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Normed proper quasilinear spaces

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Abstract

The fundamental deficiency in the theory of quasilinear spaces, introduced by Aseev [S. M. Aseev, Trudy Mat. Inst. Steklov., 167 (1985), 25–52], is the lack of a satisfactory definition of linear dependence-independence and basis notions. Perhaps, this is the most important obstacle in the progress of normed quasilinear spaces. In this work, after giving the notions of quasilinear dependence-independence and basis presented by Banazılı[H. K. Banazılı, M.Sc. Thesis, Malatya, Turkey (2014)] and Çakan [S. Çakan, Ph.D. Seminar, Malatya, Turkey (2012)], we introduce the concepts of regular and singular dimension of a quasilinear space. Also, we present a new notion namely "proper quasilinear spaces" and show that these two kind dimensions are equivalent in proper quasilinear spaces. Moreover, we try to explore some properties of finite regular and singular dimensional normed quasilinear spaces. We also obtain some results about the advantages of features of proper quasilinear spaces.

Keywords: Quasilinear spaces, Hausdorff metric, regular dimension, singular dimension, floor of an element, proper sets, proper quasilinear spaces.
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1. Introduction

Aseev [2] launched a new branch of functional analysis by introducing the concept of quasilinear spaces which is generalization of classical linear spaces. He used the partial order relation to define quasilinear spaces and gave coherent counterparts of results in linear spaces. Aseev's approach provides suitable base and necessary tools to proceed algebra and analysis on normed quasilinear spaces just as in normed spaces. So, Aseev's study brings an extended point of view to classical linear algebra and it reflects more aspects by the advantages of the order relation. Thus his treatment allows us to construct a kind of theory of quasilinear algebra. Aseev's avant garde work has motivated us to introduce some new results, [1, 3, 4, 5, 6, 7, 8, 9, 10, 11].

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The fundamental deficiency in the theory of quasilinear spaces is the lack of a satisfactory definition of linear dependence-independence and basis. Perhaps this is the most important obstacle on the improvement of quasilinear spaces. Our studies ([4] and [6]) showed that concepts of linear dependence-independence and basis directly depend on the order relation on quasilinear space.

In next section, we will give some definitions and preliminaries results about quasilinear spaces and normed quasilinear spaces. Then we introduce the concepts of "regular and singular dimension of any quasilinear space" and "floor of an element in quasilinear spaces" as new structures. Also, we introduce proper quasilinear spaces and obtain some results about features of proper quasilinear spaces with remarkable advantages.

2. Known Results About Quasilinear Spaces

Definition 2.1 ([2])**.** $(X, \preceq)$ is called a *quasilinear space (qls,* for short*)*, if a partial order relation "$\preceq$", an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any real numbers $\alpha, \beta \in \mathbb{R}$:

$$x \preceq x, \tag{2.1}$$

$$x \preceq z \text{ if } x \preceq y \text{ and } y \preceq z, \tag{2.2}$$

$$x = y \text{ if } x \preceq y \text{ and } y \preceq x, \tag{2.3}$$

$$x + y = y + x, \tag{2.4}$$

$$x + (y + z) = (x + y) + z, \tag{2.5}$$

$$\text{there exists an element } \theta \in X \text{ such that } x + \theta = x, \tag{2.6}$$

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x, \tag{2.7}$$

$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y, \tag{2.8}$$

$$1 \cdot x = x, \tag{2.9}$$

$$\theta \cdot x = \theta, \tag{2.10}$$

$$(\alpha + \beta) \cdot x \preceq \alpha \cdot x + \beta \cdot x, \tag{2.11}$$

$$x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \tag{2.12}$$

$$\alpha \cdot x \preceq \alpha \cdot y \text{ if } x \preceq y. \tag{2.13}$$

A linear space is a qls with the partial order relation “$=$”. The most popular example of qls which is not a linear space is the set of all closed intervals of real numbers with the inclusion relation “$\subseteq$”, the algebraic sum operation

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and the real-scalar multiplication

$$\lambda A = \{\lambda a : a \in A\}.$$

We denote this set by $\Omega_C(\mathbb{R})$. Another one is $\Omega(\mathbb{R})$ which is the set of all compact subsets of real numbers. In general, $\Omega(E)$ and $\Omega_C(E)$ stand for the space of all nonempty closed bounded and nonempty convex and closed bounded subsets of any normed linear space E, respectively. Both are qls (nonlinear) with the inclusion relation and with a slight modification of addition shaped

$$A + B = \overline{\{a + b : a \in A,\ b \in B\}}$$

and with the real scalar multiplication above.

Lemma 2.2 ([2]). *In a qls X the element θ is minimal, i.e., $x = \theta$ if $x \preceq \theta$.*

We note that the minimality is not only a property of θ but also is shared by the other regular elements, [11]. An element $x' \in X$ is called *inverse* of $x \in X$ if $x + x' = \theta$. Further, if an inverse element exists, then it is unique. An element x possessing inverse is called *regular*, otherwise is called *singular*. X_r and X_s stand for the sets of all regular and singular elements in X, respectively. It will be assumed in the text that $-x = (-1)x$ and an element x in a qls is regular if and only if $x - x = \theta$ equivalently $x' = -x$.

Suppose that any element x in a qls X has inverse element $x' \in X$. Then the partial order in X is determined by equality, the distributivity conditions hold and consequently, X is a linear space. In a real linear space equality is the only way to define a partial order such that the conditions (2.1)-(2.13) hold.

Let X be a qls and $Y \subseteq X$. Then Y is called a *subspace* of X whenever Y is a quasilinear space with the same partial order and the restriction of the operations on X to Y. The following characterization of subspace in a qls is surprizingly the same as in linear spaces, and its proof is similar to its classical analogue.

Theorem 2.3 ([11]). *Y is a subspace of a qls X if and only if $\alpha x + \beta y \in Y$ for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{R}$.*

Suppose that each element x in Y has inverse element $x' \in Y$ then the partial order on Y is determined by the equality. In this case the distributivity conditions hold in (2.11) on Y and Y is a linear subspace of the qls X.

An element $x \in X$ is said to be *symmetric* providing that $-x = x$, and X_d denotes the set of all symmetric elements. X_r, X_d and $X_s \cup \{0\}$ are subspaces of X and are called regular, symmetric and singular subspaces of X, respectively. For example, let $X = \Omega_C(\mathbb{R})$ and $Z = \{0\} \cup \{[a, b] : a, b \in \mathbb{R}$ and $a \neq b\}$. Z is singular subspace of X. On the other hand, the set of all singletons of real numbers $\{\{a\} : a \in \mathbb{R}\}$ is regular subspace of X.

Definition 2.4 ([2]). Let $(X, \preceq)$ be a qls. A real function $\|\cdot\|_X : X \longrightarrow \mathbb{R}$ is called a *norm* if the following conditions hold:

$$\|x\|_X > 0 \text{ if } x \neq 0, \tag{2.14}$$

$$\|x + y\|_X \leq \|x\|_X + \|y\|_X\,, \tag{2.15}$$

$$\|\alpha \cdot x\|_X = |\alpha|\,\|x\|_X\,, \tag{2.16}$$

$$\text{if } x \preceq y, \text{ then } \|x\|_X \leq \|y\|_X\,, \tag{2.17}$$

$$\begin{aligned} \text{if for any } \varepsilon &> 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that,} \\ x &\preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\|_X \leq \varepsilon \text{ then } x \preceq y. \end{aligned} \tag{2.18}$$

A qls X with a norm defined on it is called *normed quasilinear space (briefly, normed qls).* If any $x \in X$ has inverse element $x' \in X$, then the concept of normed qls coincides with the notion of a real normed linear space.

