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Fixed point theorems for $(\psi \circ \varphi)-$contractions in a fuzzy metric spaces

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Abstract

In this paper we prove some common fixed point theorems for $(\psi \circ \varphi)-$ contractions in a fuzzy metric space. We offered a generalization of $\varphi-$ contraction in fuzzy metric space. Our results generalize or improve many recent fixed point theorems in the literature.

Keywords: Fixed point theorem, fuzzy metric spaces, contractions.
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1. Introduction and Preliminaries

The concept of fuzzy metric space was introduced in different ways by some authors (see [5, 13]) and the fixed point theory in this kind of spaces has been intensively studied (see [4, 9, 10]). The notion of fuzzy metric space, introduced by Kramosil and Michalek [13] was modified by George and Veeramani [6, 7] that obtained a Hausdorff topology for this class of fuzzy metric spaces. Gregori and Sapena [10] have introduced a kind of contractive mappings in fuzzy metric spaces in the sense of George and Veeramani and proved a fuzzy Banach contraction theorem using a strong condition for completeness, now called the completeness in the sense of Grabiec, or G-completeness. Later, further studies have been done by different authors in fuzzy metric spaces (see i.e. [1, 2, 3, 16, 11, 19, 20, 21]).

Firstly we mention so called weakly contractive conditions of Alber and Guerre-Delabriere and Rhoades, altering distance functions used by Khan et al. and Boyd and Wong, as well as Meir and Keeler generalization of contractive condition.

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Cyclic representations and cyclic contractions were introduced by Kirk et al. [12] and further used by several authors to obtain various fixed point results (see [8]).

In 2010, the concept of cyclic $\varphi-$ contraction is introduced by Păcurar and Rus [15]. Meantime, they constructed a fixed point theorem for the cyclic $\varphi-$ contraction in a classical complete metric space. In addition, several problems in connection with the fixed point are investigated. Later the notion of cyclic $\varphi-$ contraction in fuzzy metric space have been proved by Y. H. Shen, D. Qıu and W. Chen [18]. H. K. Nashine, Z. Kadelburg [14] have presented some fixed point results for mappings which satisfy cyclic weaker $(\psi\circ\varphi)-$ contractions and cyclic weaker $(\psi,\varphi)-$ contractions in 0-complete partial metric spaces.

In this paper we prove some common fixed point theorems for $(\psi\circ\varphi)-$ contractions in a fuzzy metric space. We offered a generalization of $\varphi-$ contraction in fuzzy metric space. Our results generalize or improve many recent fixed point theorems in the literature.

We shall require the following definitions and lemmas in the sequel.

Definition 1.1 ([17])**.** A binary operation $T:[0,1]\times[0,1]\to[0,1]$ is called a continuous triangular norm (in short, continuous $t-$norm) if it satisfies the following conditions:
(TN-1) T is commutative and associative,
(TN-2) T is continuous,
(TN-3) $T(a,1)=a$ for every $a\in[0,1]$,
(TN-4) $T(a,b)\leq T(c,d)$ whenever $a\leq c$, $b\leq d$ and $a,b,c,d\in[0,1]$.
An arbitrary $t-$norm T can be extended (by associativity) in a unique way to an n ary operator taking for $(x_1,x_2,...,x_n)\in[0,1]^n$, $n\in\mathbb{N}$, the value $T(x_1,x_2,...,x_n)$ is defined, in Ref.[7], by

$$T^0_{I=1}x_i=1,\ T^n_{I=1}x_i=T(T^{n-1}_{I=1}x_i,x_n)=T(x_1,x_2,...,x_n).$$

Definition 1.2 ([6])**.** A fuzzy metric space is an ordered triple (X,M,T) such that X is a nonempty set, T is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$, satisfying the following conditions, for all $x,y,z\in X,\ s,t>0:$
(FM-1) $M(x,y,t)>0$,
(FM-2) $M(x,y,t)=1$ iff $x=y$,
(FM-3) $M(x,y,t)=M(y,x,t)$,
(FM-4) $T(\ M(x,y,t),M(y,z,s))\leq M(x,z,t+s)$,
(FM-5) $M(x,y,\cdot):(0,\infty)\to(0,1]$ is continuous.

