Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 740–749
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Some new Hermite–Hadamard type inequalities for geometrically quasi-convex functions on co-ordinates

Xu-Yang Guo[a], Feng Qi[b,*], Bo-Yan Xi[a]

[a] *College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, China*
[b] *Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China*

Abstract

In the paper, the authors introduce a new concept "geometrically quasi-convex function on co-ordinates" and establish some new Hermite–Hadamard type inequalities for geometrically quasi-convex functions on the co-ordinates.

Keywords: Geometrically quasi-convex function, Hermite–Hadamard type integral inequality, Hölder inequality.
2010 MSC: 26A51, 26D15.

1. Introduction

The following definitions are well known in the literature.

Definition 1.1. *A function* $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ *is said to be convex if*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

is valid for all $x, y \in I$ *and* $\lambda \in [0, 1]$.

Definition 1.2 ([6]). *A function* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be quasi-convex if*

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}$$

holds for all $x, y \in I$ *and* $\lambda \in [0, 1]$.

*Corresponding author
Email addresses: guoxuyang1991@qq.com (Xu-Yang Guo), qifeng618@gmail.com, qifeng618@hotmail.com (Feng Qi), baoyintu78@qq.com (Bo-Yan Xi)

Received 2015-2-28

Definition 1.3 ([4, 5])**.** *A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ with $a < b$ and $c < d$ if the partial mappings*

$$f_y : [a, b] \to \mathbb{R}, \quad f_y(u) = f_y(u, y) \quad \text{and} \quad f_x : [c, d] \to \mathbb{R}, \quad f_x(v) = f_x(x, v)$$

are convex for all $x \in (a, b)$ and $y \in (c, d)$.

A formal definition for co-ordinated convex functions may be restated as follows.

Definition 1.4 ([4, 5])**.** *A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ with $a < b$ and $c < d$ if*

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le t\lambda f(x, y) + t(1-\lambda)f(x, w) + (1-t)\lambda f(z, y) + (1-t)(1-\lambda)f(z, w)$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

Definition 1.5 ([9])**.** *A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be a quasi-convex on the co-ordinates on Δ with $a < b$ and $c < d$ if*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \le \max\{f(x, y), f(z, w)\}$$

holds for all $\lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

A formal definition for co-ordinated quasi-convex functions may be stated as follows.

Definition 1.6 ([10])**.** *A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ with $a < b$ and $c < d$ if*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \le \max\{f(x, y), f(x, w), f(z, y), f(z, w)\}$$

holds for all $\lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

For convex functions on the co-ordinates, there exist the following conclusions.

Theorem 1.7 ([4, 5, Theorem 2.2])**.** *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be convex on the co-ordinates on Δ with $a < b$ and $c < d$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\, x + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\, y\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x, y) \,\mathrm{d}\, x \,\mathrm{d}\, y \\ &\le \frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x, c) \,\mathrm{d}\, x + \int_a^b f(x, d) \,\mathrm{d}\, x\right) + \frac{1}{d-c}\left(\int_c^d f(a, y) \,\mathrm{d}\, y + \int_c^d f(b, y) \,\mathrm{d}\, y\right)\right] \\ &\le \frac{1}{4}[f(a, c) + f(b, c) + f(a, d) + f(b, d)]. \end{aligned}$$

Theorem 1.8 ([10, Theorem 2.1])**.** *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable function on Δ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|$ is quasi-convex on the co-ordinates on Δ, then*

$$\begin{aligned} &\left|\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} + \frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x, y) \,\mathrm{d}\, x \,\mathrm{d}\, y - A\right| \\ &\qquad \le \frac{(b-a)(d-c)}{16} \max\left\{\left|\frac{\partial^2 f(a, c)}{\partial x \partial y}\right|, \left|\frac{\partial^2 f(a, d)}{\partial x \partial y}\right|, \left|\frac{\partial^2 f(b, c)}{\partial x \partial y}\right|, \left|\frac{\partial^2 f(b, d)}{\partial x \partial y}\right|\right\}, \end{aligned}$$

where

$$A = \frac{1}{2(b-a)}\int_a^b [f(x, c) + f(x, d)] \,\mathrm{d}\, x + \frac{1}{2(d-c)}\int_c^d [f(a, y) + f(b, y)] \,\mathrm{d}\, y.$$

For more information on this topic, please refer to the papers [1, 2, 3, 7, 8, 11, 12, 13, 14, 15] and related references therein.

In this paper, we will introduce a new concept "geometrically quasi-convex function on co-ordinates" and establish some new Hermite–Hadamard type inequalities for geometrically quasi-convex functions on the co-ordinates.

2. Definition and Lemmas

Now we introduce the definition of the geometrically quasi-convex functions.

Definition 2.1. *Let $\mathbb{R}_+ = (0,\infty)$. A function $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}$ is said to be geometrically quasi-convex on the co-ordinates on Δ with $a < b$ and $c < d$ if*

$$f\bigl(x^t z^{1-t}, y^\lambda w^{1-\lambda}\bigr) \le \max\{f(x,y), f(x,w), f(z,y), f(z,w)\}$$

holds for all $t,\lambda \in [0,1]$ and $(x,y),(z,w) \in \Delta$.

