t) - l(t)/|\beta(t) + \lambda l(t)|$ and $l(t) \neq 0$. Then

$$\begin{aligned} \mathcal{E}_{\boldsymbol{\gamma}_\lambda}(t) &= \frac{\kappa_{g\lambda}(t)\boldsymbol{\gamma}_\lambda(t) + \boldsymbol{e}_\lambda(t)}{\sqrt{1 + \kappa_{g\lambda}(t)^2}} \\ &= \frac{|\beta(t) + \lambda l(t)|}{\sqrt{(-l(t)^2 + \lambda\beta(t))^2 + (\beta(t) + \lambda l(t))^2}}\left(\frac{\lambda\beta(t) - l(t)}{|\beta(t) + \lambda l(t)|}\frac{\boldsymbol{\gamma}(\boldsymbol{t}) + \lambda\boldsymbol{\nu}(t)}{\sqrt{1 + \lambda^2}} + \frac{\beta(t) + \lambda l(t)}{|\beta(t) + \lambda l(t)|}\frac{\boldsymbol{\nu}(t) - \lambda\boldsymbol{\gamma}(t)}{\sqrt{1 + \lambda^2}}\right) \\ &= \frac{1}{\sqrt{(1 + \lambda^2)(l^2(t) + \beta^2(t))}}\frac{1}{\sqrt{1 + \lambda^2}}(-l(t)(1 + \lambda^2)\boldsymbol{\gamma}(t) + \beta(t)(1 + \lambda^2)\boldsymbol{\nu}(t)) \\ &= \frac{-l(t)\boldsymbol{\gamma}(t) + \beta(t)\boldsymbol{\nu}(t)}{\sqrt{l^2(t) + \beta^2(t)}} \\ &= \mathcal{E}_{\boldsymbol{\gamma}}(t). \end{aligned} \tag{3.6}$$

If we take $\widetilde{\boldsymbol{\nu}}(t) = \boldsymbol{\mu}(t)$, then $(\mathcal{E}_{\boldsymbol{\gamma}_\lambda}(t), \widetilde{\boldsymbol{\nu}}(t))$ is a framed immersion. In fact,

$$\begin{aligned} \dot{\mathcal{E}}_{\boldsymbol{\gamma}}(t) &= \frac{\dot{\beta}(t)l(t) - \beta(t)\dot{l}(t)}{(l^2(t) + \beta^2(t))^{3/2}}(l(t)\boldsymbol{\nu}(t) - \beta(t)\boldsymbol{\gamma}(t)) \\ &= \frac{d(\beta(t)/l(t))}{dt}\frac{l^2(t)}{(l^2(t) + \beta^2(t))^{3/2}}(l(t)\boldsymbol{\nu}(t) - \beta(t)\boldsymbol{\gamma}(t)), \end{aligned} \tag{3.7}$$

we have $\langle \mathcal{E}_{\boldsymbol{\gamma}}(t), \widetilde{\boldsymbol{\nu}}(t)\rangle = \langle \dot{\mathcal{E}}_{\boldsymbol{\gamma}}(t), \widetilde{\boldsymbol{\nu}}(t)\rangle = 0$. And $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ is a framed immersion satisfying $(l(t), \beta(t)) \neq (0, 0)$. We get $\dot{\widetilde{\boldsymbol{\nu}}}(t) = -l(t)\boldsymbol{\nu}(t) - \beta(t)\boldsymbol{\gamma}(t) \neq 0$. It follows that $\mathcal{E}_{\boldsymbol{\gamma}}(t)$ is a front. This completes the proof of the Theorem. □

Remark 3.4. By the representation (3.2), we may define the evolute of a front even if $\boldsymbol{\gamma}$ have non-isolated singularities, under the condition $l(t) \neq 0$.

By Lemma 3.1 and Remark 3.4 for a framed immersion $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ with the geodesic curvature (l, β), we say that t_0 is an inflection point of the front $\boldsymbol{\gamma}$ (or, the framed immersion $(\boldsymbol{\gamma}, \boldsymbol{\nu})$) if $l(t_0) = 0$. Since $\beta(t_0) \neq 0$ and Proposition 3.1, $l(t_0) = 0$ is equivalent to the condition $\kappa_g(t_0) = 0$.