Definition 1.3 ([9])**.** Let (X,M,T) be a fuzzy metric space. Then
(i) A sequence $\{x_n\}$ in X is said to converge to x in X, denoted by $x_n\to x$, if and only if $\lim\limits_{n\to\infty}M(x_n,x,t)=1$ for all $t>0$, i.e. for each $r\in(0,1)$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that $M(x_n,x,t)>1-r$ for all $n\geq n_0$.
(ii) A sequence $\{x_n\}$ is a G-Cauchy sequence if and only if $\lim\limits_{n\to\infty}M(x_n,x_{n+p},t)=1$ for any $p>0$ and $t>0$.
(iii) The fuzzy metric space (X,M,T) is called G-complete if every G-Cauchy sequence is convergent.

Definition 1.4 ([18])**.** Let (X,M,T) be a fuzzy metric space and let $\{f_n\}$ be a sequence of self mapping on X. $f_0:X\to X$ is a given mapping. The sequence $\{f_n\}$ is said to converge uniformly to f_0 if for each $\epsilon\in(0,1)$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that

$$M(f_n(x),f_0(x),t)>1-\epsilon$$

for all $n\geq n_0$ and $x\in X$.

Definition 1.5 ([18])**.** A function $\psi:[0,1]\longrightarrow[0,1]$ is called a comparison function if it satisfies
(i) ψ is left continuous and non-decreasing,
(ii) $\psi(t)>t$ for all $t\in(0,1)$.

Lemma 1.6 ([18])**.** *If ψ be a comparison function, then*

(i) $\psi(1) = 1$,
(ii) $\lim_{n\to+\infty} \psi^n(t) = 1$ *for all* $t \in (0,1)$, *where* $\psi^n(t)$ *denotes the composition of* $\psi(t)$ *with itself* n *times.*

In 2003 Kirk et al. introduced the following notion of cyclic representation.

Definition 1.7 ([15])**.** Let X be a nonempty set, m a positive integer and $f : X \longrightarrow X$ an operator. Then $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to f if
(a) A_i, $i = 1, .., m$ are non-empty subsets of X,
(b) $f(A_1) \subset A_2$, $f(A_2) \subset A_{3,}..., f(A_{m-1}) \subset A_m$, $f(A_m) \subset A_1$.

Definition 1.8 ([19])**.** Two mappings f and g of a fuzzy metric space $(X, M, *)$ into itself are said to be weakly commuting if

$$M(fgx, gfx, t) \geq M(fx, gx, t) \quad \forall x \in X.$$

2. Main Results

Definition 2.1. Let (X, M, T) be a G-complete fuzzy metric space and $P_{cl}(X)$ denotes the collection of nonempty closed subsets of X. m a positive integer, $A_1, A_2, ..., A_m \in P_{cl}(X)$, $Y = \cup_{i=1}^m A_i$, and $f : Y \to Y$ an operator. If,
(i) $Y = \cup_{i=1}^m A_i$ is a cyclic represention of Y with respect to f,
(ii) There exists a function $\varphi : [0,1] \longrightarrow [0,1]$ such that
(φ_1) φ is non-decreasing and continuous function,
(φ_2) $\varphi(t) > 0$ for $t > 0$, $\varphi(1) = 1$ and $\varphi(0) = 0$,
(φ_3) $\varphi(t) \leq t$ for all $t \in (0,1)$,
(iii) $\varphi(M(fx, fy, t)) \geq \psi(\varphi(M(x, y, t)))$ where ψ is a function as in Definition 1.5,
for any $x \in A_i$, $y \in A_{i+1}$ and $t > 0$,where $A_{m+1} = A_1$, then f is called cyclic weaker $(\psi \circ \varphi)-$ contraction in the fuzzy metric space (X, M, T).