Remark 2.2. If $f : \Delta \subseteq \mathbb{R}_+^2 \to \mathbb{R}$ is increasing and convex on the co-ordinates on Δ, then it is geometrically quasi-convex on the co-ordinates on Δ. If $f : \Delta \subseteq \mathbb{R}_+^2 \to \mathbb{R}$ is decreasing and geometrically quasi-convex on the co-ordinates on Δ, then it is quasi-convex on the co-ordinates on Δ.

Proof. By Definitions 1.4, 1.6, and 2.1, we have

$$\begin{aligned} f\bigl(x^t z^{1-t}, y^\lambda w^{1-\lambda}\bigr) &\le f(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ &\le t\lambda f(x,y) + t(1-\lambda) f(x,w) + (1-t)\lambda f(z,y) + (1-t)(1-\lambda) f(z,w) \\ &\le \max\{f(x,y), f(x,w), f(z,y), f(z,w)\} \end{aligned}$$

and

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \le f\bigl(x^\lambda z^{1-\lambda}, y^\lambda w^{1-\lambda}\bigr) \le \max\{f(x,y), f(x,w), f(z,y), f(z,w)\}.$$

This completes the required proof. □

In order to prove our main results, we need the following integral identity.

Lemma 2.3. *Let $f : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R}_+^2 \to \mathbb{R}$ have partial derivatives of the second order with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$, then*

$$\begin{aligned} S(f) &\triangleq \frac{16}{(\ln b - \ln a)(\ln d - \ln c)} \biggl[f\bigl(\sqrt{ab}, \sqrt{cd}\bigr) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f\bigl(x, \sqrt{cd}\bigr)}{x} \,\mathrm{d}\,x \\ &\quad - \frac{1}{\ln d - \ln c} \int_c^d \frac{f\bigl(\sqrt{ab}, y\bigr)}{y} \,\mathrm{d}\,y + \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x,y)}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y \biggr] \\ &= \int_0^1 \int_0^1 (1-t)(1-\lambda) a^{\frac{1-t}{2}} b^{\frac{1+t}{2}} c^{\frac{1-\lambda}{2}} d^{\frac{1+\lambda}{2}} \frac{\partial^2}{\partial x \partial y} f\Bigl(a^{\frac{1-t}{2}} b^{\frac{1+t}{2}}, c^{\frac{1-\lambda}{2}} d^{\frac{1+\lambda}{2}}\Bigr) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\ &\quad - \int_0^1 \int_0^1 (1-t)(1-\lambda) a^{\frac{1-t}{2}} b^{\frac{1+t}{2}} c^{\frac{1+\lambda}{2}} d^{\frac{1-\lambda}{2}} \frac{\partial^2}{\partial x \partial y} f\Bigl(a^{\frac{1-t}{2}} b^{\frac{1+t}{2}}, c^{\frac{1+\lambda}{2}} d^{\frac{1-\lambda}{2}}\Bigr) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\ &\quad - \int_0^1 \int_0^1 (1-t)(1-\lambda) a^{\frac{1+t}{2}} b^{\frac{1-t}{2}} c^{\frac{1-\lambda}{2}} d^{\frac{1+\lambda}{2}} \frac{\partial^2}{\partial x \partial y} f\Bigl(a^{\frac{1+t}{2}} b^{\frac{1-t}{2}}, c^{\frac{1-\lambda}{2}} d^{\frac{1+\lambda}{2}}\Bigr) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\ &\quad + \int_0^1 \int_0^1 (1-t)(1-\lambda) a^{\frac{1+t}{2}} b^{\frac{1-t}{2}} c^{\frac{1+\lambda}{2}} d^{\frac{1-\lambda}{2}} \frac{\partial^2}{\partial x \partial y} f\Bigl(a^{\frac{1+t}{2}} b^{\frac{1-t}{2}}, c^{\frac{1+\lambda}{2}} d^{\frac{1-\lambda}{2}}\Bigr) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda. \end{aligned}$$

Proof. Integrating by parts gives

$$\begin{aligned}
&\int_0^1\int_0^1(1-t)(1-\lambda)a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\frac{\partial^2}{\partial x\partial y}f\Big(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\Big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
&=\frac{2}{\ln b-\ln a}\int_0^1(1-\lambda)c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\bigg[(1-t)\frac{\partial}{\partial y}f\Big(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\Big)\Big|_0^1\\
&\quad+\int_0^1\frac{\partial}{\partial y}f\Big(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\Big)\,\mathrm{d}\,t\bigg]\,\mathrm{d}\,\lambda\\
&=\frac{2}{\ln a-\ln b}\bigg[\int_0^1(1-\lambda)c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\frac{\partial}{\partial y}f\Big(\sqrt{ab}\,,c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\Big)\,\mathrm{d}\,\lambda\\
&\quad-\int_0^1\int_0^1(1-\lambda)c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\frac{\partial}{\partial y}f\Big(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\Big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\bigg]\\
&=\frac{4}{(\ln b-\ln a)(\ln d-\ln c)}\bigg[f\big(\sqrt{ab}\,,\sqrt{cd}\,\big)-\int_0^1f\Big(\sqrt{ab}\,,c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\Big)\,\mathrm{d}\,\lambda\\
&\quad-\int_0^1f\Big(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},\sqrt{cd}\Big)\,\mathrm{d}\,t+\int_0^1\int_0^1f\Big(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\Big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\bigg].
\end{aligned}$$