Let $(X, \preceq)$ be a normed qls. *Hausdorff metric* or *norm metric* on X is defined by the equality

$$h_X(x, y) = \inf\{r \geq 0 : x \preceq y + a_1^r,\, y \preceq x + a_2^r, \|a_i^r\| \leq r\}\,.$$

Since $x \preceq y + (x - y)$ and $y \preceq x + (y - x)$, the quantity $h_X(x, y)$ is well defined. It is not hard to see that this function $h_X(x, y)$ satisfies all of the metric axioms and we should note that $h_X(x, y)$ may not equal to $\|x - y\|_X$ if X is a nonlinear qls. Further, for any elements $x, y \in X$, and $h_X(x, y) \leq \|x - y\|_X$. Therefore, we use the metric to discuss a topological property in normed quasilinear spaces instead of the norm. For example, $x_n \to x$ if and only if $h_X(x_n, x) \to 0$ in a normed qls. Although, allways $\|x_n - x\|_X \to 0$ implies $x_n \to x$ in normed quasilinear spaces, $x_n \to x$ may not imply $\|x_n - x\|_X \to 0$.

Proposition 2.5 ([2]). *The following conditions hold with respect to Hausdorff metric:*

$$h_X(\alpha \cdot x, \alpha \cdot y) = |\alpha| \, h_X(x, y) \ \text{, for any } \alpha \in \mathbb{R}, \tag{2.19}$$

$$h_X(x + y, z + v) = h_X(x, z) + h_X(y, v), \tag{2.20}$$

$$\|x\| = h_X(x, 0). \tag{2.21}$$

Lemma 2.6 ([2]). *The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is continuous function with respect to the Hausdorff metric.*

Lemma 2.7 ([2]). *Suppose that $x_n \to x_0$ and $y_n \to y_0$ and that $x_n \preceq y_n$ for any positive integer n. Then $x_0 \preceq y_0$.*

Let X be a real complete normed linear space (a real Banach space). Then X is a complete normed qls with partial order given by equality. Conversely, if X is a complete normed qls and any $x \in X$ has inverse element $x^{'} \in X$, then X is a real Banach space, and the partial order on X is equality. In this case $h_X(x, y) = \|x - y\|_X$.

Let E be a real normed linear space. The norm on $\Omega(E)$ is defined by

$$\|A\|_{\Omega(E)} = \sup_{a \in A} \|a\|_E .$$

Then $\Omega(E)$ and $\Omega_C(E)$ are normed quasilinear spaces. In this case, the Hausdorff metric is defined as usual:

$$h_\Omega(A, B) = \inf\{r \geq 0 : A \subseteq B + S(\theta, r),\ B \subseteq A + S(\theta, r)\},$$

where $S(\theta, r)$ is the closed ball of radius r and centered at $\theta \in X$.

Definition 2.8 ([4]). Let X be a qls, $\{x_k\}_{k=1}^{n} \subset X$ and $\{\alpha_k\}_{k=1}^{n} \subset \mathbb{R}$. The element

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \sum_{k=1}^{n} \alpha_k x_k$$

of X is said to be a *quasilinear combination (qs-combination, for short)* of $\{x_k\}_{k=1}^{n}$.

Let $(X, \preceq)$ be a qls and $A = \{x_1, x_2, ..., x_n\} \subset X$. The set

$$QspA = \{x \in X : \sum_{k=1}^{n} \alpha_k x_k \preceq x,\ \ x_1, x_2, ..., x_n \in A and\ \alpha_1, \alpha_2, ...,\ \alpha_n \in \mathbb{R}\}$$

is said to be *(quasi) span of* A and is denoted by $QspA$. One can see easily that $QspA$ is subspace of X.

It is clear that $SpanA \subseteq QspA$. If X is a linear space, then $QspA = SpanA$.

Definition 2.9 ([4]). Let $(X, \preceq)$ be a qls, $\{x_k\}_{k=1}^{n} \subset X$ and $\{\alpha_k\}_{k=1}^{n} \subset \mathbb{R}$. If

$$\theta_X \preceq \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

implies $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ then $\{x_k\}_{k=1}^{n}$ is said to be *quasilinear independent (briefly, qs-independent)*, otherwise $\{x_k\}_{k=1}^{n}$ is said to be *quasilinear dependent (qs-dependent, for short)*.

Theorem 2.10 ([4]). *Any set A which has $n+1$ elements has to be qs-dependent in $\Omega_C(\mathbb{R}^n)$.*

Definition 2.11 ([4]). Let X be a qls. If $A \subset X$ is qs-independent and $QspA = X$ then the set A is called a *basis* for X.

Lemma 2.12 ([11]). *The linear subspace X_r of a normed qls X is closed.*

Lemma 2.13 ([5]). *Let X be a qls. For every $x, y \in X$, $x + y \in X_r$ implies $x \in X_r$ and $y \in X_r$.*

3. Main results

Combining Lemma 2.12 and Lemma 2.13 we obtain the following result.

Corollary 3.1. *Let X be a qls, $x \in X_r$ and $y \in X_s$. Then $x + y \in X_s$.*

Theorem 3.2. *Let $(X, \preceq)$ be a qls and $x_0 \in X_r$. If no there exist $y \in X_s$ such that $x_0 \preceq y$, then X is a (pure) linear space.*

Proof. Let $x_0 \in X_r$. Suppose that $x_0 \npreceq x$ for all $x \in X_s$. Let $z \neq x_0$ and $z \in X_r$.

Now let us assume that there exists at least element $y \in X_s$ such that $z \preceq y$. Let S be a basis for X_r. Then, each $x_0 \in X_r$ has a representation

$$x_0 = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n$$

by aid of the elements $a_1, a_2, ..., a_n$ in S and real scalars $\lambda_1, \lambda_2, ..., \lambda_n$.

Thus, for $z \in X_r$ we write

$$z = \beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_m b_m$$

by aid of the elements $b_1, b_2, ..., b_m$ in S and real scalars $\beta_1, \beta_2, ..., \beta_m$.

(Where the elements $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_m$ may be the same, but it does not a problem to proof.)

By (2.12) from qls axioms, we obtain

$$x_0 + z = \sum_{k=1}^{n} \lambda_k a_k + \sum_{k=1}^{m} \beta_k b_k \tag{3.1}$$

Since $z \preceq y$ we write $-z \preceq -y$ by (2.13). By using (3.1)

$$x_0 \preceq -y + \sum_{k=1}^{n} \lambda_k a_k + \sum_{k=1}^{m} \beta_k b_k. \tag{3.2}$$

In (3.2), the elements $\sum_{k=1}^{n} \lambda_k a_k$ and $\sum_{k=1}^{m} \beta_k b_k$ are reguler, $-y$ is a singular element. By Corollary 3.1, the element $-y + \sum_{k=1}^{n} \lambda_k a_k + \sum_{k=1}^{m} \beta_k b_k$ is singular. This result contradicts with the hypothesis. So our assumption is wrong.

Thus, for every $x \in X_r$ and for all $y \in X_s$, we obtain $x \npreceq y$. On the other hand, since every $x \in X_r$ is minimal, we say $y \npreceq x$ for all $y \in X_s$. Hence, X has not any singular element and $X = X_r$. This complete the proof. □

The following important comment is a result of the above theorem.

Corollary 3.3. *If X is a nonlinear qls, then for every $x \in X_r$ there exists at least one $y \in X_s$ such that $x \preceq y$.*

3.1. Dimension in Quasilinear Spaces

In this section, we introduce the definitions of regular and singular dimension of any qls as new concepts. We note that these concepts are redundant in linear spaces. Also, in next section, after introducing proper quasilinear spaces, we show that the notions of regular and singular dimension are coincide in a proper qls and we use only the name of "dimension" in proper quasilinear spaces.

Definition 3.4. *Singular dimension* of a qls X is defined as maximum number of qs-independent elements in X_s. If this number is finite then X is called finite singular dimensional, otherwise infinite singular dimensional. Further the dimension of regular subspace of X is called *regular dimension* of X. Regular and singular dimension of X are denoted by $s - \dim X$ and $r - \dim X$, respectively.