Theorem 2.2. *Let (X, M, T) be a G-complete fuzzy metric space, m a positive integer, $A_1, A_2, ..., A_m \in P_{cl}(X)$, $Y = \cup_{i=1}^m A_i$, and $f : Y \to Y$ an operator. Assume that*
(1) $Y = \cup_{i=1}^m A_i$ is a cyclic represention of Y with respect to f,
(2) $f : Y \to Y$ is a cyclic weaker $(\psi \circ \varphi)-$ contraction.
Then f has a unique fixed point $x \in \cap_{i=1}^m A_i$ and the iterative sequence $\{x_n\}_{n\geq 0}$ ($x_n = f(x_{n-1})$, $n \in \mathbb{N}$) converges to x for any starting point $x_0 \in Y$.

Proof. Fix $x_0 \in X$ and define the sequence (x_n) by
$x_1 = f(x_0)$, $x_2 = f(x_1), ...,$ $x_{2n+1} = f(x_{2n})$, $x_{2n+2} = f(x_{2n+1}), ...$
For any $n \geq 0$, there exists $i_n \in \{1, 2, ..., m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$. Therefore, we can obtain

$$\varphi(M(x_n, x_{n+1}, t)) = \varphi(M(f(x_{n-1}), f(x_n), t)) \geq \psi(\varphi(M(f(x_{n-1}), f(x_n), t)))$$

Consider the definition of ψ, we get by induction that

$$M(x_n, x_{n+1}, t) \geq \varphi(M(x_n, x_{n+1}, t)) \geq \psi^n(\varphi(M(x_0, x_1, t)) > 0.$$

Thus, for any $p > 0$,we have

$$\begin{aligned} M(x_n, x_{n+p}, t) &\geq T(M(x_n, x_{n+1}, \frac{t}{p}), M(x_{n+1}, x_{n+2}, \frac{t}{p}), ..., M(x_{n+p-1}, x_{n+p}, \frac{t}{p})) \\ &\geq T(\psi^n(\varphi(M(x_0, x_1, \frac{t}{p}))), \psi^{n+1}(\varphi(M(x_0, x_1, \frac{t}{p}))), ..., \psi^{n+p-1}(\varphi(M(x_0, x_1, \frac{t}{p})))) \\ &= T_{i=0}^{p-1} \psi^{n+i}(M(x_0, x_1, \frac{t}{p})). \end{aligned}$$

By Lemma 1.6, for every $i \in \{0, 1, ..., p-1\}$, we obtain that

$$\lim_{n\to\infty} \psi^{n+i}(M(x_0, x_1, \frac{t}{p})) = 1.$$

According to the continuitiy of t-norm T, it can easily be verified that $M(x_n, x_{n+p}, t) \to 1$ as $n \to \infty$. It shows that $\{x_n\}_{n\geq 0}$ is a G-Cauchy. Since Y is G-complete, then there exists $x \in Y$ such that $x_n \to x$.

On the other hand, by the condition (1), it follows that the iterative sequence $\{x_n\}_{n\geq 0}$ has an infinite number of terms in each A_i, $i = 1, 2, ..., m$. Since Y is G-complete, from each A_i, $i = 1, 2, ..., m$, one can extract a subsequence of $\{x_n\}_{n\geq 0}$ which converges to x as well. Because each A_i, $i = 1, 2, ..., m$ is closed, we conclude that $x \in \cap_{i=1}^m A_i$ and thus $\cap_{i=1}^m A_i \neq \varnothing$. Set $Z = \cap_{i=1}^m A_i$. Obviously, Z is also closed and G-complete. Consider the restriction of f to Z, that is, $f|_Z : Z \to Z$. Next, we will prove that $f|_Z$ has a unique fixed point in $Z \subset Y$.