Choosing in the above identity $x=a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}$ and $y=c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}$ for $t,\lambda\in[0,1]$ yields

$$\begin{aligned}
&\int_0^1\int_0^1(1-t)(1-\lambda)a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\frac{\partial^2}{\partial x\partial y}f\Big(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\Big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
&=\frac{4}{(\ln b-\ln a)(\ln d-\ln c)}\bigg[f\big(\sqrt{ab}\,,\sqrt{cd}\,\big)-\frac{2}{\ln b-\ln a}\int_{\sqrt{ab}}^b\frac{f\big(x,\sqrt{cd}\,\big)}{x}\,\mathrm{d}\,x\\
&\quad-\frac{2}{\ln d-\ln c}\int_{\sqrt{cd}}^d\frac{f\big(\sqrt{ab}\,,y\big)}{y}\,\mathrm{d}\,y+\frac{4}{(\ln b-\ln a)(\ln d-\ln c)}\int_{\sqrt{cd}}^d\int_{\sqrt{ab}}^b\frac{f(x,y)}{xy}\,\mathrm{d}\,x\,\mathrm{d}\,y\bigg].
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1(1-t)(1-\lambda)a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}c^{\frac{1+\lambda}{2}}d^{\frac{1-\lambda}{2}}\frac{\partial^2}{\partial x\partial y}f\Big(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},c^{\frac{1+\lambda}{2}}d^{\frac{1-\lambda}{2}}\Big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
&=-\frac{4}{(\ln b-\ln a)(\ln d-\ln c)}\bigg[f\big(\sqrt{ab}\,,\sqrt{cd}\,\big)-\frac{2}{\ln b-\ln a}\int_{\sqrt{ab}}^b\frac{f\big(x,\sqrt{cd}\,\big)}{x}\,\mathrm{d}\,x\\
&\quad-\frac{2}{\ln d-\ln c}\int_c^{\sqrt{cd}}\frac{f\big(\sqrt{ab}\,,y\big)}{y}\,\mathrm{d}\,y+\frac{4}{(\ln b-\ln a)(\ln d-\ln c)}\int_c^{\sqrt{cd}}\int_{\sqrt{ab}}^b\frac{f(x,y)}{xy}\,\mathrm{d}\,x\,\mathrm{d}\,y\bigg],
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1(1-t)(1-\lambda)a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\frac{\partial^2}{\partial x\partial y}f\Big(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}},c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\Big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
&=-\frac{4}{(\ln b-\ln a)(\ln d-\ln c)}\bigg[f\big(\sqrt{ab}\,,\sqrt{cd}\,\big)-\frac{2}{\ln b-\ln a}\int_a^{\sqrt{ab}}\frac{f\big(x,\sqrt{cd}\,\big)}{x}\,\mathrm{d}\,x\\
&\quad-\frac{2}{\ln d-\ln c}\int_{\sqrt{cd}}^d\frac{f\big(\sqrt{ab}\,,y\big)}{y}\,\mathrm{d}\,y+\frac{4}{(\ln b-\ln a)(\ln d-\ln c)}\int_{\sqrt{cd}}^d\int_a^{\sqrt{ab}}\frac{f(x,y)}{xy}\,\mathrm{d}\,x\,\mathrm{d}\,y\bigg],
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1(1-t)(1-\lambda)a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}c^{\frac{1+\lambda}{2}}d^{\frac{1-\lambda}{2}}\frac{\partial^2}{\partial x\partial y}f\Big(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}},c^{\frac{1+\lambda}{2}}d^{\frac{1-\lambda}{2}}\Big)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
&=\frac{4}{(\ln b-\ln a)(\ln d-\ln c)}\bigg[f\big(\sqrt{ab}\,,\sqrt{cd}\,\big)-\frac{2}{\ln b-\ln a}\int_a^{\sqrt{ab}}\frac{f\big(x,\sqrt{cd}\,\big)}{x}\,\mathrm{d}\,x\\
&\quad-\frac{2}{\ln d-\ln c}\int_c^{\sqrt{cd}}\frac{f\big(\sqrt{ab}\,,y\big)}{y}\,\mathrm{d}\,y+\frac{4}{(\ln b-\ln a)(\ln d-\ln c)}\int_c^{\sqrt{cd}}\int_a^{\sqrt{ab}}\frac{f(x,y)}{xy}\,\mathrm{d}\,x\,\mathrm{d}\,y\bigg].
\end{aligned}$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *Let $u,v>0$, $h\in\mathbb{R}$, and $h\ne 0$. Then*

$$Q(h;u,v)=\int_0^1(1-t)u^{\frac12+ht}v^{\frac12-ht}\,\mathrm{d}\,t=\begin{cases}\dfrac{u^{\frac12}v^{\frac12-h}\big[v^h-L\big(u^h,v^h\big)\big]}{h(\ln v-\ln u)}, & u\ne v,\\ \dfrac12 u, & u=v,\end{cases}\tag{2.1}$$

where $L(u,v)$ is the logarithmic mean

$$L(u,v)=\begin{cases}\dfrac{v-u}{\ln v-\ln u}, & u\ne v,\\ u, & u=v.\end{cases}$$

Proof. This follows from integration by parts. □

3. Main Results

Now we start out to prove some new inequalities of Hermite–Hadamard type for geometrically quasi-convex functions on the co-ordinates.

