Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 776–786
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Stability of functional inequalities associated with the Cauchy-Jensen additive functional equalities in non-Archimedean Banach spaces

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Abstract

In this article, we prove the generalized Hyers-Ulam stability of the following Pexider functional inequalities

$$\|f(x)+g(y)+kh(z)\| \leq \left\|kp\left(\frac{x+y}{k}+z\right)\right\|,$$

$$\|f(x)+g(y)+h(z)\| \leq \left\|kp\left(\frac{x+y+z}{k}\right)\right\|$$

in non-Archimedean Banach spaces.

Keywords: Hyers-Ulam stability, Pexider Cauchy-Jensen functional inequality, non-Archimedean space, additive mapping.
2010 MSC: 39B05, 39B52, 39B62, 39B82.

1. Introduction and Preliminaries

We recall some basic facts concerning non-Archimedean spaces. By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ to $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$ and $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

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Received 2014-12-22

Definition 1.1. Let X be a vector space over a non-Archimedean scalar field $\mathbb{K}$ with a valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is a *non-Archimedean norm* if it satisfies for all $r \in \mathbb{K},\ x, y \in X$

(i) $\|x\| = 0$ if and only if $x = 0$,
(ii) $\|rx\| = |r|\|x\|$,
(iii) $\|x + y\| \le \max\{\|x\|, \|y\|\}$ (the strong triangle inequality).

Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

(1) $\{x_n\}$ *converges* to $x \in X$ if, for any $\epsilon \ge 0$ there exists an integer N such that $\|x_n - x\| \le \epsilon$ for $n \ge N$. Then the point x is called the *limit* of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} x_n = x$.
(2) $\{x_n\}$ is a *Cauchy sequence* if the sequence $\{x_{n+1} - x_n\}$ converges to zero.
(3) X is called a *non-Archimedean Banach space* if every Cauchy sequence in X is convergent.

The stability problem of functional equations originated from a question of Ulam [16] in 1940, concerning the stability of group homomorphisms. In 1941, Hyers [9] gave the first affirmative answer to the problem of Ulam for Banach spaces. Hyers' result was generalized by Aoki [1] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. Generalizations of the Rassias' theorem were obtained by Forti [5] and Găvruta [6] who permitted the Cauchy difference to become arbitrary unbounded.

During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings. A large list of references concerning the stability of various functional equations can be found e.g., in the books [3, 10, 11].

Gilányi [7] and Rätz [15] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\|$$

then f satisfies the Jordan-von Neumann functional equation $2f(x) + 2f(y) = f(xy) + f(xy^{-1})$. Gilányi [8] and Fechner [4] investigated the Hyers-Ulam stability of the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \le \|f(x + y)\|.$$

Park et al. [13] investigated the following inequalities:

$$\|f(x) + f(y) + f(z)\| \le \left\|2f\left(\frac{x+y+z}{2}\right)\right\|,$$
$$\|f(x) + f(y) + f(z)\| \le \|f(x + y + z)\|,$$
$$\|f(x) + f(y) + 2f(z)\| \le \left\|2f\left(\frac{x+y}{2} + z\right)\right\|$$

in Banach spaces. Recently, Cho et al. [2] investigated the following inequality

$$\|f(x) + f(y) + f(z)\| \le \left\|kf\left(\frac{x+y+z}{k}\right)\right\|, \qquad (0 < |k| < 3)$$

in non-Archimedean Banach spaces. Lu and Park [12] investigated the following functional inequalities

$$\|f(x) + f(y) + f(z)\| \le \left\|kf\left(\frac{x+y+z}{k}\right)\right\|,$$

$$\|f(x) + f(y) + kf(z)\| \le \left\|kf\left(\frac{x+y}{k} + z\right)\right\|$$

in Banach spaces.

In this paper we investigate the generalized Hyers-Ulam stability of the following Pexider functional inequalities

$$\|f(x)+g(y)+kh(z)\| \le \left\|kp\left(\frac{x+y}{k}+z\right)\right\| \tag{1.1}$$

$$\|f(x)+g(y)+h(z)\| \le \left\|kp\left(\frac{x+y+z}{k}\right)\right\| \tag{1.2}$$

in non-Archimedean Banach spaces.