On the other hand, if $s - \dim X = r - \dim X = a$ then a is said to be *dimension* of X and it is written as $\dim X = a$.

Corollary 3.5. *For every linear space $s-\dim X=0$. If $s-\dim X>0$, then X is a nonlinear qls.*

We note that X may not be a linear space if $s-\dim X=0$. The following example reflects this situation.

Example 3.6. For the symetric subspace

$$(\Omega_C(\mathbb{R}))_d=\{[-a,a]:a\in\mathbb{R}\}$$

of $\Omega_C(\mathbb{R})$, we have $r-\dim\left((\Omega_C(\mathbb{R}))_d\right)=s-\dim\left((\Omega_C(\mathbb{R}))_d\right)=0$.

Example 3.7. Regular and singular dimension of the quasilinear spaces $\mathbb{R}$, $\Omega_C(\mathbb{R})$ and $(\Omega_C(\mathbb{R}))_s$ are as follows:

$$\begin{aligned}
r-\dim(\mathbb{R}) &= 1 \text{ and } s-\dim(\mathbb{R})=0,\\
r-\dim(\Omega_C(\mathbb{R})) &= 1 \text{ and } s-\dim(\Omega_C(\mathbb{R}))=1,\\
r-\dim\left((\Omega_C(\mathbb{R}))_s\right) &= 0 \text{ and } s-\dim\left((\Omega_C(\mathbb{R}))_s\right)=1,
\end{aligned}$$

respectively.

Similarly, regular and singular dimension of the quasilinear spaces $\mathbb{R}^2$, $\Omega_C(\mathbb{R}^2)$ and $\left(\Omega_C(\mathbb{R}^2)\right)_s$ are as follows:

$$\begin{aligned}
r-\dim\left(\mathbb{R}^2\right) &= 2 \text{ and } s-\dim\left(\mathbb{R}^2\right)=0,\\
r-\dim\left(\Omega_C(\mathbb{R}^2)\right) &= 2 \text{ and } s-\dim\left(\Omega_C(\mathbb{R}^2)\right)=2,\\
r-\dim\left(\left(\Omega_C(\mathbb{R}^2)\right)_s\right) &= 0 \text{ and } s-\dim\left(\left(\Omega_C(\mathbb{R}^2)\right)_s\right)=2,
\end{aligned}$$

respectively.

Example 3.8. Let us consider the subspace

$$W=\left(\Omega_C(\mathbb{R}^2)\right)_s\cup\{\{(x,0)\}:x\in\mathbb{R}\}$$

of $\Omega_C(\mathbb{R}^2)$ and the elements

$$w_1=\{(0,y):1\le y\le 2\}$$

and

$$w_2=\{(x,0):1\le x\le 2\}$$

of W_s. The set $\{w_1,w_2\}$ is qs-independent in W_s since no there exist non-zero scalars λ_1 and λ_2 satisfying the inclusion $\{(0,0)\}\subseteq\lambda_1w_1+\lambda_2w_2$. Hence singular dimension of W must be greater than or equal to 2. Remember that W is a subspace of $\Omega_C(\mathbb{R}^2)$ and Theorem 2.10. Then $s-\dim W=2$. Obviously W_r is equivalent to $\mathbb{R}$ and so $r-\dim W=1$.

Example 3.9. Let us recall that $\Omega_C(c_0)$ is a qls with the partial order relation " $\subseteq$ ". If we take

$$X=(\Omega_C(c_0))_s\cup\{\theta\},\ \text{where } \theta=(0,0,0,...)\in c_0.$$

Then $r-\dim X=0$ and $s-\dim X=\infty$. The quantity of qs-independent elements in X is not finite. Indeed, the family

$$\{\{(t,0,0,...):1\le t\le 2\},\ \{(0,t,0,0,...):1\le t\le 2\},...\}=\{[1,2]\,e_1,[1,2]\,e_2,...\}$$

is qs-independent. Let us show that any finite subset of this family is qs-independent. This also implies that this family is qs-independent:

Assume that $(n_k)\subset\mathbb{Z}^+$ is a increasing sequence and

$$\theta=\{(0,0,...)\}\subseteq\lambda_1\left([1,2]\,e_{n_1}\right)+\lambda_2\left([1,2]\,e_{n_2}\right)+\cdots+\lambda_k\left([1,2]\,e_{n_k}\right).$$

Then

$$\begin{aligned}
\{(0,0,...)\} &\subseteq \left\{ \begin{array}{c} \lambda_1 t \left(0,0,...,0,\overset{n_1.term}{1},0,0,...\right) + \lambda_2 t \left(0,0,...,0,\overset{n_2.term}{1},0,0,...\right) + \\ \cdots + \lambda_k t \left(0,0,...,0,\overset{n_k.term}{1},0,0,...\right) : 1 \leq t \leq 2 \end{array} \right\} \\
&= \left\{ \left(0,0,...,0,\overset{n_1.term}{\lambda_1 t},0,0,...,0,\overset{n_2.term}{\lambda_2 t},0,0,...,0,\overset{n_k.term}{\lambda_k t},0,0,...\right) : 1 \leq t \leq 2 \right\} \\
&\Leftrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_k = 0.
\end{aligned}$$

Also, for the qls

$$X = \Omega_C(c_0)$$

$r - \dim X = \infty$ and $s - \dim X = \infty$.

As another example, we can say that $r - \dim X = 2$ and $s - \dim X = \infty$, for the qls

$$X = \left(\Omega_C(l_\infty)\right)_s \cup \{(0,0,...,0,k,0,0,...,0,l,0,0,...) : k,l \in \mathbb{R}\}.$$

Further, since the regular subspace of the qls $\Omega_C(\mathbb{R}^n)$ is $\mathbb{R}^n$ and the maximum numbers of qs-independent elements in $\left(\Omega_C(\mathbb{R}^n)\right)_s$ is n, we have $r - \dim \Omega_C(\mathbb{R}^n) = s - \dim \Omega_C(\mathbb{R}^n) = n$.

3.2. Proper Quasilinear Spaces

The main purpose this section is to introduce the notion of proper quasilinear spaces. Before giving the definition of proper quasilinear spaces, we must present some new definitions.

Definition 3.10. Let $(X, \preceq)$ be a qls, $M \subseteq X$ and $x \in M$. The set

$$F_x^M = \{z \in M_r : z \preceq x\}$$

is called *floor in* M *of* x . In the case of $M = X$ it is called briefly *floor of* x and written briefly F_x instead of F_x^X.

Floor of an element x in linear spaces is $\{x\}$. Therefore, it is nothing to discuss the notion of floor of an element in a linear space.

Definition 3.11. Let $(X, \preceq)$ be a qls and $M \subseteq X$. Then the union set

$$\bigcup_{x \in M} F_x^M$$

is called *floor of* M and is denoted by $\mathcal{F}_M$. In the case of $M = X$, $\mathcal{F}_X$ is called *floor of the qls* X.

On the other hand, the set

$$\mathcal{F}_M^X = \bigcup_{x \in M} F_x^X$$

is called *floor in* X *of* M and is denoted by $\mathcal{F}_M^X$.

Let X be a qls, $M \subseteq X$ and $x \in M$. Then $F_x^M \subseteq F_x$. Indeed, let z be an arbitrary element in F_x^M. Then $z \preceq x$ and $z \in M_r$. Since $M_r \subseteq X_r$ we say $z \in X_r$. Thus $z \in F_x$.

Also, it is not surprising that $F_M \subseteq F_X$.

We note that $\mathcal{F}_X$ is equal to X_r. Also, floor of an element x in a qls X may not be subspace of X. For example, for $x = [2,3] \in \Omega_C(\mathbb{R})$, we have $\{2\}, \{3\} \in F_x$, but $\{2\} + \{3\} = \{5\} \notin F_x$. Further for some set $M \subseteq X$, $\mathcal{F}_M$ may not be subspace of X.