Theorem 3.1. *Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}_+^2\to\mathbb{R}$ be a partial differentiable function on Δ with $a<b$, $c<d$, and $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ is a geometrically quasi-convex function on the co-ordinates on Δ for $q\ge 1$, then*

$$\begin{aligned}\big|S(f)\big|\le\bigg[&Q\Big(-\frac12;a,b\Big)Q\Big(-\frac12;c,d\Big)+Q\Big(-\frac12;a,b\Big)Q\Big(\frac12;c,d\Big)\\&+Q\Big(\frac12;a,b\Big)Q\Big(-\frac12;c,d\Big)+Q\Big(\frac12;a,b\Big)Q\Big(\frac12;c,d\Big)\bigg]M(f),\end{aligned}\tag{3.1}$$

where $Q(h;u,v)$ is defined by (2.1) and

$$M(f)=\max\bigg\{\bigg|\frac{\partial^2 f(a,c)}{\partial x\partial y}\bigg|,\bigg|\frac{\partial^2 f(a,d)}{\partial x\partial y}\bigg|,\bigg|\frac{\partial^2 f(b,c)}{\partial x\partial y}\bigg|,\bigg|\frac{\partial^2 f(b,d)}{\partial x\partial y}\bigg|\bigg\}.\tag{3.2}$$

Proof. By Lemma 2.3, we have

$$\begin{aligned}\big|S(f)\big|\le&\int_0^1\int_0^1(1-t)(1-\lambda)a^{\frac{1-t}2}b^{\frac{1+t}2}c^{\frac{1-\lambda}2}d^{\frac{1+\lambda}2}\bigg|\frac{\partial^2}{\partial x\partial y}f\Big(a^{\frac{1-t}2}b^{\frac{1+t}2},c^{\frac{1-\lambda}2}d^{\frac{1+\lambda}2}\Big)\bigg|\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\&+\int_0^1\int_0^1(1-t)(1-\lambda)a^{\frac{1-t}2}b^{\frac{1+t}2}c^{\frac{1+\lambda}2}d^{\frac{1-\lambda}2}\bigg|\frac{\partial^2}{\partial x\partial y}f\Big(a^{\frac{1-t}2}b^{\frac{1+t}2},c^{\frac{1+\lambda}2}d^{\frac{1-\lambda}2}\Big)\bigg|\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\&+\int_0^1\int_0^1(1-t)(1-\lambda)a^{\frac{1+t}2}b^{\frac{1-t}2}c^{\frac{1-\lambda}2}d^{\frac{1+\lambda}2}\bigg|\frac{\partial^2}{\partial x\partial y}f\Big(a^{\frac{1+t}2}b^{\frac{1-t}2},c^{\frac{1-\lambda}2}d^{\frac{1+\lambda}2}\Big)\bigg|\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\&+\int_0^1\int_0^1(1-t)(1-\lambda)a^{\frac{1+t}2}b^{\frac{1-t}2}c^{\frac{1+\lambda}2}d^{\frac{1-\lambda}2}\bigg|\frac{\partial^2}{\partial x\partial y}f\Big(a^{\frac{1+t}2}b^{\frac{1-t}2},c^{\frac{1+\lambda}2}d^{\frac{1-\lambda}2}\Big)\bigg|\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\&\triangleq I_1+I_2+I_3+I_4.\end{aligned}\tag{3.3}$$

Using Hölder's integral inequality, from the co-ordinated geometrically quasi-convexity of $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ on Δ, and by Lemma 2.4, we have

$$
\begin{aligned}
I_1 &\le \left(\int_0^1\int_0^1 (1-t)(1-\lambda)a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}\\
&\quad\times\left[\int_0^1\int_0^1 (1-t)(1-\lambda)a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\left|\frac{\partial^2}{\partial x\partial y}f\left(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\right)\right|^q\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}\\
&\le \left(\int_0^1\int_0^1 (1-t)(1-\lambda)a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}\\
&\quad\times\left[\int_0^1\int_0^1 (1-t)(1-\lambda)a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}\left[\int_0^1\int_0^1 M^q(f)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}\\
&= \left(\int_0^1\int_0^1 (1-t)(1-\lambda)a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)M(f)\\
&= Q\left(-\frac{1}{2};a,b\right)Q\left(-\frac{1}{2};c,d\right)M(f).
\end{aligned}
\tag{3.4}
$$