For the foregoing $x \in Z$, since $f|_Z(x) \in Z$ and $x_n \in A_{i_n}$, we can choose A_{i_n+1} such that $f|_Z(x) \in A_{i_n+1}$. Hence, for any $t > 0$, we have

$$\begin{aligned} M(f|_Z(x), x, t) &= M(f(x), x, t) \\ &\geq T(M(f(x), f(x_n), \frac{t}{2}), M(x_{n+1}, x, \frac{t}{2})) \\ &\geq T(\psi(\varphi(M(x, x_n, \frac{t}{2})), M(x_{n+1}, x, \frac{t}{2}) \end{aligned}$$

from (TN-3) we get $n \to \infty$, $T(1, 1) = 1$. Clearly, we get $f|_Z(x) = x$, namely, x a fixed point, which is obtained by iteration from starting point x_0. To show uniqueness, we assume that $z \in \cap_{i=1}^m A_i$ is another fixed point of $f|_Z$. Since $x, z \in A_i$ for all $i \in \mathbb{N}$, we can obtain

$$\varphi(M(x, z, t)) = \varphi(M((f|_Z(x), f|_Z(z), t)) = \varphi(M(f(x), f(z), t)) \geq \psi(\varphi(M(x, z, t))) > \varphi(M(x, z, t)).$$

This leads to a contradiction. Thus, x is the unique fixed of $f|_Z$ for any starting point $x_0 \in Z \subset Y$.

Now, we still have to prove that iterative sequence $\{x_n\}_{n\geq 0}$ converges to x for any initial point $x_0 \in Y$. Let $y \in Y = \cup_{i=1}^m A_i$, there exists $i_0 \in \{0, 1, ..., m\}$ such that $y \in A_{i_0}$. As $x \in \cap_{i=1}^m A_i$, it follows that $x \in A_{i_0+1}$ as well. Then, for any $t > 0$, we have

$$\varphi(M(f(y), f(x), t)) \geq \psi(\varphi(M(y, x, t)).$$

By induction, we can obtain

$$\begin{aligned} \varphi(M(x_n, x, t)) &= \varphi(M(f^n(x_0), x, t)) = \varphi(M(f^n(x_0), f(x), t)) \\ &= \varphi(M(f(f^{n-1}(x_0), f(x), t)) \\ &\geq \psi(\varphi(f^{n-1}(x_0), x, t) \\ &\geq .... \\ &\geq \psi^n(\varphi(x_0, x, t)). \end{aligned}$$

Supposing $x_0 \neq x$, it follows immediately that $x_n \to x$ as $n \to \infty$. So the iterative sequence $\{x_n\}_{n\geq 0}$ converges to the unique fixed point x of f for any starting point $x_0 \in Y$.

Now, for all $t > 0$,

$$\varphi(M(x_{2n+1}, f(x), t)) = \varphi(M(f(x_{2n}), f(x), t)) \geq \psi(\varphi(M(x_{2n}, x, t)))$$

and since f is continuous and property of φ, letting $n \to \infty$ it follows,

$$M(x, f(x), t) \geq \varphi(M(x, f(x), t)) \geq \psi(1) = 1$$

hence $f(x) = x$. Then, x is a fixed point of f.

Now we prove the uniqueness of the fixed points of f . Assume that $x, y \in X$ are two common fixed points of f . If $x \neq y$, then there exists $t > 0$ such that $0 < M(x, y, t) < 1$ and hence

$$\begin{aligned}\varphi(M(x,y,t)) &= \varphi(M(f(x),f(y),t)) \geq \psi(\varphi(M(x,y,t)) \\ &= \psi(\varphi(M(x,y,t))) > \varphi(M(x,y,t)),\end{aligned}$$

which is a contradiction. Therefore $x = y$. □

Example 2.3. Let X be the subset of $\mathbb{R}$ defined by $X = \{1, 2, 3, 4, 5\}$. $\psi(\lambda) = \sqrt{\lambda}$, $\varphi(\lambda) = \lambda$ for all $\lambda \in [0, 1]$. Define $M(x, y, t) = e^{-\dfrac{2d(x,y)}{t}}$, where $d(x, y) = |x - y|$. Clearly (X, M, T) is a G-complete fuzzy metric space with respect to $t-$norm $T(a, b) = ab$. Let $f : X \to X$ be given by