Remark 3.5. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be a framed immersion, so does $(\boldsymbol{\gamma}, -\boldsymbol{\nu})$. However, $\mathcal{E}_{\boldsymbol{\gamma}}(t)$ does not change. It follows that we can define an evolute of a non-orientable front, by taking double covering of $\boldsymbol{\gamma}$.

Remark 3.6. By Definition 2.8, the evolute of a front is independent on the parametrisation of $(\boldsymbol{\gamma}, \nu)$. The geodesic curvature (l, β) depends on the parametrisation of $(\boldsymbol{\gamma}, \boldsymbol{\nu})$. If $s = s(t)$ is a parameter changing on I to $\overline{I}$, then $l(t) = l(s(t))\dot{s}(t)$ and $\beta(t) = \beta(s(t))\dot{s}(t)$. It also follows from the representation (1) that the evolute of a front is independent on the parametrisation of $(\boldsymbol{\gamma}, \boldsymbol{\nu})$.

If t_0 is a singular point of $\boldsymbol{\gamma}$, then $\beta(t_0) = 0$. As a corollary of Theorem 3.3, we have following.

Corollary 3.7. *If t_0 is a singular point of $\boldsymbol{\gamma}$, then $\mathcal{E}_{\boldsymbol{\gamma}}(t_0) = \boldsymbol{\gamma}(t_0)$.*

Proposition 3.8. *Suppose that t_0 is a singular point of $\boldsymbol{\gamma}$.*

(1) *t_0 is a regular point of $\mathcal{E}_{\boldsymbol{\gamma}}(t)$ if and only if $\boldsymbol{\gamma}$ is diffeomorphic to the $3/2$ cusp.*

(2) *t_0 is a singular point of $\mathcal{E}_{\boldsymbol{\gamma}}(t)$ if and only if $\ddot{\boldsymbol{\gamma}}(t_0) = 0$.*

Proof. (1) Let t_0 is a regular point of $\mathcal{E}_{\boldsymbol{\gamma}}(t)$. Since $\beta(t_0)=0$ and $l(t_0) \neq 0$, $\dot{\beta}(t_0) \neq 0$. By the differentiate of $\dot{\boldsymbol{\gamma}}(t) = \beta(t)\boldsymbol{\mu}(t)$, we have

$$\ddot{\boldsymbol{\gamma}}(t) = \dot{\beta}(t)\boldsymbol{\mu}(t) - l(t)\beta(t)\boldsymbol{\nu}(t), \tag{3.8}$$

$$\dddot{\boldsymbol{\gamma}}(t) = (\ddot{\beta}(t) - \beta(t)l(t)^2)\boldsymbol{\mu}(t) - (2\dot{\beta}(t)l(t) + \beta(t)\dot{l}(t))\boldsymbol{\nu}(t). \tag{3.9}$$

It follows that

$$\dot{\boldsymbol{\gamma}}(t_0) = 0, \quad \ddot{\boldsymbol{\gamma}}(t_0) = \dot{\beta}(t_0)\boldsymbol{\mu}(t_0), \tag{3.10}$$

$$\dddot{\boldsymbol{\gamma}}(t_0) = \ddot{\beta}(t_0)\boldsymbol{\mu}(t_0) - 2\dot{\beta}(t_0)l(t_0)\boldsymbol{\nu}(t_0), \tag{3.11}$$

$$\|\ddot{\boldsymbol{\gamma}}(t_0) \wedge \dddot{\boldsymbol{\gamma}}(t_0)\| = 2\dot{\beta}(t_0)^2 l(t_0). \tag{3.12}$$

(2) By the proof of (1), $\dot{\beta}(t_0) = 0$ if and only if $\ddot{\boldsymbol{\gamma}}(t_0) = 0$. □

As a well-known result, a singular point of $E_{\boldsymbol{\gamma}}(t)$ of a regular plane curve $\boldsymbol{\gamma}$ is corresponding to a vertex of $\boldsymbol{\gamma}$, namely $\dot{\kappa_g}(t) = 0$.

We extend the notion of vertex. For a framed immersion $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ with the geodesic curvature (l, β), t_0 is a vertex of the front $\boldsymbol{\gamma}$ (or a framed immersion $(\boldsymbol{\gamma}, \boldsymbol{\nu})$) if $\frac{d}{dt}(\beta/l)(t_0) = 0$, namely, $\frac{d}{dt}\mathcal{E}_{\boldsymbol{\gamma}}(t_0) = 0$. Note that if t_0 is a regular point of $\boldsymbol{\gamma}$, the definition of the vertex coincides with usual vertex for regular curves. Therefore, this is a generalisation of the notion of the vertex of a regular plane curve.