2. Hyers-Ulam stability of (1.1)

In what follows we assume that X is a non-Archimedean normed space, Y is a non-Archimedean Banach space and k is a nonzero scalar.

Proposition 2.1. *Let $f, g, h, p : X \to Y$ be mappings such that $g(0) = h(0) = p(0) = 0$ and*

$$\|f(x)+g(y)+kh(z)\| \le \left\|kp\left(\frac{x+y}{k}+z\right)\right\| \tag{2.1}$$

for all $x, y, z \in X$. Then f, g and h are additive, $f(x) = g(x) = kh(\frac{x}{k})$ for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.1), we have $f(0) = 0$.

Replacing (x, y, z) by $(x, -x, 0)$ in (2.1),

$$f(x)+g(-x)=0 \tag{2.2}$$

for all $x \in X$. Replacing (x, y, z) by $(x, 0, -\frac{x}{k})$ in (2.1),

$$f(x)+kh\left(-\frac{x}{k}\right)=0 \tag{2.3}$$

for all $x \in X$.

Replacing (x, y, z) by $\left(x, y, -\frac{x+y}{k}\right)$ in (2.1),

$$f(x)+g(y)+kh\left(-\frac{x+y}{k}\right)=0 \tag{2.4}$$

for all $x \in X$.

By (2.3) and (2.4), we have

$$f(x)+g(y)-f(x+y)=0, \tag{2.5}$$

so that

$$f(x)+g(y)=f(x+y) \tag{2.6}$$

for all $x, y \in X$. Letting $x = 0$ in (2.6), it follows that $f(y) = g(y)$, and hence

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in X$. Since f is additive it is clear that h is additive and $f(x) = kh\left(\frac{x}{k}\right)$ for all $x \in X$. This completes the proof. □

We prove the generalized Hyers-Ulam stability of the functional inequality (1.1).

Theorem 2.2. *Let $f, g, h, p : X \to Y$ be mappings such that $g(0) = h(0) = p(0) = 0$ and*

$$\|f(x) + g(y) + kh(z)\| \le \left\|kp\left(\frac{x+y}{k} + z\right)\right\| + \varphi(x, y, z), \tag{2.7}$$

where $\varphi : X^3 \to [0, \infty)$ satisfies $\varphi(0, 0, 0) = 0$ and

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{2.8}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned} \|f(x) - A(x)\| &\le \psi_1(x), \\ \|g(x) - A(x)\| &\le \psi_2(x), \\ \left\|h(x) - \frac{1}{k}A(kx)\right\| &\le \psi_3(x) \end{aligned} \tag{2.9}$$

for all $x \in X$. Here,

$$\begin{aligned} \psi_1(x) &= \sup_{j\ge 0}\left\{|2|^j \varphi\left(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, 0\right), |2|^j \varphi\left(\frac{x}{2^{j+1}}, 0, -\frac{x}{2^{j+1}k}\right), |2|^j \varphi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}k}\right)\right\}, \\ \psi_2(x) &= \sup_{j\ge 0}\left\{|2|^j \varphi\left(-\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right), |2|^j \varphi\left(0, \frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}k}\right), |2|^j \varphi\left(-\frac{x}{2^{j+1}}, \frac{x}{2^j}, -\frac{x}{2^{j+1}k}\right)\right\}, \\ \psi_3(x) &= \frac{1}{|k|}\sup_{j\ge 0}\left\{|2|^j \varphi\left(-\frac{kx}{2^{j+1}}, 0, \frac{x}{2^{j+1}}\right), |2|^j \varphi\left(0, -\frac{kx}{2^{j+1}}, \frac{x}{2^{j+1}}\right), |2|^j \varphi\left(-\frac{kx}{2^{j+1}}, -\frac{kx}{2^{j+1}}, \frac{x}{2^j}\right)\right\} \end{aligned}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.7), we get $f(0) = 0$. Replacing (x, y, z) by $(x, -x, 0)$ in (2.7), we have