$$f(1) = f(2) = f(3) = f(4) = 2, \qquad f(5) = 1.$$

Set $A_1 = \{1, 2, 3, 4\}$, $A_2 = \{2, 4, 5\}$. $f(A_1) = \{2\} \subseteq A_2$, $f(A_2) = \{1, 2\} \subseteq A_1$. According to Definition 1.7, $X = A_1 \cup A_2$ is a cyclic representation of X with respect to f. In addition, it can easily be verified that $\varphi(M(fx, fy, t)) \geq \psi(\varphi(M(x, y, t)))$ for every $x \in A_1$, $y \in A_2$ and $t > 0$. This shows that f a cyclic weaker $(\psi \circ \varphi)-$ contraction. Hence all the conditions of Theorem 2.2 are satisfied and then f has a unique fixed point, that is, $x = 2$.

Theorem 2.4. *Let (X, M, T) be a G-complete fuzzy metric space, m a positive integer, $A_1, A_2, ..., A_m \in P_{cl}(X)$, $Y = \cup_{i=1}^{m} A_i$, and $f, g : Y \to Y$ two operators. Assume that*
(1) $Y = \cup_{i=1}^{m} A_i$ is a cyclic represention of Y with respect to f and g,
(2) f and g are two cyclic weaker $(\psi \circ \varphi)-$ contractions,
(3) $\varphi(M(fx, gy, t)) \geq \psi(\varphi(\min\{M(x, y, t), M(fx, x, t), M(x, gx, t)\}))$ for all $x, y \in X$ and $t > 0$.
Then f and g have a unique common fixed point $x \in \cap_{i=1}^{m} A_i$.

Proof. Fix $x_0 \in X$ and define the sequence (x_n) by
$x_1 = f(x_0)$, $x_2 = g(x_1), ...,$ $x_{2n+1} = f(x_{2n})$, $x_{2n+2} = g(x_{2n+1}), ...$
Similarly from the proof of Theorem 2.2, (x_n) is a G-Cauchy. Since X is G-complete, then there exists $x \in X$ such that $x_n \to x$. Now, for all $t > 0$,

$$\varphi(M(x_{2n+1}, f(x), t)) = \varphi(M(f(x_{2n}), f(x), t)) \geq \psi(\varphi(M(x_{2n}, x, t)))$$

and since f is continuous and property of φ, letting $n \to \infty$ it follows,

$$M(x, f(x), t) \geq \varphi(M(x, f(x), t)) \geq \psi(1) = 1$$

hence $f(x) = x$. Then, x is a fixed point of f. Analogously, we obtain that $g(x) = x$ and x is a common fixed point of f and g.

Now we prove the uniqueness of the fixed points of f and g . Assume that $x, y \in X$ are two common fixed points of f and g. If $x \neq y$, then there exists $t > 0$ such that $0 < M(x, y, t) < 1$ and hence

$$\begin{aligned}\varphi(M(x,y,t)) &= \varphi(M(f(x),g(y),t)) \geq \psi(\varphi(\min\{M(x,y,t),M(fx,x,t),M(x,gx,t)\}) \\ &= \psi(\varphi(M(x,y,t))) > \varphi(M(x,y,t)),\end{aligned}$$

which is a contradiction. Therefore $x = y$. □

Corollary 2.5. *If we get $f = g$ in Theorem 2.4, we obtain Theorem 2.2.*

Theorem 2.6. *Let (X, M, T) be a G-complete fuzzy metric space, m a positive integer, $A_1, A_2, ..., A_m \in P_{cl}(X)$, $Y = \cup_{i=1}^{m} A_i$, and f and g satisfying the following conditions:*
(1) f is a cyclic weaker $(\psi \circ \varphi)-$ contraction and g, s are two continuous mappings,
(2) $f(X) \subset g(X) \cap s(X)$ and (f, g), (f, s) are weakly commuting,
(3) $\varphi(M(fx, gy, t)) \geq \psi(\varphi(\min\{M(gx, sy, t), M(gx, fx, t), M(gx, fy, t), M(sy, fy, t)\}))$
for all $x, y \in X$ and $t > 0$.
Then f, g and s have a unique common fixed point $x \in \cap_{i=1}^{m} A_i$.