Using simple techniques of integration shows

$$
I_2 \le Q\left(-\frac{1}{2};a,b\right)Q\left(\frac{1}{2};c,d\right)M(f),\quad I_3 \le Q\left(\frac{1}{2};a,b\right)Q\left(-\frac{1}{2};c,d\right)M(f),
$$

and

$$
I_4 \le Q\left(\frac{1}{2};a,b\right)Q\left(\frac{1}{2};c,d\right)M(f). \tag{3.5}
$$

Substituting the inequalities (3.4) to (3.5) into the inequality (3.3) yields (3.1). Theorem 3.1 is proved. □

Theorem 3.2. *Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}_+^2\to\mathbb{R}$ be a partial differentiable function on Δ with $a<b$, $c<d$, and $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ is a geometrically quasi-convex function on the co-ordinates on Δ for $q\ge 1$, then*

$$
\begin{aligned}
\bigl|S(f)\bigr| \le 2^{2(1/q-1)}\Biggl\{&\left[Q\left(-\frac{1}{2};a^q,b^q\right)Q\left(-\frac{1}{2};c^q,d^q\right)\right]^{1/q}+\left[Q\left(-\frac{1}{2};a^q,b^q\right)Q\left(\frac{1}{2};c^q,d^q\right)\right]^{1/q}\\
&+\left[Q\left(\frac{1}{2};a^q,b^q\right)Q\left(-\frac{1}{2};c^q,d^q\right)\right]^{1/q}+\left[Q\left(\frac{1}{2};a^q,b^q\right)Q\left(\frac{1}{2};c^q,d^q\right)\right]^{1/q}\Biggr\}M(f),
\end{aligned}
\tag{3.6}
$$

where $Q(h;u,v)$ is defined by (2.1) and $M(f)$ is defined by (3.2).

Proof. Using the inequality (3.3), by Hölder's integral inequality, and from the co-ordinated geometrically quasi-convexity of $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ on Δ, we have

$$
\begin{aligned}
I_1 &\le \left(\int_0^1\int_0^1 (1-t)(1-\lambda)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right)^{1-1/q}\\
&\quad\times\left[\int_0^1\int_0^1 (1-t)(1-\lambda)a^{q\frac{1-t}{2}}b^{q\frac{1+t}{2}}c^{q\frac{1-\lambda}{2}}d^{q\frac{1+\lambda}{2}}\left|\frac{\partial^2}{\partial x\partial y}f\left(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}},c^{\frac{1-\lambda}{2}}d^{\frac{1+\lambda}{2}}\right)\right|^q\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}\\
&\le 2^{-2(1-1/q)}\left[\int_0^1\int_0^1 (1-t)(1-\lambda)a^{q\frac{1-t}{2}}b^{q\frac{1+t}{2}}c^{q\frac{1-\lambda}{2}}d^{q\frac{1+\lambda}{2}}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\right]^{1/q}M(f)\\
&= 2^{-2(1-1/q)}\left[Q\left(-\frac{1}{2};a^q,b^q\right)Q\left(-\frac{1}{2};c^q,d^q\right)\right]^{1/q}M(f).
\end{aligned}
$$

Similarly, we have

$$I_2 \le 2^{-2(1-1/q)}\left[Q\left(-\frac{1}{2}; a^q, b^q\right)Q\left(\frac{1}{2}; c^q, d^q\right)\right]^{1/q} M(f),$$
$$I_3 \le 2^{-2(1-1/q)}\left[Q\left(\frac{1}{2}; a^q, b^q\right)Q\left(-\frac{1}{2}; c^q, d^q\right)\right]^{1/q} M(f),$$

and

$$I_4 \le 2^{-2(1-1/q)}\left[Q\left(\frac{1}{2}; a^q, b^q\right)Q\left(\frac{1}{2}; c^q, d^q\right)\right]^{1/q} M(f).$$

This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ be a partial differentiable function on Δ with $a < b$, $c < d$, and $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is a geometrically quasi-convex function on the co-ordinates on Δ for $q > 1$ and $q \ge r \ge 0$, then*

$$\begin{aligned}\left|S(f)\right| \le \Bigg\{&\left[Q\left(-\frac{1}{2}; a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}\right)Q\left(-\frac{1}{2}; c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}\right)\right]^{1-1/q}\left[Q\left(-\frac{1}{2}; a^r, b^r\right)Q\left(-\frac{1}{2}; c^r, d^r\right)\right]^{1/q}\\
&+\left[Q\left(-\frac{1}{2}; a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}\right)Q\left(\frac{1}{2}; c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}\right)\right]^{1-1/q}\left[Q\left(-\frac{1}{2}; a^r, b^r\right)Q\left(\frac{1}{2}; c^r, d^r\right)\right]^{1/q}\\
&+\left[Q\left(\frac{1}{2}; a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}\right)Q\left(-\frac{1}{2}; c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}\right)\right]^{1-1/q}\left[Q\left(\frac{1}{2}; a^r, b^r\right)Q\left(-\frac{1}{2}; c^r, d^r\right)\right]^{1/q}\\
&+\left[Q\left(\frac{1}{2}; a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}\right)Q\left(\frac{1}{2}; c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}\right)\right]^{1-1/q}\left[Q\left(\frac{1}{2}; a^r, b^r\right)Q\left(\frac{1}{2}; c^r, d^r\right)\right]^{1/q}\Bigg\}M(f),\end{aligned}$$