$$\|f(x) + g(-x)\| \le \varphi(x, -x, 0) \qquad \forall x \in X. \tag{2.10}$$

Replacing (x, y, z) by $\left(x, 0, -\frac{x}{k}\right)$ in (2.7), we have

$$\left\|f(x) + kh\left(-\frac{x}{k}\right)\right\| \le \varphi\left(x, 0, -\frac{x}{k}\right) \qquad \forall x \in X. \tag{2.11}$$

From (2.10) and (2.11) we have

$$\|2f(x) + g(-x) + kh\left(-\frac{x}{k}\right)\| \le \max\left\{\varphi(x, -x, 0), \varphi\left(x, 0, -\frac{x}{k}\right)\right\} \qquad \forall x \in X. \tag{2.12}$$

Replacing (x, y, z) by $\left(2x, -x, -\frac{x}{k}\right)$ in (2.7), we have

$$\left\|f(2x) + g(-x) + kh\left(-\frac{x}{k}\right)\right\| \le \varphi\left(2x, -x, -\frac{x}{k}\right). \tag{2.13}$$

By (2.12) and (2.13), it follows that

$$\|2f(x) - f(2x)\| \le \max\left\{\varphi(x, -x, 0), \varphi\left(x, 0, -\frac{x}{k}\right), \varphi\left(2x, -x, -\frac{x}{k}\right)\right\}, \tag{2.14}$$

so that

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le \max\left\{\varphi\left(\frac{x}{2}, -\frac{x}{2}, 0\right), \varphi\left(\frac{x}{2}, 0, -\frac{x}{2k}\right), \varphi\left(x, -\frac{x}{2}, -\frac{x}{2k}\right)\right\} \qquad \forall x \in X. \tag{2.15}$$

Replacing x by $\frac{x}{2^j}$ and multiplying $|2|^j$ on both sides of (2.15), we have

$$\begin{aligned}&\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \max\left\{|2|^j\varphi\left(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, 0\right), |2|^j\varphi\left(\frac{x}{2^{j+1}}, 0, -\frac{x}{2^{j+1}k}\right), |2|^j\varphi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}k}\right)\right\} \to 0 \\ &\text{as } j \to \infty\end{aligned} \tag{2.16}$$

for all $x \in X$. Hence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence in Y. Since Y is complete, we can define the map $A : X \to Y$ such that

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right).$$

For nonnegative integers $l < m$, we have for all $x \in X$

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \max_{l\leq j\leq m-1}\left\{\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|\right\} \\ &\leq \max_{l\leq j\leq m-1}\left\{|2|^j\varphi\left(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, 0\right), |2|^j\varphi\left(\frac{x}{2^{j+1}}, 0, -\frac{x}{2^{j+1}k}\right),\right. \\ &\qquad\qquad \left. |2|^j\varphi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}k}\right)\right\}.\end{aligned} \tag{2.17}$$

Letting $l = 0$ and taking the limit as $m \to \infty$ in (2.17), we have

$$\begin{aligned}&\|f(x) - A(x)\| \\ &\leq \sup_{j\geq 0}\left\{|2|^j\varphi\left(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, 0\right), |2|^j\varphi\left(\frac{x}{2^{j+1}}, 0, -\frac{x}{2^{j+1}k}\right), |2|^j\varphi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}k}\right)\right\} \\ &= \psi_1(x)\end{aligned} \tag{2.18}$$

for all $x \in X$.

Similarly, there exists a mapping $B : X \to Y$ such that $B(x) = \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right)$ and

$$\begin{aligned}&\|g(x) - B(x)\| \\ &\leq \sup_{j\geq 0}\left\{|2|^j\varphi\left(-\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right), |2|^j\varphi\left(0, \frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}k}\right), |2|^j\varphi\left(-\frac{x}{2^{j+1}}, \frac{x}{2^j}, -\frac{x}{2^{j+1}k}\right)\right\} \\ &= \psi_2(x)\end{aligned} \tag{2.19}$$

for all $x \in X$.