Proof. Let $x_0 \in X$ be any arbitrary point. Since $f(X) \subset g(X)$ then there exists a point $x_1 \in X$ such that $fx_0 = gx_1$. Also, since $f(X) \subset s(X)$, there exists another point $x_2 \in X$ such that $fx_1 = sx_2$.
In general, we get a sequence (y_n) recursively as

$$y_n = gx_{n+1} = fx_n \quad \text{and} \quad y_{n+1} = sx_{n+2} = fx_{n+1}, \quad n \in \mathbb{N}.$$

Let $M_n = M(y_{n+1}, y_n, t) = M(fx_{n+1}, fx_n, t)$ and $M(y_0, y_1, t) > 0$. Then,
$M_{n+1} = M(y_{n+2}, y_{n+1}, t) = M(fx_{n+2}, fx_{n+1}, t)$.
Using inequality (3), we get,

$$\begin{aligned} \varphi(M_{n+1}) &= \varphi(M(fx_{n+2}, fx_{n+1}, t)) \geq \psi(\varphi(\min\{M(gx_{n+2}, sx_{n+1}, t), M(gx_{n+2}, fx_{n+2}, t), \\ &\quad M(gx_{n+2}, fx_{n+1}, t), M(sx_{n+1}, fx_{n+1,}t)\})) \\ &= \psi(\varphi(\min\{M(fx_{n+1}, fx_n, t), M(fx_{n+1}, fx_{n+2}, t), M(fx_{n+1}, fx_{n+1}, t), \\ &\quad M(fx_n, fx_{n+1,}t)\})) \\ &= \psi(\varphi \min\{M_n, M_{n+1}, 1, M_n\})) \end{aligned}$$

If $M_n > M_{n+1}$, then by definition of ψ and φ we have

$$\varphi(M_{n+1}) \geq \psi(\varphi(M_{n+1})) > \varphi(M_{n+1})$$

a contradiction. So,

$$\varphi(M_{n+1}) \geq \psi(\varphi(M_n)).$$

Thus, we get,

$$M(y_{n+2}, y_{n+1}, t) \geq \psi(\varphi(M(y_{n+1}, y_n, t))) \quad \forall n \in \mathbb{N}, \quad t > 0.$$

Hence, repeating this inequality n times we obtain,

$$M(y_n, y_{n+1}, t) \geq \psi^n(\varphi(M(y_0, y_1, t)))$$

Letting $n \to \infty$, from Lemma 1.6 we get,

$$\lim_{n \to \infty} M(y_n, y_{n+1}, t) = 1.$$

That is, from Definition 1.3 we get that (y_n) is a G-Cauchy. Since X is G-complete, then there exists $z \in X$ such that $y_n \to z$. Hence $(fx_n) \to z \in X$. Since f is a cyclic weaker $(\psi \circ \varphi)-$ contraction and by definition of ψ and φ ,

$$M(y_n, fz, t) \geq \varphi(M(y_n, fz, t)) = \varphi(M(fx_n, fz, t)) \geq \psi(\varphi(M(x_n, z, t))),$$