Remark 3.9. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be a framed immersion. If t_0 is a singular point of $\boldsymbol{\gamma}$ which degenerate more than $3/2$ cusp, then t_0 is a vertex of a front $\boldsymbol{\gamma}$. In fact,

$$\beta(t_0) = \dot{\beta}(t_0) = 0, \quad \frac{d}{dt}(\frac{\beta(t)}{l(t)})(t_0) = \frac{\dot{\beta(t)}l(t) - \beta(t)\dot{l}(t)}{l(t)^2} = 0. \tag{3.13}$$

Proposition 3.10. *Let $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : [0, 2\pi) \to \mathrm{S}^2 \times \mathrm{S}^2$ be a closed framed immersion without inflection points.*

(1) *$\boldsymbol{\gamma}$ has at least two singular points which generate more than $3/2$ cusp, then $\boldsymbol{\gamma}$ has at least four vertices.*

(2) *If $\boldsymbol{\gamma}$ at least four singular points, then $\boldsymbol{\gamma}$ has at least four vertices.*

Proof. (1) Let t_i be a singular point of $\boldsymbol{\gamma}$ for each $i = 1, 2, ..., n$. Suppose that at least two of them are degenerate more than 3/2 cusp. By Remark 3.9, there singularities are vertices of $\boldsymbol{\gamma}$, therefore it is sufficient to show that these is at least one vertex between two adjacent singular points. Since $\boldsymbol{\gamma}$ has no inflection points, the sign of the geodesic curvature of $\boldsymbol{\gamma}$ on regular points is constant. Therefore, either $\lim_{t\to t_i} \kappa_g(t) = \infty$ for all $i = 1, 2, ..., n$ or $\lim_{t\to t_i} \kappa_g(t) = -\infty$ for all $i = 1, 2, ..., n$. This concludes there exists $t \in (t_i, t_{i+1})$ such that $\dot{\kappa}_g(t) = 0$ for all $i = 1, 2, ..., n$. Moreover, since $\boldsymbol{\gamma}$ is closed, there exists a point $t \in [0, t_1) \cup (t_n, 2\pi)$ such that $\dot{\kappa}_g(t) = 0$. Therefore, $\boldsymbol{\gamma}$ has at least four vertices.

(2) We can also prove by the same method of (1). □

4. Evolutes of the evolutes of fronts

By Theorem 3.3, the evolute of a front is also a front. We consider a repeated evolute of an evolute of a front and give its properties at singular points.

Theorem 4.1. *Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be a framed immersion with the geodesic curvature (l, β). The evolute of an evolute of a front is given by*

$$\mathcal{E}_{\mathcal{E}_{\boldsymbol{\gamma}}}(t) = \frac{-(l^2(t) + \beta^2(t))^{3/2}\mathcal{E}_{\boldsymbol{\gamma}}(t) + (\dot{\beta}(t)l(t) - \beta(t)\dot{l}(t))\boldsymbol{\mu}(t)}{\sqrt{(l(t) + \beta(t))^3 + (\dot{\beta}(t)l(t) - \beta(t)\dot{l}(t))^2}}. \tag{4.1}$$

Proof. We denote $\widetilde{\boldsymbol{\gamma}}(t) = \mathcal{E}_{\boldsymbol{\gamma}}(t)$. And $(\widetilde{\boldsymbol{\gamma}}(t), \widetilde{\boldsymbol{\nu}}(t)) = (\mathcal{E}_{\boldsymbol{\gamma}}(t), \boldsymbol{\mu}(t))$ is a framed immersion. Since $\widetilde{\boldsymbol{\mu}}(t) = \boldsymbol{\mu}(t) \wedge \mathcal{E}_{\boldsymbol{\gamma}}(t)$, we have

$$\widetilde{\beta}(t) = \frac{\dot{\beta}(t)l(t) - \beta(t)\dot{l}(t)}{\sqrt{l(t)^2 + \beta(t)^2}}, \tag{4.2}$$