Now we consider the mapping h. Replacing (x, y, z) by $\left(x, 0, -\frac{x}{k}\right)$ in (2.7), we have

$$\left\|f(x) + kh\left(-\frac{x}{k}\right)\right\| \leq \varphi\left(x, 0, -\frac{x}{k}\right) \qquad \forall x \in X. \tag{2.20}$$

Replacing (x, y, z) by $\left(0, x, -\frac{x}{k}\right)$ in (2.7), we have

$$\left\|g(x) + kh\left(-\frac{x}{k}\right)\right\| \leq \varphi\left(0, x, -\frac{x}{k}\right) \qquad \forall x \in X. \tag{2.21}$$

By (2.20),(2.21),

$$\left\|f(x) + g(x) + 2kh\left(-\frac{x}{k}\right)\right\| \leq \max\left\{\varphi\left(x, 0, -\frac{x}{k}\right), \varphi\left(0, x, -\frac{x}{k}\right)\right\} \qquad \forall x \in X. \tag{2.22}$$

Replacing (x, y, z) by $\left(x, x, -\frac{2x}{k}\right)$ in (2.7), we have

$$\left\|f(x) + g(x) + kh\left(-\frac{2x}{k}\right)\right\| \leq \varphi\left(x, x, -\frac{2x}{k}\right) \qquad \forall x \in X. \tag{2.23}$$

From (2.22) and (2.23), it follows that

$$\left\|2kh\left(-\frac{x}{k}\right)-kh\left(-\frac{2x}{k}\right)\right\|\le \max\left\{\varphi\left(x,0,-\frac{x}{k}\right),\varphi\left(0,x,-\frac{x}{k}\right),\varphi\left(x,x,-\frac{2x}{k}\right)\right\},$$

so that

$$\|2h(x)-h(2x)\|\le \frac{1}{|k|}\max\{\varphi(-kx,0,x),\varphi(0,-kx,x),\varphi(-kx,-kx,2x)\}\qquad \forall x\in X.$$

Then we have

$$\left\|2h\left(\frac{x}{2}\right)-h(x)\right\|\le \frac{1}{|k|}\max\left\{\varphi\left(-\frac{kx}{2},0,\frac{x}{2}\right),\varphi\left(0,-\frac{kx}{2},\frac{x}{2}\right),\varphi\left(-\frac{kx}{2},-\frac{kx}{2},x\right)\right\}\quad \forall x\in X. \tag{2.24}$$

Then by the same argument, there exists a mapping $C : X \to Y$ such that $C(x) = \lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right)$ and

$$\begin{aligned}&\|h(x)-C(x)\|\\ &\le \frac{1}{|k|}\sup_{j\ge 0}\left\{|2|^j\varphi\left(-\frac{kx}{2^{j+1}},0,\frac{x}{2^{j+1}}\right),|2|^j\varphi\left(0,-\frac{kx}{2^{j+1}},\frac{x}{2^{j+1}}\right),|2|^j\varphi\left(-\frac{kx}{2^{j+1}},-\frac{kx}{2^{j+1}},\frac{x}{2^j}\right)\right\}\\ &=\psi_3(x)\end{aligned} \tag{2.25}$$

for all $x\in X$.

Next, we show that A, B, C are additive and $A = B$, $A(x) = kC(\frac{x}{k})$ for all $x \in X$.
Replacing (x, y, z) by $\left(\frac{x}{2^n},-\frac{x}{2^n},0\right)$ in (2.7), we have

$$|2|^n\left\|f\left(\frac{x}{2^n}\right)+g\left(-\frac{x}{2^n}\right)\right\|\le |2|^n\varphi\left(\frac{x}{2^n},-\frac{x}{2^n},0\right),$$

so that

$$A(x)+B(-x)=0\qquad \forall x\in X. \tag{2.26}$$

Replacing (x, y, z) by $\left(\frac{x}{2^n},0,-\frac{x}{2^nk}\right)$ in (2.7), we have for all $x\in X$