By taking the limit as $n \to \infty$ we obtain,

$$M(z, fz, t) \geq \psi(1) = 1.$$

hence $fz = z$. Since $(fx_n) \to z \in X$, hence the subsequences (gx_n) and (sx_n) of (fx_n) have the same limit. Since g is continuous, in this case we have $gfx_n \to gz$, $ggx_n \to gz$. Also (f, g) is weakly commuting, we have $fgx_n \to gz$. Let $x = gx_n$, $y = x_n$ in (3) we get,

$$\begin{aligned} \varphi(M(fgx_n, fx_n, t)) &\geq \psi(\varphi(\min\{M(ggx_n, sx_n, t), M(ggx_n, fgx_n, t), \\ &\quad M(ggx_n, fx_n, t), M(sx_n, fx_n, t)\})). \end{aligned}$$

Taking limit $n \to \infty$,

$$\begin{aligned}\varphi(M(gz,z,t)) &\geq \psi(\varphi(\min\{M(gz,z,t),M(gz,gz,t),M(gz,z,t),M(z,z,t)\}))\\ &= \psi(\varphi(\min\{M(gz,z,t),1,M(gz,z,t),1\}))\\ &= \psi(\varphi(M(gz,z,t)))\\ &> \varphi(M(gz,z,t))\end{aligned}$$

So, we get $gz = z$. Since s is continuous, in this case we have $ssx_n \to sz$, $sfx_n \to sz$. Also (f,s) is weakly commuting, we have $fsx_n \to sz$. Now, let $x = x_n, y = sx_n$ in (3) we get,

$$\begin{aligned}\varphi(M(fx_n,fsx_n,t)) \geq\ & \psi(\varphi(\min\{M(gx_n,ssx_n,t),M(gx_n,fx_n,t),\\ & M(gx_n,fsx_n,t),M(ssx_n,fsx_n,t)\}))\end{aligned}$$

Taking limit $n \to \infty$,

$$\begin{aligned}\varphi(M(z,sz,t)) &\geq \psi(\varphi(\min\{M(z,sz,t),M(z,z,t),M(z,sz,t),M(sz,sz,t)\}))\\ &= \psi(\varphi(\min\{M(z,sz,t),1,M(z,sz,t),1\}))\\ &= \psi(\varphi M(sz,z,t)))\\ &> \varphi(M(z,sz,t))\end{aligned}$$

So, we get $sz = z$. Thus, we have $fz = gz = sz = z$. Hence z is a common fixed point of f, g and s.
Now we prove the uniqueness of the common fixed points of f, g and s. Let v be another common fixed point of f, g and s, then $fv = gv = sv = v$. Put $x = z$, $y = v$ in (3), we get,

$$\begin{aligned}\varphi(M(z,v,t)) &\geq \psi(\varphi(\min\{M(gz,sv,t),M(gz,fz,t),M(gz,fv,t),M(sv,fz,t)\}))\\ &= \psi(\varphi(M(z,v,t)))\\ &> \varphi(M(z,v,t))\end{aligned}$$

which gives $z = v$. Therefore z is a unique common fixed point of f, g and s.
If we take $s = g$, then we get following corollary: □

Corollary 2.7. *Let (X,M,T) be a G-complete fuzzy metric space, m a positive integer, $A_1, A_2, ..., A_m \in P_{cl}(X)$, $Y = \cup_{i=1}^m A_i$, and f and g satisfying the following conditions:*
(1) f is a cyclic weaker $(\psi \circ \varphi)-$ contraction and g is a continuous mapping,
(2) $f(X) \subset g(X)$ and (f,g) is weakly commuting,
(3) $\varphi(M(fx,gy,t)) \geq \psi(\varphi(\min\{M(gx,gy,t),M(gx,fx,t),M(gx,fy,t),M(gy,fy,t)\}))$
for all $x,y \in X$ and $t > 0$.
Then f and g have a unique common fixed point in Y.

In this paper, we presented the notion of cyclic weaker $(\psi \circ \varphi)-$ contraction in a fuzzy metric space and proved a fixed point theorem for this type of mapping in a G-complete fuzzy metric space. In our next research, we intend to establish a fixed point theorem for cyclic weaker $(\psi \circ \varphi)-$ contraction in an M-complete fuzzy metric space.

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