Lemma 3.12. *Let X be a normed qls and $x \in X$. Then F_x is closed and bounded in X.*

Proof. Assume that (z_n) is a sequence in F_x such that $(z_n) \to z$ and $z \in X$. We should prove $z \in F_x$ to show that F_x is closed. Since $z_n \in F_x$ for all $n \in \mathbb{N}$, $z_n \preceq x$ and $z_n \in X_r$. Now let us consider fixed sequence (x) in X. Since $(z_n) \to z$, $(x) \to x$ and $z_n \preceq x$ for any positive integer n we get $z \preceq x$ by the Lemma 2.7. Further since $(z_n) \to z$ and $z_n \in X_r$ for all $n \in \mathbb{N}$, we have $z \in X_r$ by Lemma 2.12. So we obtain $z \in F_x$.

On the other hand, because of $z \preceq x$ for all $z \in F_x$, we say $\|z\| \leq \|x\|$ by the normed qls axioms. Hence F_x is a bounded set. □

Proposition 3.13. *Let $s - \dim X \neq 0$ that is X be a nonlinear quasilinear space. Then*

$$r - \dim X \leq s - \dim X.$$

Proof. Assume that $s - \dim X \neq 0$ and $s - \dim X < r - \dim X$. Let $r - \dim X = n$ and $s - \dim X = m$. Then $m < n$. To show that for any $v \in X_r$, no there exists $u \in X_s$ such that $v \preceq u$ will be finished the proof.

Now let us consider the set $F_{X_s}^X$. We will show $\dim \left(F_{X_s}^X\right) < n$. To achieve this, we take arbitrary elements $x_1, x_2, ..., x_n$ in $F_{X_s}^X$. Let

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0. \tag{3.3}$$

There exist $y_1 \in X_s$ such that $x_1 \preceq y_1$ since $x_1 \in F_{X_s}^X$, $y_2 \in X_s$ such that $x_2 \preceq y_2$ since $x_2 \in F_{X_s}^X$, ..., $y_n \in X_s$ such that $x_n \preceq y_n$ since $x_n \in F_{X_s}^X$.

By the qls axioms (2.12) and (2.13), we obtain

$$\lambda_1 x_1 \preceq \lambda_1 y_1,\ \lambda_2 x_2 \preceq \lambda_2 y_2,\ \cdots,\ \lambda_n x_n \preceq \lambda_n y_n$$

and

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \preceq \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n \tag{3.4}$$

respectively.

From (3.3) and (3.4), we have

$$0 \preceq \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n. \tag{3.5}$$

The set $\{y_1, y_2, ..., y_n\}$ is qs-dependent in X_s since $y_1, y_2, ..., y_n \in X_s$ and $s - \dim X = m < n$.

Then, at least one of scalars $\lambda_1, \lambda_2, ..., \lambda_n$ in (3.5) is not zero. Hence, the set $\{x_1, x_2, ..., x_n\}$ is linear dependence, since at least one of scalars $\lambda_1, \lambda_2, ..., \lambda_n$ in the equality (3.3) is not also zero. Consequently we obtain that $\dim \left(F_{X_s}^X\right) < n$.

Since $F_{X_s}^X \subset X_r$, $F_{X_s}^X$ is a linear subspace of X_r and $\dim \left(F_{X_s}^X\right) < \dim (X_r)$ then there exists an element v in X_r such that $v \notin F_{X_s}^X$. Then for the regular element v, no there exists any element $u \in X_s$ such that $v \preceq u$.

Accordingly, we say that X is a linear space by Theorem 3.2. From Corollary 3.5, we write $s - \dim X = 0$. This result contradicts with the hypothesis. So our assumption is wrong.

Thus if $s - \dim X \neq 0$ that is X be a nonlinear quasilinear space, then

$$r - \dim X \leq s - \dim X.$$

□

Corollary 3.14. *If $s - \dim X \neq 0$ and $s - \dim X < r - \dim X$, X is not a qls.*

Definition 3.15. Let X be a qls, $M \subseteq X$ and $x, y \in M$. M is called *proper set* if the following two conditions hold:

(i) $F_x^M \neq \emptyset$ for all $x \in M$,

(ii) $F_x^M \neq F_y^M$ for each pair of points x, y with $x \neq y$.

Otherwise M is called *improper set.*

Especially if X is proper set, then it is called *proper quasilinear space (briefly, proper qls).*

Now let us deal with the condition ii).

If $x \neq y$, there is three different cases:

♦ Case of $x \npreceq y$: Then it should be that $F_x \subsetneq F_y$ to hold the condition (ii). (We note that $F_y \subseteq F_x$ may be.) This means that there exists at least one element $z \in X_r$ such that $z \preceq x$ and $z \npreceq y$.

♦ Case of $y \npreceq x$: Then it should be that $F_y \subsetneq F_x$ to hold the condition (ii). (We note that $F_y \subseteq F_x$ may be.) This means that there exists at least one element $m \in X_r$ such that $m \preceq y$ and $m \npreceq x$.

♦ Case of that there is not a comparison between x and y. Then it should be that $F_x \subsetneq F_y$ and $F_y \subsetneq F_x$ to hold the condition (ii). This means that there exist at least two elements $z, m \in X_r$ such that $z \preceq x$, $z \npreceq y$ and $m \preceq y$, $m \npreceq x$.

It is obvious that every linear space is a proper qls with relation of "=".

Also, trivial space $X = \{\theta\}$ is a proper space.

Example 3.16. Let E be a normed linear space. Then $\Omega(E)$ and $\Omega_C(E)$ are proper quasilineear spaces. We will show that $\Omega_C(E)$ is a proper qls.

It is obvious that $F_A \neq \emptyset$ for every $A \in \Omega_C(E)$.

Let us take arbitrary elements $A, B \in \Omega_C(E)$ such that $A \neq B$. Then there is three cases.

♦ If $A \nsubseteq B$, then there is at least $a \in A$ such that $a \notin B$. So $\{a\} \subseteq A$ and $\{a\} \nsubseteq B$.

♦ If $B \nsubseteq A$, then there is at least $b \in B$ such that $b \notin A$. Hence $\{b\} \subseteq B$ and $\{b\} \nsubseteq A$.

♦ If there is not a comparison between A and B, then there exist two elements a and b such that $a \in A$, $a \notin B$ and $b \in B$, $b \notin A$. Thus $\{a\} \subseteq A$, $\{a\} \nsubseteq B$ and $\{b\} \subseteq B$, $\{b\} \nsubseteq A$. So $\Omega_C(E)$ is a proper qls.

It can be similarly shown that $\Omega(E)$ is proper qls.

Example 3.17. The singular subspace of $\Omega_C(\mathbb{R})$

$$(\Omega_C(\mathbb{R}))_s \cup \{0\} = \{[a, b] : a < b,\ a, b \in \mathbb{R}\} \cup \{0\}$$

is improper. Because, in this space, floors of some elements may be empty set and floors of any two different elements may be same. For example, we have $F_{[a,b]} = F_{[c,d]} = \emptyset$ while $a, b > 0$ ($c, d > 0$) or $a, b < 0$ ($c, d < 0$) with $[a, b] \neq [c, d]$. Further we have $F_{[a,b]} = F_{[c,d]} = \{0\}$ for $[a, b] \neq [c, d]$ such that $a < 0 < b$ and $c < 0 < d$.

Example 3.18. The symetric subspace of $\Omega_C(\mathbb{R})$

$$(\Omega_C(\mathbb{R}))_d = \{[-a, a] : a \in \mathbb{R}\}$$

is improper. Because, floors of every two different elements in this space is $\{0\}$. From the same reason, the subspace of $\Omega_C(\mathbb{R})$

$$A = \{[a, b] : a \leq 0 \leq b \text{ and } a, b \in \mathbb{R}\}$$

is improper.