$$|2|^n\left\|f\left(\frac{x}{2^n}\right)+kh\left(-\frac{x}{2^nk}\right)\right\|\le |2|^n\varphi\left(\frac{x}{2^n},0,-\frac{x}{2^nk}\right),$$

so that

$$A(x)+kC\left(-\frac{x}{k}\right)=0\qquad \forall x\in X. \tag{2.27}$$

Replacing (x, y, z) by $\left(\frac{x}{2^n},\frac{y}{2^n},-\frac{x+y}{2^nk}\right)$ in (2.7), we have

$$|2|^n\left\|f\left(\frac{x}{2^n}\right)+g\left(\frac{y}{2^n}\right)+kh\left(-\frac{x+y}{2^nk}\right)\right\|\le |2|^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},-\frac{x+y}{2^nk}\right).$$

Hence

$$A(x)+B(y)+kC\left(-\frac{x+y}{k}\right)=0\qquad \forall x,y\in X. \tag{2.28}$$

Then by (2.26) and (2.27),

$$A(x)-A(-y)-A(x+y)=0\qquad \forall x,y\in X. \tag{2.29}$$

Letting $x = y = 0$ in (2.29), it follows that $A(0) = 0$. Letting $x = 0$ in (2.29), it follows that $A(-y) = -A(y)$, so that by (2.29) again,

$$A(x+y)=A(x)+A(y)\qquad \forall x,y\in X.$$

Letting $x = 0$ in (2.28), we have by (2.27)

$$B(y) - A(y) = B(y) + kC\left(-\frac{y}{k}\right) = 0,$$

so that $A = B$. Since A is additive, it follows by (2.27) that $A(x) = kC(\frac{x}{k})$ and C is additive. By (2.18),(2.19) and (2.25), the inequalities (2.9) hold true.

Next, we show the uniqueness of A. Assume that $T : X \to Y$ is another additive map satisfying (2.9). Then $\|f(x) - T(x)\| \leq \psi_1(x)$ for all $x \in X$. So, we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \lim_{n\to\infty} |2|^n \left\|A\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty} \max\left\{|2|^n \left\|A\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|, |2|^n \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right\} \\
&\leq \lim_{n\to\infty} \sup_{j\geq 0} \left\{|2|^{n+j}\varphi\left(\frac{x}{2^{n+j+1}}, -\frac{x}{2^{n+j+1}}, 0\right), |2|^{n+j}\varphi\left(\frac{x}{2^{n+j+1}}, 0, -\frac{x}{2^{n+j+1}k}\right),\right. \\
&\qquad\qquad \left. |2|^{n+j}\varphi\left(\frac{x}{2^{n+j}}, -\frac{x}{2^{n+j+1}}, -\frac{x}{2^{n+j+1}k}\right)\right\} \\
&= 0
\end{aligned}$$

for all $x \in X$. Hence it follows that $A = T$. This completes the proof. □

Corollary 2.3. *Let $f, g, h, p : X \to Y$ be mappings such that $g(0) = h(0) = p(0) = 0$ and $|2| < 1, |k| < 1$. Assume that*

$$\|f(x) + g(y) + kh(z)\| \leq \left\|kp\left(\frac{x+y}{k} + z\right)\right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in X$, where θ and r are constants with $\theta > 0$ and $0 \leq r < 1$. Then there exists a unique additive mapping $A : X \to Y$ such that for all $x \in X$

$$\|f(x) - A(x)\| \leq \left(1 + \frac{1}{|2|^r} + \frac{1}{|2k|^r}\right)\theta\|x\|^r,$$

$$\|g(x) - A(x)\| \leq \left(1 + \frac{1}{|2|^r} + \frac{1}{|2k|^r}\right)\theta\|x\|^r,$$

$$\left\|h(x) - \frac{1}{k}A(kx)\right\| \leq \begin{cases} \frac{1}{|k|}\left(1 + \frac{2\cdot|k|^r}{|2|^r}\right)\theta\|x\|^r & \text{if} \quad |k|^r + |2|^r \geq 1, \\ \frac{1}{|k|}\frac{1+|k|^r}{|2|^r}\theta\|x\|^r & \text{if} \quad |k|^r + |2|^r < 1. \end{cases}$$