where $Q(h; u, v)$ is defined by (2.1) *and $M(f)$ is defined by* (3.2)*.*

Proof. Using the inequality (3.3), by Hölder's integral inequality, and from the co-ordinated geometrically quasi-convexity of $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ on Δ, then

$$\begin{aligned}I_1 &\le \left(\int_0^1\int_0^1 (1-t)(1-\lambda)\left(a^{\frac{1-t}{2}} b^{\frac{1+t}{2}} c^{\frac{1-\lambda}{2}} d^{\frac{1+\lambda}{2}}\right)^{(q-r)/(q-1)} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda\right)^{1-1/q}\\
&\quad\times\left[\int_0^1\int_0^1 (1-t)(1-\lambda)\left(a^{\frac{1-t}{2}} b^{\frac{1+t}{2}} c^{\frac{1-\lambda}{2}} d^{\frac{1+\lambda}{2}}\right)^r \left|\frac{\partial^2}{\partial x \partial y} f\left(a^{\frac{1-t}{2}} b^{\frac{1+t}{2}}, c^{\frac{1-\lambda}{2}} d^{\frac{1+\lambda}{2}}\right)\right|^q \,\mathrm{d}\,t \,\mathrm{d}\,\lambda\right]^{1/q}\\
&\le \left(\int_0^1\int_0^1 (1-t)(1-\lambda)\left(a^{\frac{1-t}{2}} b^{\frac{1+t}{2}} c^{\frac{1-\lambda}{2}} d^{\frac{1+\lambda}{2}}\right)^{(q-r)/(q-1)} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda\right)^{1-1/q}\\
&\quad\times\left[\int_0^1\int_0^1 (1-t)(1-\lambda)\left(a^{\frac{1-t}{2}} b^{\frac{1+t}{2}} c^{\frac{1-\lambda}{2}} d^{\frac{1+\lambda}{2}}\right)^r \,\mathrm{d}\,t \,\mathrm{d}\,\lambda\right]^{1/q} M(f)\\
&= \left[Q\left(-\frac{1}{2}; a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}\right)Q\left(-\frac{1}{2}; c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}\right)\right]^{1-1/q}\left[Q\left(-\frac{1}{2}; a^r, b^r\right)Q\left(-\frac{1}{2}; c^r, d^r\right)\right]^{1/q} M(f).\end{aligned}$$

Similarly, we have

$$I_2 \le \left[Q\left(-\frac{1}{2}; a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}\right)Q\left(\frac{1}{2}; c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}\right)\right]^{1-1/q}\left[Q\left(-\frac{1}{2}; a^r, b^r\right)Q\left(\frac{1}{2}; c^r, d^r\right)\right]^{1/q} M(f),$$
$$I_3 \le \left[Q\left(\frac{1}{2}; a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}\right)Q\left(-\frac{1}{2}; c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}\right)\right]^{1-1/q}\left[Q\left(\frac{1}{2}; a^r, b^r\right)Q\left(-\frac{1}{2}; c^r, d^r\right)\right]^{1/q} M(f),$$

and

$$I_4 \le \left[Q\left(\frac{1}{2}; a^{\frac{q-r}{q-1}}, b^{\frac{q-r}{q-1}}\right) Q\left(\frac{1}{2}; c^{\frac{q-r}{q-1}}, d^{\frac{q-r}{q-1}}\right)\right]^{1-1/q} \left[Q\left(\frac{1}{2}; a^r, b^r\right) Q\left(\frac{1}{2}; c^r, d^r\right)\right]^{1/q} M(f).$$

This completes the required proof. □

Remark 3.4. Under the conditions of Theorem 3.3, if $r = q$, then (3.6) holds.

Corollary 3.5. *Under the conditions of Theorem 3.3, when $r = 0$, we have*

$$\begin{aligned}\left|S(f)\right| \le 2^{-2/q}\Bigg\{&\left[Q\left(-\frac{1}{2}; a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}\right) Q\left(-\frac{1}{2}; c^{\frac{q}{q-1}}, d^{\frac{q}{q-1}}\right)\right]^{1-1/q} \\ &+ \left[Q\left(-\frac{1}{2}; a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}\right) Q\left(\frac{1}{2}; c^{\frac{q}{q-1}}, d^{\frac{q}{q-1}}\right)\right]^{1-1/q} + \left[Q\left(\frac{1}{2}; a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}\right) Q\left(-\frac{1}{2}; c^{\frac{q}{q-1}}, d^{\frac{q}{q-1}}\right)\right]^{1-1/q} \\ &+ \left[Q\left(\frac{1}{2}; a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}\right) Q\left(\frac{1}{2}; c^{\frac{q}{q-1}}, d^{\frac{q}{q-1}}\right)\right]^{1-1/q}\Bigg\} M(f),\end{aligned}$$

where $Q(h; u, v)$ is defined by (2.1) *and $M(f)$ is defined by* (3.2).