Corollary 3.19. *If regular subspace of a qls X is $\{\theta_X\}$, then X is an improper space. Therefore singular subspace of a qls is improper.*

The following example shows that a proper qls may has improper subspaces.

Example 3.20. Let $X = \Omega_C\left(\mathbb{R}^2\right)$ and

$$V = X_s \cup \{\{(x, 0)\} : x \in \mathbb{R}\}.$$

It is obvious that V is a subspace of X and

$$V_s = X_s \text{ and } V_r = \{\{(x, 0)\} : x \in \mathbb{R}\}.$$

Also, since $F_{v_1} = F_{v_2} = \emptyset$ for

$$v_1 = \{\{(0, y)\} : 1 \leq y \leq 2\}$$

and

$$v_2 = \{\{(0, y)\} : 3 \leq y \leq 4\},$$

V is an improper qls.

On the other hand, an improper qls may has proper subspaces:

Example 3.21. Taking into account the set V in above example, the set

$$H = \{x \in V : (x_1, 0) \in x,\ a \leq x_1 \leq b \text{ and } a, b \in \mathbb{R}\}$$

is a proper subspace of qls V.

Lemma 3.22. *Regular subspace of a nontrivial proper qls is nonempty.*

Proof. Let X be a nontrivial proper qls. Then $F_x \neq \emptyset$ for each $x \in X$ and $F_x \neq F_y$ for every $x, y \in X$ such that $x \neq y$.

♦ If $x \npreceq y$, then there exists at least $z \in X_r$ such that $z \preceq x$ and $z \npreceq y$.

♦ If $y \npreceq x$, then there exists at least $m \in X_r$ such that $m \preceq y$ and $m \npreceq x$.

♦ If there is not a comparison between x and y, then there exist $z, m \in X_r$ such that $z \preceq x$, $z \npreceq y$ and $m \preceq y$, $m \npreceq x$.

Thus regular subspace X_r of proper qls X has at least element $z(m)$. □

Remark 3.23. A qls X may has a regular element such that y_x such that $y_x \preceq x$ for every element x. But this case does not require that X is a proper qls.

The next example reflects this situation.

Example 3.24. We consider the set

$$U = \{\{(x, 0)\} : x \in \mathbb{R}\} \subset \Omega_C\left(\mathbb{R}^2\right).$$

Let $V = X_s \cup U$ and

$$W = \{x \in V : \text{bir } z \in U \text{ için } z \subseteq x\}.$$

Although for every $x \in W$, there exists a $z \in W_r$ such that $z \subseteq x$, since

$$F_{w_1} = F_{w_2} = \{\{(x, 0)\} : 1 \leq x \leq 2\}$$

and $w_1 \neq w_2$ for

$$w_1 = \{\{(x, 0)\} : 1 \leq x \leq 2\}$$

and

$$w_2 = \{\{(x, y)\} : -1 \leq y \leq 0,\ 1 \leq x \leq 2\},$$

W is an improper subspace of proper qls $\Omega_C\left(\mathbb{R}^2\right)$.

Theorem 3.25. *In a nonlinear proper qls X, $s - \dim X = r - \dim X$.*

Proof. Let $r - \dim X = n\ (= \dim X_r)$. Now let us assume that $y_1, y_2, ..., y_n, y_{n+1}$ are qs-independent vectors in X_s. Since $X_s \subset X$, we have $y_1, y_2, ..., y_n, y_{n+1} \in X$. Because of the fact that X is a proper qls, for every $x \in X$, there exist $y \in X_r$ such that $y \preceq x$. Then there exist $a_1, a_2, ..., a_n, a_{n+1} \in X_r$ such that

$$a_1 \preceq y_1,\ a_2 \preceq y_2,\ ..., a_n \preceq y_n,\ a_{n+1} \preceq y_{n+1}. \tag{3.6}$$

Since X_r is $n-$dimensional, the set $\{a_1, a_2, ..., a_n, a_{n+1}\}$ is linear dependent. Then we can find scalars γ_k $(1 \leq k \leq n)$ such that

$$a_{n+1} = \sum_{k=1}^{n} \gamma_k a_k.$$

From (3.6), by using the axiom (2.12) and (2.13), we get

$$a_{n+1} = \sum_{k=1}^{n} \gamma_k a_k \preceq \sum_{k=1}^{n} \gamma_k y_k.$$

Since $a_{n+1} \in X_r$, we have

$$\theta \preceq \sum_{k=1}^{n} \gamma_k y_k - a_{n+1}. \tag{3.7}$$

Taking into account qls axioms, we get

$$\sum_{k=1}^{n} \gamma_k y_k \preceq \sum_{k=1}^{n} \gamma_k y_k \text{ and } -a_{n+1} \preceq -y_{n+1}.$$

Then $\sum\limits_{k=1}^{n} \gamma_k y_k - a_{n+1} \preceq \sum\limits_{k=1}^{n} \gamma_k y_k - y_{n+1}$. Using this relation and the relation (3.7), we obtain

$$\theta \preceq \sum_{k=1}^{n} \gamma_k y_k - y_{n+1}. \tag{3.8}$$

Since it is supposed that $\{y_1, y_2, ..., y_n, y_{n+1}\}$ is qs-independent, γ_k must be zero for all k with $1 \leq k \leq n+1$. On the other hand, (3.8) hold for $\gamma_{n+1} = -1$. This contradiction shows that $\{y_1, y_2, ..., y_n, y_{n+1}\}$ is qs-dependent in X_s. This means that

$$s - \dim X < n + 1,$$

that is

$$s - \dim X \leq r - \dim X. \tag{3.9}$$

On the other hand, by Proposition 3.13, we have prove that

$$r - \dim X \leq s - \dim X. \tag{3.10}$$

for a nonlinear qls X.

From (3.9) and (3.10), we can say that $s - \dim X = r - \dim X$ when X is a nonlinear proper qls. □

Remark 3.26. The converse of Theorem 3.25 may not be correct. For example, consider

$$X = (\Omega_C(l_\infty))_s \cup \{(0, t_1, t_2, t_3, ...) : \forall k \in \mathbb{N} \text{ için } t_k \in \mathbb{R}\}.$$

Then $r - \dim X = s - \dim X = \infty$. But X is improper.

As another example, $r - \dim((\Omega_C(\mathbb{R}))_d) = s - \dim((\Omega_C(\mathbb{R}))_d) = 0$ for the subspace $(\Omega_C(\mathbb{R}))_d = \{[-a, a] : a \in \mathbb{R}\}$ of $\Omega_C(\mathbb{R})$, however $(\Omega_C(\mathbb{R}))_d$ is improper.

Remark 3.27. Since $s - \dim X = r - \dim X$ in a nonlinear proper qls X, we write only $\dim X$ instead of $r - \dim X$ $(s - \dim X)$ and say that dimension of X is $\dim X$.

Theorem 3.28. *Let X be a proper normed qls. Then the closed unit ball of X*

$$S(\theta, 1) = \{x \in X : \|x\| \leq 1\}$$

is a proper set.

Proof. Since X is proper, $F_x \neq F_y$ for all $x, y \in X$ such that $x \neq y$. If $x \neq y$ then there is three cases:

♦ If $x \npreceq y$, then there exists at least one element $z_1 \in X_r$ such that $z_1 \preceq x$ and $z_1 \npreceq y$.

♦ If $y \npreceq x$, then there exists at least one element $m_1 \in X_r$ such that $m_1 \preceq y$ and $m_1 \npreceq x$.

♦ If there is not a comparison between x and y, then there exist at least two elements $z_1, m_1 \in X_r$ such that $z_1 \preceq x$, $z_1 \npreceq y$ and $m_1 \preceq y$, $m_1 \npreceq x$.