Corollary 2.4. *Let $f, g, h, p : X \to Y$ be mappings such that $g(0) = h(0) = p(0) = 0$ and*

$$\|f(x) + g(y) + kh(z)\| \leq \left\|kp\left(\frac{x+y}{k} + z\right)\right\| + \theta\|x\|^r \cdot \|y\|^r \cdot \|z\|^r$$

for all $x, y, z \in X$, where θ and r are constants with $\theta > 0$ and $r < \frac{1}{3}$. If $|2| \neq 1$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|4k|^r}\theta\|x\|^{3r},$$

$$\|g(x) - A(x)\| \leq \frac{1}{|4k|^r}\theta\|x\|^{3r},$$

$$\left\|h(x) - \frac{1}{k}A(kx)\right\| \leq \frac{|k|^{2r-1}}{|4|^r}\theta\|x\|^{3r}$$

for all $x \in X$.

3. Hyers-Ulam stability of (1.2)

Proposition 3.1. *Let $f, g, h, p : X \to Y$ be mappings such that $g(0) = h(0) = p(0) = 0$ and*

$$\|f(x) + g(y) + h(z)\| \le \left\| kp\left(\frac{x+y+z}{k}\right)\right\| \tag{3.1}$$

for all $x, y, z \in X$. Then $f = g = h$ and they are additive.

Proof. Replacing (x, y, z) by $(x, -x, 0)$ in (3.1), we get

$$f(x) + g(-x) = 0 \qquad \forall x \in X. \tag{3.2}$$

Replacing (x, y, z) by $(x, 0, -x)$ in (3.1), we get

$$f(x) + h(-x) = 0 \qquad \forall x \in X,$$

and so

$$g(x) = h(x) \qquad \forall x \in X.$$

Replacing (x, y, z) by $(x + y, -x, -y)$ in (3.1), we have

$$f(x + y) + g(-x) + g(-y) = 0 \qquad \forall x, y \in X,$$

so that by (3.2)

$$f(x + y) - f(x) - f(y) = 0 \qquad \forall x, y \in X.$$

That is, f is additive. Since $f(-x) + g(x) = 0$ by (3.2), we have $-f(x) + g(x) = 0$ for all $x \in X$. Hence $f = g$. This completes the proof. □

We now prove the Hyers-Ulam stability of the functional inequality (1.2).

Theorem 3.2. *Let $f, g, h, p : X \to Y$ be mappings such that $g(0) = h(0) = p(0) = 0$ and*

$$\|f(x) + g(y) + h(z)\| \le \left\| kp\left(\frac{x+y+z}{k}\right)\right\| + \varphi(x, y, z), \tag{3.3}$$

where $\varphi : X^3 \to [0, \infty)$ satisfies $\varphi(0, 0, 0) = 0$ and

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{3.4}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \psi_1(x), \tag{3.5}$$

$$\|g(x) - A(x)\| \le \psi_2(x), \tag{3.6}$$

$$\|h(x) - A(x)\| \le \psi_3(x). \tag{3.7}$$

Here,

$$\psi_1(x) = \sup_{j\ge 0}\left\{|2^j|\varphi\left(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, 0\right), |2^j|\varphi\left(\frac{x}{2^{j+1}}, 0, -\frac{x}{2^{j+1}}\right), |2^j|\varphi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}\right)\right\},$$

$$\psi_2(x) = \sup_{j\ge 0}\left\{|2^j|\varphi\left(-\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right), |2^j|\varphi\left(0, \frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}\right), |2^j|\varphi\left(-\frac{x}{2^{j+1}}, \frac{x}{2^j}, -\frac{x}{2^{j+1}}\right)\right\},$$

$$\psi_3(x) = \sup_{j\ge 0}\left\{|2^j|\varphi\left(-\frac{x}{2^{j+1}}, 0, \frac{x}{2^{j+1}}\right), |2^j|\varphi\left(0, -\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), |2^j|\varphi\left(-\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, \frac{x}{2^j}\right)\right\}.$$