where $\dot{\widetilde{\boldsymbol{\gamma}}}(t) = \widetilde{\beta}(t)\widetilde{\boldsymbol{\mu}}(t)$. Moreover $\widetilde{l}(t) = \sqrt{l(t)^2 + \beta(t)^2}$ by the Frenet formula of a front. It follows that

$$\begin{aligned}\mathcal{E}_{\mathcal{E}_{\boldsymbol{\gamma}}}(t) &= \mathcal{E}_{\widetilde{\boldsymbol{\gamma}}}(t) \\ &= \frac{-\widetilde{l}(t)\mathcal{E}_{\boldsymbol{\gamma}}(t) + \widetilde{\beta}(t)\widetilde{\boldsymbol{\nu}}(t)}{\sqrt{\widetilde{l}(t)^2 + \widetilde{\beta}(t)^2}} \\ &= \frac{-\sqrt{l(t)^2 + \beta(t)^2}\mathcal{E}_{\boldsymbol{\gamma}}(t) + \frac{\dot{\beta}(t)l(t) - \beta(t)\dot{l}(t)}{\sqrt{l(t)^2+\beta(t)^2}}\boldsymbol{\mu}(t)}{\sqrt{l(t)^2 + \beta(t)^2 + \frac{(\beta(t)\dot{l}(t) - \dot{\beta}(t)l(t))^2}{l(t)^2+\beta(t)^2}}} \\ &= \frac{-(l^2(t) + \beta^2(t))^{3/2}\mathcal{E}_{\boldsymbol{\gamma}}(t) + (\dot{\beta}(t)l(t) - \beta(t)\dot{l}(t))\boldsymbol{\mu}(t)}{\sqrt{(l(t) + \beta(t))^3 + (\dot{\beta}(t)l(t) - \beta(t)\dot{l}(t))^2}}.\end{aligned} \tag{4.3}$$

□

Proposition 4.2. *Suppose that t_0 is singular point of both $\boldsymbol{\gamma}$ and $\mathcal{E}_{\boldsymbol{\gamma}}$.*

(1) t_0 is a regular point of $\mathcal{E}_{\mathcal{E}_{\boldsymbol{\gamma}}}$ if and only if $\boldsymbol{\gamma}$ is diffeomorphic to the 4/3 cusp at t_0.

(2) t_0 is a singular point of $\mathcal{E}_{\mathcal{E}_{\boldsymbol{\gamma}}}$ if and only if $\dddot{\boldsymbol{\gamma}}(t_0) = 0$.

Proof. (1) Let t_0 be a regular point of $\mathcal{E}_{\mathcal{E}_{\boldsymbol{\gamma}}}$. By Proposition 3.8,

$$\beta(t_0) = \dot{\beta}(t_0) = 0, \qquad l(t_0) \neq 0. \tag{4.4}$$

Then $\dot{\boldsymbol{\gamma}}(t_0) = \ddot{\boldsymbol{\gamma}}(t_0) = 0$. Since

$$\frac{d}{dt}\mathcal{E}_{\mathcal{E}_{\boldsymbol{\gamma}}}(t_0) = -\ddot{\beta}(t_0)l(t_0)^{-2} \neq 0 \tag{4.5}$$

if and only if $\ddot{\beta}(t_0) \neq 0$. By the differentiate $\dot{\boldsymbol{\gamma}}(t) = \beta(t)\boldsymbol{\mu}(t)$, It follows that

$$\dddot{\boldsymbol{\gamma}}(t_0) = \ddot{\beta}(t_0)\boldsymbol{\mu}(t_0), \quad \boldsymbol{\gamma}^4(t_0) = \dddot{\beta}(t_0)\boldsymbol{\mu}(t_0) - 3\ddot{\beta}(t_0)l(t_0)\boldsymbol{\mu}(t_0). \tag{4.6}$$

Hence

$$\|\dddot{\boldsymbol{\gamma}}(t_0) \wedge \boldsymbol{\gamma}^{(4)}(t_0)\| = 3\ddot{\beta}(t_0)^2 l(t_0) \neq 0. \tag{4.7}$$

This condition follows that $\boldsymbol{\gamma}$ is diffeomorphic to the 4/3 cusp at t_0.

(2) By the proof of (1), $\ddot{\beta}(t_0) = 0$ if and only if $\dddot{\boldsymbol{\gamma}}(t_0) = 0$.

□

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