We want to show that $S(\theta, 1)$ is a proper set. To do this, let us take elements u and v from $S(\theta, 1)$ such that $u \neq v$. Because of the fact that X is a proper normed qls, we have $F_u \neq F_v$. Hence, we have:

♦ If $u \npreceq v$, then there exists at least one element $z_2 \in X_r$ such that $z_2 \preceq u$ and $z_2 \npreceq v$. So we can say that $\|z_2\| \leq \|u\| \leq 1$ by using the norm axioms and the fact that $u \in S(\theta, 1)$. Hence $z_2 \in S(\theta, 1)$.

♦ If $v \npreceq u$, then there exists at least one element $m_2 \in X_r$ such that $m_2 \preceq v$ and $m_2 \npreceq u$. Thus we can say that $\|m_2\| \leq \|v\| \leq 1$ by using the norm axioms and the fact that $v \in S(\theta, 1)$. So $m_2 \in S(\theta, 1)$.

♦ If there is not a comparison between u and v, then there is at least $z_2, m_2 \in X_r$ such that $z_2 \preceq u$, $z_2 \npreceq v$ and $m_2 \preceq v$, $m_2 \npreceq u$. Therefore we can say that $\|z_2\| \leq \|u\| \leq 1$ and $\|m_2\| \leq \|v\| \leq 1$ by using the norm axioms and the facts that $u \in S(\theta, 1)$ and $v \in S(\theta, 1)$, and so $z_2 \in S(\theta, 1)$, $m_2 \in S(\theta, 1)$, respectively.

On the other hand, since X is proper , $S(\theta, 1) \subset X$ and $F_u^{S(\theta,1)} \subset F_u$, we have $F_u^{S(\theta,1)} \neq \emptyset$ for every $u \in S(\theta, 1)$.

Consequently we obtain that z_2, $m_2 \in (S(\theta, 1))_r$. Hence, $S(\theta, 1)$ is a proper subset of X. □

Theorem 3.29. *Every proper qls has a Hamel basis.*

Proof. Let $\mathcal{S}$ be the family of all qs-independent subsets of X. Now let us consider this family with the partial order relation "$\subseteq$" and assume that $\mathcal{C}$ is a chain in $\mathcal{S}$. Define

$$\mathcal{H}_{\mathcal{S}} = \bigcup_{A \in \mathcal{C}} A.$$

We claim that $\mathcal{H}_{\mathcal{S}}$ is qs-independent subset of X. To see this, let us suppose that $\mathcal{H}_{\mathcal{S}}$ is qs-dependent. Then we can take a qs-dependent subset $\{v_1, v_2, ..., v_n\}$ of $\mathcal{H}_{\mathcal{S}}$. By the definition of $\mathcal{H}_{\mathcal{S}}$, for every $k = 1, 2, ..., n$, there exist an $A_k \in \mathcal{C}$ with $v_k \in A_k$. Since $\mathcal{C}$ is a chain, there exists a $k_0 \in \{1, 2, ..., n\}$, such that $A_k \subset A_{k_0}$ for $k = 1, 2, ..., n$. Thus $v_1, v_2, ..., v_n \in A_{k_0}$. Since A_{k_0} is qs-independent, this contradicts with qs-dependence of $\{v_1, v_2, ..., v_n\}$. This contradiction shows that $\mathcal{H}_{\mathcal{S}}$ is qs-independent and therefore $\mathcal{H}_{\mathcal{S}} \in \mathcal{S}$. Obviously $\mathcal{H}_{\mathcal{S}}$ is an upper bound for $\mathcal{C}$. By Zorn's lemma, there exists a maximal element $\mathcal{B}_{\mathcal{S}}$ of $\mathcal{S}$, as required. It remains to show that $Qspan\mathcal{B}_{\mathcal{S}} = X$.

Let $v_0 \in X \backslash \mathcal{B}_{\mathcal{S}}$. Then $\mathcal{B}_{\mathcal{S}} \cup \{v_0\}$ is not qs-independent, hence there exists certain elements $v_1, v_2, ..., v_n \in \mathcal{B}_{\mathcal{S}}$ and scalars $\alpha_1, \alpha_2, ..., \alpha_n$ with $(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_n) \neq (0, 0, ..., 0)$ such that

$$\theta \preceq \alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n.$$

Since $\mathcal{B}_{\mathcal{S}}$ is qs-independent, we have $\alpha_0 \neq 0$. This implies implicitly that $v_0 \in Qspan\mathcal{B}_{\mathcal{S}}$ and hence X is spanned by $\mathcal{B}_{\mathcal{S}}$. □

Now let us express how we can represent an element in a proper qls $(X, \preceq)$:

Let us consider the floor of $y \in X$. Since $x \preceq y$ for every $x \in F_y$, F_y is bounded from above with respect to the partial order relation "$\preceq$" on X.

Now we claim that

$$y = \sup_{(\preceq)} \{x \in X_r : x \in F_y\},$$

that is

$$y = \sup_{(\preceq)} \{x \in X_r : x \preceq y\}.$$

Where supremum is considered according to relation "$\preceq$".

Theorem 3.30. *Let $(X, \preceq)$ be a proper qls and $y \in X$. Then*

$$\sup_{(\preceq)} \{x \in X_r : x \preceq y\} = y. \tag{3.11}$$

Proof. Suppose that $s \in X$ and $x \preceq s$ for each $x \in F_y$. We should prove $y \preceq s$ to complete the proof. Assume that $y \npreceq s$. Since X is a proper qls, there exists $z \in X_r$ such that $z \preceq y$ and $z \npreceq s$. So $z \in F_y$. Therefore we can write $z \preceq s$ because of the fact that s is an upper bound for F_y. This situation contradicts with $z \npreceq s$. Then $y \preceq s$. □

Further we emphasize that this representation is unique.

Now, to illustrate of representation of an element we will give the following example.

Example 3.31. Let us consider the proper qls $X = \Omega_C(\mathbb{R}^2)$ and the element

$$A = \{(x_1, x_2) : x_1^2 + x_2^2 \leq p\}$$

in X and recall that any element $\{x\} = \{(x_1, x_2)\} \in X_r$ has the unique representation as

$$\{x\} = x_1\{(1,0)\} + x_2\{(0,1)\} \tag{3.12}$$

by the standard basis of X_r. Also we can write $x_1^2 + x_2^2 \leq p$ because of the fact that $\{x\} \in X_r$ and $\{x\} = \{(x_1, x_2)\} \subseteq A$. Now we take the set

$$F_A = \{\{x\} \in X_r : \{x\} \subseteq A\}$$

This set is bounded from above since A is an upper bound for F_A. Also we can write

$$\begin{aligned}\sup_{(\subseteq)}\{\{x\} \in X_r : \{x\} \subseteq A\} &= \sup_{(\subseteq)}\left\{\{x\} = \{(x_1, x_2)\} \in X_r : x_1^2 + x_2^2 \leq p,\ x \subseteq A\right\} \\ &= \sup_{(\subseteq)}\{x_1\{(1,0)\} + x_2\{(0,1)\} : x_1^2 + x_2^2 \leq p,\ x \subseteq A\} \\ &= A.\end{aligned}$$

Taking into account uniqueness of supremum and the representation given by (3.12) of each element $x = \{(x_1, x_2)\}$ in X_r, we can say that the representation of A is unique.

Remark 3.32. As is known, in classical linear algebra every element is represented with the basis of the space. Let us recall that every linear space is a qls with the partial order relation " = " and consider the element $u = (u_1, u_2)$ in the qls $(R^2, =)$. Then the representation of u is obtained as

$$\sup_{(=)}\left\{u \in \mathbb{R}^2 : u = (u_1, u_2)\right\} \quad = \quad \sup_{(=)}\{u_1(1,0) + u_2(0,1)\} = (u_1, u_2)$$

by using the method is described above.

3.3. Finite Regular and Singular Dimensional Normed Quasilinear Spaces

We firstly give the following preparatory lemma.