Theorem 3.6. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ be a partial differentiable function on Δ with $a < b$, $c < d$, and $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ is a geometrically quasi-convex function on the co-ordinates on Δ for $q > 1$, then*

$$\left|S(f)\right| \le \left(\frac{q-1}{2q-1}\right)^{2(1-1/q)} \left[(ac)^{\frac{1}{2}} + (ad)^{\frac{1}{2}} + (bc)^{\frac{1}{2}} + (bd)^{\frac{1}{2}}\right] \left[L\left(a^{\frac{q}{2}}, b^{\frac{q}{2}}\right) L\left(c^{\frac{q}{2}}, d^{\frac{q}{2}}\right)\right]^{1/q} M(f),$$

where $L(u, v)$ is the logarithmic mean and $M(f)$ is defined by (3.2).

Proof. Using the inequality (3.3), by Hölder's integral inequality, and from the co-ordinated geometrically quasi-convexity of $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^q$ on Δ, we have

$$\begin{aligned} I_1 &\le \left(\int_0^1 \int_0^1 [(1-t)(1-\lambda)]^{q/(q-1)} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda\right)^{1-1/q} \\ &\quad \times \left[\int_0^1 \int_0^1 a^{q\frac{1-t}{2}} b^{q\frac{1+t}{2}} c^{q\frac{1-\lambda}{2}} d^{q\frac{1+\lambda}{2}} \left|\frac{\partial^2}{\partial x \partial y} f\left(a^{\frac{1-t}{2}} b^{\frac{1+t}{2}}, c^{\frac{1-\lambda}{2}} d^{\frac{1+\lambda}{2}}\right)\right|^q \mathrm{d}\,t \,\mathrm{d}\,\lambda\right]^{1/q} \\ &\le \left(\frac{q-1}{2q-1}\right)^{2(1-1/q)} \left[\int_0^1 \int_0^1 a^{q\frac{1-t}{2}} b^{q\frac{1+t}{2}} c^{q\frac{1-\lambda}{2}} d^{q\frac{1+\lambda}{2}} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda\right]^{1/q} M(f) \\ &= \left(\frac{q-1}{2q-1}\right)^{2(1-1/q)} (bd)^{\frac{1}{2}} \left[L\left(a^{\frac{q}{2}}, b^{\frac{q}{2}}\right) L\left(c^{\frac{q}{2}}, d^{\frac{q}{2}}\right)\right]^{1/q} M(f). \end{aligned}$$

Similarly, we have

$$I_2 \le \left(\frac{q-1}{2q-1}\right)^{2(1-1/q)} (bc)^{\frac{1}{2}} \left[L\left(a^{\frac{q}{2}}, b^{\frac{q}{2}}\right) L\left(c^{\frac{q}{2}}, d^{\frac{q}{2}}\right)\right]^{1/q} M(f),$$

$$I_3 \le \left(\frac{q-1}{2q-1}\right)^{2(1-1/q)} (ad)^{\frac{1}{2}} \left[L\left(a^{\frac{q}{2}}, b^{\frac{q}{2}}\right) L\left(c^{\frac{q}{2}}, d^{\frac{q}{2}}\right)\right]^{1/q} M(f),$$

and

$$I_4 \le \left(\frac{q-1}{2q-1}\right)^{2(1-1/q)} (ac)^{\frac{1}{2}} \left[L\left(a^{\frac{q}{2}}, b^{\frac{q}{2}}\right) L\left(c^{\frac{q}{2}}, d^{\frac{q}{2}}\right)\right]^{1/q} M(f).$$

The proof of Theorem 3.6 is complete. □

Theorem 3.7. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ be integrable on Δ with $a < b$ and $c < d$. If f is a geometrically quasi-convex function on the co-ordinates on Δ, then*

$$\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y \le \max\{f(a, c), f(a, d), f(b, c), f(b, d)\}.$$

Proof. Letting $x = a^t b^{1-t}$ and $y = c^\lambda d^{1-\lambda}$ for $t, \lambda \in [0, 1]$. By the co-ordinated geometrically quasi-convexity of f on Δ, we have

$$\begin{aligned}\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y &= \int_0^1 \int_0^1 f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\ &\le \int_0^1 \int_0^1 \max\{f(a, c), f(a, d), f(b, c), f(b, d)\} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\ &= \max\{f(a, c), f(a, d), f(b, c), f(b, d)\}.\end{aligned}$$

This completes the proof of Theorem 3.7. □

Theorem 3.8. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ be integrable on Δ with $a < b$ and $c < d$. If f is a geometrically quasi-convex function on the co-ordinates on Δ, then*

$$\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b f(x, y) \,\mathrm{d}\,x \,\mathrm{d}\,y \le L(a, b)L(c, d) \max\{f(a, c), f(a, d), f(b, c), f(b, d)\},$$

where $L(u, v)$ is the logarithmic mean.