Proof. Replacing (x, y, z) by $(0, 0, 0)$ in (3.3), we get $f(0) = 0$. Replacing (x, y, z) by $(x, -x, 0)$ in (3.3), we have

$$\|f(x) + g(-x)\| \leq \varphi(x, -x, 0).$$

Replacing (x, y, z) by $(x, 0, -x)$ in (3.3), we have

$$\|f(x) + h(-x)\| \leq \varphi(x, 0, -x).$$

Then

$$\|2f(x) + g(-x) + h(-x)\| \leq \max\{\varphi(x, -x, 0), \varphi(x, 0, -x)\}. \tag{3.8}$$

Replacing (x, y, z) by $(2x, -x, -x)$ in (3.3), we have

$$\|f(2x) + g(-x) + h(-x)\| \leq \varphi(2x, -x, -x). \tag{3.9}$$

Hence by (3.8) and (3.9),

$$\|2f(x) - f(2x)\| \leq \max\{\varphi(x, -x, 0), \varphi(x, 0, -x), \varphi(2x, -x, -x)\},$$

and so

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq \max\left\{\varphi\left(\frac{x}{2}, -\frac{x}{2}, 0\right), \varphi\left(\frac{x}{2}, 0, -\frac{x}{2}\right), \varphi\left(x, -\frac{x}{2}, -\frac{x}{2}\right)\right\} \tag{3.10}$$

for all $x \in X$. Replacing x by $\frac{x}{2^j}$ and multiplying by $|2^j|$ on both sides of (3.10) for every nonnegative integer j, we have

$$\begin{aligned} &\left\|2^{j+1}f\left(\frac{x}{2^{j+1}}\right) - 2^j f\left(\frac{x}{2^j}\right)\right\| \\ &\quad \leq \max\left\{|2^j|\varphi\left(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, 0\right), |2^j|\varphi\left(\frac{x}{2^{j+1}}, 0, -\frac{x}{2^{j+1}}\right), |2^j|\varphi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}\right)\right\} \end{aligned} \tag{3.11}$$

for all $x \in X$. Hence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence in Y. Since Y is complete, we can define the mapping $A : X \to Y$ such that

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right).$$

For nonnegative integers $l < m$, we have

$$\begin{aligned} &\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \\ &\quad \leq \max_{l\leq j\leq m-1}\left\{\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|\right\} \\ &\quad \leq \max_{l\leq j\leq m-1}\left\{|2^j|\varphi\left(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, 0\right), |2^j|\varphi\left(\frac{x}{2^{j+1}}, 0, -\frac{x}{2^{j+1}}\right), |2^j|\varphi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}\right)\right\} \end{aligned} \tag{3.12}$$

for all $x \in X$. Letting $l = 0$ and taking the limit as $m \to \infty$ in (3.12), we have

$$\begin{aligned} &\|f(x) - A(x)\| \\ &\quad \leq \sup_{j\geq 0}\left\{|2|^j\varphi\left(\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, 0\right), |2|^j\varphi\left(\frac{x}{2^{j+1}}, 0, -\frac{x}{2^{j+1}}\right), |2|^j\varphi\left(\frac{x}{2^j}, -\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}\right)\right\} = \psi_1(x) \end{aligned} \tag{3.13}$$

for all $x \in X$.