Lemma 3.33. *Let $\{x_1, x_2, ..., x_n\}$ be a quasilinear independent set of elements in a normed qls X (of any dimension). Then there exists a positive constant c such that*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|) \tag{3.13}$$

for all scalars $\alpha_1, \alpha_2, ..., \alpha_n$.

Proof. Since (3.13) holds for any c if $|\alpha_1|+|\alpha_2|+\cdots+|\alpha_n| = 0$, we can suppose that $|\alpha_1|+|\alpha_2|+\cdots+|\alpha_n| > 0$. Then (3.13) is equivalent to

$$\|\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n\| \geq c\ ,\ \beta_j = \frac{\alpha_j}{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|}\ ,\ \sum_{j=1}^{n} |\beta_j| = 1. \tag{3.14}$$

Hence it suffices to prove the existence of a $c > 0$ such that (3.14) holds for all $n-$tuple of scalars $\beta_1, \beta_2, ..., \beta_n$ with $\sum_{j=1}^{n} |\beta_j| = 1$. Suppose that this is false. Then there exists a sequence (y_m) of elements

$$y_m = \beta_1^m x_1 + \beta_2^m x_2 + \cdots + \beta_n^m x_n,\ \sum_{j=1}^{n} \left|\beta_j^m\right| = 1$$

such that $\|y_m\| \to 0$ as $m \to \infty$. Since $\sum\limits_{j=1}^{n} \left|\beta_j^m\right| = 1$, we have $\left|\beta_j^m\right| \leq 1$. In this manner, for each fixed j, the sequence

$$(\beta_j^m) = (\beta_j^1, \beta_j^2, ...)$$

is bounded. As a result, (β_1^m) has a convergent subsequence by the Bolzano-Weierstrass theorem. Let β_1 denote the limit of that subsequence and let $(y_{1,m})$ denote the corresponding subsequence of (y_m). With the same idea $(y_{1,m})$ has a subsequence $(y_{2,m})$ for which the corresponding subsequence of scalars β_2^m converges, let β_2 denote the limit. Continuing in this way, after n steps we obtain a subsequence $(y_{n,m}) = (y_{n,1}, y_{n,2}, ...)$ of (y_m) such that

$$y_{n,m} = \sum_{j=1}^{n} \gamma_j^m x_j, \;\; \sum_{j=1}^{n} \left|\gamma_j^m\right| = 1$$

with scalars γ_j^m satisfying $\gamma_j^m \to \beta_j$ as $m \to \infty$. Then, as $m \to \infty$

$$y_{n,m} \to y = \sum_{j=1}^{n} \beta_j x_j = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n$$

where $\sum\limits_{j=1}^{n} |\beta_j| = 1$. So, all β_j can not be zero. Since $\{x_1, x_2, ..., x_n\}$ is a qs-independent set, we say that

$$0 \npreceq \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = y \tag{3.15}$$

On the other hand, because of continuity of norm function $y_{n,m} \to y$ implies $\|y_{n,m}\| \to \|y\|$. Since $\|y_m\| \to 0$ by assumption and $(y_{n,m})$ is a subsequence of (y_m), we must have

$$\|y_{n,m}\| \to 0. \tag{3.16}$$

Hence $\|y\| = 0$ and then $y = 0$. This contradicts with (3.15). □

A subset M of a normed qls X is called *compact* if every sequence in M contains a convergent subsequence whose limit belongs to M.

Since every normed qls is a metric space with the Hausdorff metric, we have following theorem.

Theorem 3.34. *Compact sets in normed qls are closed and bounded.*

The converse of this theorem may not be correct. The closed unit ball of normed qls $\Omega_C(c_0)$ is closed and bounded, but it is not compact. This example shows that closed and bounded sets in infinite regular dimensional normed quasilinear spaces need not be compact.

However, for a finite regular dimensional proper normed qls we say:

Theorem 3.35. *Any subset M in a finite dimensional proper normed qls X is compact if and only if M is closed and bounded.*

Proof. Since compactness implies closedness and boundedness by Lemma 3.34 we only prove the converse. Let M be a closed and bounded and $dimX = n$. Then $r - \dim X = \dim X_r = n$. Let $\{b_1, b_2, ..., b_n\}$ a basis for regular subspace X_r. We consider any sequence (x_m) in M. Each x_m has a representation

$$x_m = \sup_{(\preceq)}\{y_m \in X_r \;:\; y_m \preceq x_m\}$$

by aid of the elements $y_m \in X_r$ which has unique representation such that

$$y_m = \alpha_1^{(m)} b_1 + \alpha_2^{(m)} b_2 + \cdots + \alpha_n^{(m)} b_n,$$

where $\alpha_j^{(m)}$ $(j = 1, 2, ..., n)$ are real scalars.

The sequence (x_m) is bounded since M is bounded. Then there exists $K > 0$ such that $\|x_m\| \leq K$ for all $m \in \mathbb{N}$. Taking into account the normed qls axioms and Lemma 3.33,

$$K \geq \|x_m\| \geq \|y_m\| = \left\| \sum_{i=1}^{n} \alpha_i^{(m)} b_i \right\| \geq c \sum_{i=1}^{n} \left| \alpha_i^{(m)} \right|$$

where $c > 0$. So the sequence of numbers $(\alpha_i^{(m)})$ (i fixed such that $1 \leq i \leq n$) is bounded and has a accumulation point α_i. With similar thought in the proof of Lemma 3.33, we get that (x_m) has a subsequence (z_m) which converges to $z = \sum\limits_{i=1}^{n} \alpha_i b_i$. Since M is closed, $z \in M$. This shows that the arbitrary sequence (x_m) in M has a subsequence which converges in M. Thus M is compact. □

Remark 3.36. Teorem 3.35 can be given for finite regular dimensional proper (finite-dimensional) normed qls. This situation may not be valid in an improper normed quasilinear spaces although it is finite regular dimensional.

The following example reflects this situation.

Example 3.37. We consider singular subspace of the normed qls $\Omega_C(c_0)$ with the partial order relation "$\subseteq$". This subspace is an improper normed qls which has 0 regular and ∞ singular dimension. Now let us consider the closed ball $S\left(z, \frac{1}{4}\right)$ such that

$$z = \{(t, 0, 0, ...) : 0 \leq t \leq 1\} \in (\Omega_C(c_0))_s \cup \{\theta\}$$

We claim that this ball is closed and bounded in the finite regular dimensional improper normed qls $(\Omega_C(c_0))_s \cup \{\theta\}$, but it is not compact.

Firstly, we show that the closed ball $S\left(z, \frac{1}{4}\right)$ is subset of $(\Omega_C(c_0))_s \cup \{\theta\}$. For this, it is enough to show that this ball can not contain any regular element. We immediately note that $(\Omega_C(c_0))_r = \{\{u\} : u \in c_0\}$ and consider any singleton $\{u\} \subset c_0$. Now we show that $\{u\} \notin S\left(z, \frac{1}{4}\right)$ for arbitrary element $\{u\}$.

$$\begin{aligned} h_X(\{u\}, z) &= \inf\{r \geq 0 : \{u\} \subseteq z + S(\theta, r),\ z \subseteq \{u\} + S(\theta, r)\} \\ &= \inf \left\{ \begin{array}{c} r \geq 0 : \{(u_1, u_2, ...)\} \subseteq \{(t, 0, 0, ...) : 0 \leq t \leq 1\} + S(\theta, r), \\ \{(t, 0, 0, ...) : 0 \leq t \leq 1\} \subseteq \{(u_1, u_2, ...)\} + S(\theta, r) \end{array} \right\}, \end{aligned}$$

where $S(\theta, r)$ indicates the ball of radius r, centered at θ in $(\Omega_C(c_0))_s \cup \{\theta\}$.