Proof. Similarly as in Theorem 3.7, by the co-ordinated geometrically quasi-convexity of f on Δ, we have

$$\begin{aligned}\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b f(x, y) \,\mathrm{d}\,x \,\mathrm{d}\,y &= \int_0^1 \int_0^1 a^t b^{1-t} c^\lambda d^{1-\lambda} f\big(a^t b^{1-t}, c^\lambda d^{1-\lambda}\big) \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\ &\le \max\{f(a, c), f(a, d), f(b, c), f(b, d)\} \int_0^1 \int_0^1 a^t b^{1-t} c^\lambda d^{1-\lambda} \,\mathrm{d}\,t \,\mathrm{d}\,\lambda \\ &= L(a, b)L(c, d) \max\{f(a, c), f(a, d), f(b, c), f(b, d)\}.\end{aligned}$$

The proof of Theorem 3.8 is complete. □

We proceed similarly as in the proof of Theorems 3.7 and 3.8, we can obtain the following theorem.

Theorem 3.9. *Let $f, g : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2_+ \to \mathbb{R}_0 = [0, \infty)$ be integrable on Δ with $a < b$ and $c < d$. If f, g are geometrically quasi-convex functions on the co-ordinates on Δ, then*

$$\begin{aligned}&\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)g(x, y)}{xy} \,\mathrm{d}\,x \,\mathrm{d}\,y \\ &\qquad \le \max\{f(a, c), f(a, d), f(b, c), f(b, d)\} \max\{g(a, c), g(a, d), g(b, c), g(b, d)\}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b f(x, y)g(x, y) \,\mathrm{d}\,x \,\mathrm{d}\,y \\ &\qquad \le [L(a, b)L(c, d)]^2 \max\{f(a, c), f(a, d), f(b, c), f(b, d)\} \max\{g(a, c), g(a, d), g(b, c), g(b, d)\},\end{aligned}$$

where $L(u, v)$ is the logarithmic mean.

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China under Grant No. 11361038, by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant Grant No. NJZY14192, by the Science Research Fund of the Inner Mongolia University for Nationalities under Grant No. NMD1302, and by the Scientific Innovation Project for Graduates at the Inner Mongolia University for Nationalities under Grant No. NMDSS1419, China.

The authors thank anonymous referees for their valuable comments on and careful corrections to the original version of this paper.

References

[1] M. Alomari, M. Darus, *On the Hadamard's inequality for log-convex functions on the coordinates*, J. Inequal. Appl., **2009** (2009), 13 pages. 1

[2] S. P. Bai, F. Qi, *Some inequalities for* (s_1, m_1)*-*(s_2, m_2)*-convex functions on co-ordinates*, Glob. J. Math. Anal., **1** (2013), 22–28. 1

[3] S. P. Bai, S. H. Wang, F. Qi, *Some new integral inequalities of Hermite–Hadamard type for* $(\alpha, m; P)$*-convex functions on co-ordinates*, J. Appl. Anal. Comput., (2015), in press. 1

[4] S. S. Dragomir, *On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese J. Math., **5** (2001), 775–788. 1.3, 1.4, 1.7

[5] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite–Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, (2000). 1.3, 1.4, 1.7

[6] S. S. Dragomir, J. Pečarić, L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math., **21** (1995), 335–341. 1.2

[7] M. A. Latif, S. S. Dragomir, *On some new inequalities for dfferentiable co-ordinated convex functions*, J. Inequal. Appl., **2012** (2012), 13 pages. 1

[8] M. E. Özdemir, A. O. Akdemir, H. Kavurmaci, M. Avci, *On the Simpson's inequality for co-ordinated convex functions*, arXive, (2010), 8 pages. 1

[9] M. E. Özdemir, A. O. Akdemir, Ç. Yıldız, *On co-ordinated quasi-convex functions*, Czechoslovak Math. J., **62** (2012), 889–900. 1.5

[10] M. E. Özdemir, Ç. Yıldız, A. O. Akdemir,*on some new Hadamard-type inequalities for co-ordinated quasi-convex functions*, Hacet. J. Math. Stat., **41** (2012), 697–707. 1.6, 1.8

[11] B. Y. Xi, R.-F. Bai, F. Qi, *Hermite–Hadamard type inequalities for the* m*- and* (α, m)*-geometrically convex functions*, Aequationes Math., **84** (2012), 261–269. 1

[12] B. Y. Xi, S. P. Bai, F. Qi, *Some new inequalities of Hermite–Hadamard type for* (α, m_1)*-*(s, m_2)*-convex functions on co-ordinates*, ResearchGate. 1

[13] B. Y. Xi, J. Hua, F. Qi, *Hermite–Hadamard type inequalities for extended* s*-convex functions on the co-ordinates in a rectangle*, J. Appl. Anal., **20** (2014), 29–39. 1

[14] B. Y. Xi, F. Qi, *Some Hermite–Hadamard type inequalities for differentiable convex functions and applications*, Hacet. J. Math. Stat., **42** (2013), 243–257. 1

[15] B. Y. Xi, F. Qi, *Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means*, J. Funct. Spaces Appl., **2012** (2012), 14 pages. 1