Similarly, there exists a mapping $B : X \to Y$ such that

$$B(x) := \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right),$$

and

$$\begin{aligned} &\|g(x) - B(x)\| \\ &\quad \leq \sup_{j\geq 0}\left\{|2^j|\varphi(-\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0), |2^j|\varphi(0, \frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}), |2^j|\varphi(-\frac{x}{2^{j+1}}, \frac{x}{2^j}, -\frac{x}{2^{j+1}})\right\} = \psi_2(x). \end{aligned}$$

We also obtain a mapping $C : X \to Y$ such that

$$C(x) := \lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right),$$

and

$$\|h(x) - C(x)\| \\ \leq \sup_{j\geq 0} \left\{|2^j|\varphi\left(-\frac{x}{2^{j+1}}, 0, \frac{x}{2^{j+1}}\right), |2^j|\varphi\left(0, -\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), |2^j|\varphi\left(-\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}, \frac{x}{2^j}\right)\right\} = \psi_3(x).$$

Next, we show that $A = B = C$ and they are additive. Replacing (x, y, z) by $\left(\frac{x}{2^n}, -\frac{x}{2^n}, 0\right)$ in (3.3), we have

$$|2^n| \left\|f\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right)\right\| \leq |2^n|\varphi\left(\frac{x}{2^n}, -\frac{x}{2^n}, 0\right),$$

and so

$$A(x) + B(-x) = 0 \tag{3.14}$$

for all $x \in X$. Similarly $A(x) + C(-x) = 0$ for all $x \in X$. Hence $B = C$.

Replacing (x, y, z) by $\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{-(x+y)}{2^n}\right)$ in (3.3), we have

$$|2^n| \left\|f\left(\frac{x}{2^n}\right) + g\left(\frac{y}{2^n}\right) + h\left(\frac{-(x+y)}{2^n}\right)\right\| \leq |2^n|\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{-(x+y)}{2^n}\right),$$

and so

$$A(x) + B(y) + C\big(-(c+y)\big) = 0$$

for all $x, y \in X$. Then

$$A(x) - A(-y) - A(x+y) = 0,$$

so that

$$A(x+y) = A(x) - A(-y) \tag{3.15}$$

for all $x, y \in X$. Letting $x = y = 0$ in (3.15), we have $A(0) = 0$. Letting $x = 0$ in (3.15), $A(-y) = -A(y)$, so that

$$A(x+y) = A(x) + A(y)$$

for all $y \in X$. Then it follows by (3.14) that

$$A(-x) = -A(x) = B(-x)$$

for all $x \in X$. Hence $A = B = C$ and A is additive. Therefore the inequalities (3.5),(3.6) and (3.7) hold.

Since the uniqueness of A can be proved similarly as in the proof of Theorem 2.2, we omit it. This completes the proof. □

Corollary 3.3. *Let $f, g, h, p : X \to Y$ be mappings such that $g(0) = h(0) = p(0) = 0$ and*

$$\|f(x) + g(y) + h(z)\| \leq \left\|kp\left(\frac{x+y+z}{k}\right)\right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in X$, where θ and r are constants with $\theta > 0$ and $r < 1$. If $|2| \neq 1$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq (2|2|^{-r} + 1)\theta\|x\|^r,$$

$$\|g(x) - A(x)\| \leq (2|2|^{-r} + 1)\theta\|x\|^r,$$

$$\|h(x) - A(x)\| \leq (2|2|^{-r} + 1)\theta\|x\|^r$$

for all $x \in X$.

Corollary 3.4. *Let $f, g, h, p : X \to Y$ be mappings such that $g(0) = h(0) = p(0) = 0$ and*

$$\|f(x) + g(y) + h(z)\| \leq \left\| kp\left(\frac{x+y+z}{k}\right)\right\| + \theta\|x\|^r \cdot \|y\|^r \cdot \|z\|^r$$

for all $x, y, z \in X$, where θ and r are constants with $\theta > 0$ and $r < \frac{1}{3}$. If $|2| \neq 1$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq |2|^{-2r}\theta\|x\|^{3r},$$

$$\|g(x) - A(x)\| \leq |2|^{-2r}\theta\|x\|^{3r},$$

$$\|h(x) - A(x)\| \leq |2|^{-2r}\theta\|x\|^{3r}$$

for all $x \in X$.

Acknowledgements:

The first author was supported by Hallym University Research Fund, 2014 (HRF-201403-008).

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