On the other hand, infimum of numbers r satisfying the includings

$$\{(u_1, u_2, ...)\} \subseteq \{(t, 0, 0, ...) : 0 \leq t \leq 1\} + S(\theta, r) \tag{3.17}$$

and

$$\{(t, 0, 0, ...) : 0 \leq t \leq 1\} \subseteq \{(u_1, u_2, ...)\} + S(\theta, r) \tag{3.18}$$

is obtained as 1/2. In other words, the includings (3.17) and (3.18) hold for sets $S(\theta, r)$ with $r \geq 1/2$. The reason of this is explained in the following discuss:

Taking into account

$$\begin{aligned} \{(u_1, u_2, ...)\} &\subseteq \{(t, 0, 0, ...) : 0 \leq t \leq 1\} + S(\theta, r) \\ &\iff (u_1, u_2, ...) \in \{(t, 0, 0, ...) : 0 \leq t \leq 1\} + S(\theta, r) \end{aligned}$$

and

$$\begin{aligned} \{(t, 0, 0, ...) : 0 \leq t \leq 1\} &\subseteq \{(u_1, u_2, ...)\} + S(\theta, r) \\ &\iff (t, 0, 0, ...) \in \{(u_1, u_2, ...)\} + S(\theta, r), \text{ for all } t \in [0, 1], \end{aligned}$$

for the includings (3.17) and (3.18) hold. Then $S(\theta, r)$ must contain the element $w = (u_1 - t', u_2, u_3, ...)$ for a fixed real number $t' \in [0,1]$ and the elements $w_t = (t - u_1, -u_2, -u_3, ...)$ for every $t \in [0,1]$. Therefore, it must be $h(w, \theta) = \|w\|_{c_0} \leq r$ and $h(w_t, \theta) = \|w_t\|_{c_0} \leq r$ for every $t \in [0,1]$. So

$$r \geq \max\left\{\|w\|_{c_0}, \sup_{t\in[0,1]} \|w_t\|_{c_0}\right\}.$$

Since

$$\|w\|_{c_0} = \max\left\{\left|u_1 - t'\right|, \sup_{n\geq 2}|u_n|\right\}$$

and

$$\|w_t\|_{c_0} = \max\left\{|t - u_1|, \sup_{n\geq 2}|-u_n|\right\},\ t \in [0,1]$$

then we obtain

$$\begin{aligned}
h_X(\{u\}, z) &= \inf\{r \geq 0 : \{u\} \subseteq z + S(\theta, r)),\ z \subseteq \{u\} + S(\theta, r)\} \\
&= \inf\left\{r \geq 0 : r \geq \max\left\{\|w\|_{c_0}, \sup_{t\in[0,1]}\left\{\|w_t\|_{c_0}\right\}\right\}\right\} \\
&= \inf\left\{r \geq 0 : r \geq \max\left\{\max\left\{\left|u_1 - t'\right|, \sup_{n\geq 2}|u_n|\right\}, \sup_{t\in[0,1]}\left\{\max\left\{|t - u_1|, \sup_{n\geq 2}|-u_n|\right\}\right\}\right\}\right\} \\
&\geq \inf\left\{r \geq 0 : r \geq \max\left\{\left|u_1 - t'\right|, \sup_{t\in[0,1]}|t - u_1|\right\}\right\} \\
&\geq \frac{1}{2}.
\end{aligned}$$

So we can say $\{u\} \notin S\left(z, \frac{1}{4}\right)$ and $S\left(z, \frac{1}{4}\right) \subset (\Omega_C(c_0))_s \cup \{\theta\}$.

On the other hand, boundedness of closed ball $S\left(z, \frac{1}{4}\right)$ is obvious.

Since a closed ball is closed set in any metric space, $S\left(z, \frac{1}{4}\right)$ is closed.

Now we show that $S\left(z, \frac{1}{4}\right)$ is not compact. Consider the sequence (z_n) defined by the formula

$$z_n = z + \frac{1}{4}\{e_n\}.$$

The first three terms of this sequence are as follows

$$\begin{aligned}
z_1 &= z + \frac{1}{4}\{e_1\} = \left\{\left(t + \frac{1}{4}, 0, 0, ...\right) : 0 \leq t \leq 1\right\}, \\
z_2 &= z + \frac{1}{4}\{e_2\} = \left\{\left(t, \frac{1}{4}, 0, 0, ...\right) : 0 \leq t \leq 1\right\}, \\
z_3 &= z + \frac{1}{4}\{e_3\} = \left\{\left(t, 0, \frac{1}{4}, 0, ...\right) : 0 \leq t \leq 1\right\}.
\end{aligned}$$

On the other hand, by Proposition 2.5 - (2.20) we have

$$\begin{aligned}
h_X\left(z + \frac{1}{4}\{e_n\}, z\right) &\leq h_X(z, z) + h_X\left(\frac{1}{4}\{e_n\}, 0\right) \\
&= 0 + \left\|\frac{1}{4}\{e_n\}\right\|_{\Omega_C(c_0)} \\
&= \sup_{a\in\frac{1}{4}\{e_n\}} \|a\|_{c_0} \\
&= \sup_{a\in\frac{1}{4}\{e_n\}} \sup_{b\in a}|b| = \frac{1}{4}.
\end{aligned}$$

So $(z_n) \subset S\left(z, \frac{1}{4}\right)$.

Now we prove that (z_n) can not has a convergent subsequence. To do this, we show that any subsequence of (z_n) is not a Cauchy sequence. Let $(z_{k_n}) = \left(z + \frac{1}{4}\{e_{k_n}\}\right)$ be a subsequence of (z_n). Then

$$\begin{aligned} h_X(z_{k_n}, z_{k_m}) &= h_X\left(z + \frac{1}{4}\{e_{k_n}\}, z + \frac{1}{4}\{e_{k_m}\}\right) \\ &= \inf\left\{r \geq 0 : z + \frac{1}{4}\{e_{k_n}\} \subseteq z + \frac{1}{4}\{e_{k_m}\} + S(\theta, r),\ z + \frac{1}{4}\{e_{k_m}\} \subseteq z + \frac{1}{4}\{e_{k_n}\} + S(\theta, r)\right\} \end{aligned}$$

$$= \inf\left\{\begin{array}{c} r \geq 0 : \left\{\left(t, 0, 0, \cdots, 0, 0, \overset{k_n.\ term}{\tfrac{1}{4}}, 0, 0, \cdots\right) : 0 \leq t \leq 1\right\} \\ \subseteq \left\{\left(t, 0, 0, \cdots, 0, 0, \overset{k_m.\ term}{\tfrac{1}{4}}, 0, 0, \cdots\right) : 0 \leq t \leq 1\right\} + S(\theta, r), \\ \left\{\left(t, 0, 0, \cdots, 0, 0, \overset{k_m.\ term}{\tfrac{1}{4}}, 0, 0, \cdots\right) : 0 \leq t \leq 1\right\} \\ \subseteq \left\{\left(t, 0, 0, \cdots, 0, 0, \overset{k_n.\ term}{\tfrac{1}{4}}, 0, 0, \cdots\right) : 0 \leq t \leq 1\right\} + S(\theta, r) \end{array}\right\}$$

and the includings

$$\begin{aligned} \left\{\left(t, 0, 0, \cdots, 0, 0, \overset{k_n.\ term}{\frac{1}{4}}, 0, 0, \cdots\right) : 0 \leq t \leq 1\right\} \subseteq &\left\{\left(t, 0, 0, \cdots, 0, 0, \overset{k_m.\ term}{\frac{1}{4}}, 0, 0, \cdots\right) : 0 \leq t \leq 1\right\} \\ &+ S(\theta, r) \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \left\{\left(t, 0, 0, \cdots, 0, 0, \overset{k_m.\ term}{\frac{1}{4}}, 0, 0, \cdots\right) : 0 \leq t \leq 1\right\} \subseteq &\left\{\left(t, 0, 0, \cdots, 0, 0, \overset{k_n.\ term}{\frac{1}{4}}, 0, 0, \cdots\right) : 0 \leq t \leq 1\right\} \\ &+ S(\theta, r) \end{aligned} \tag{3.20}$$

hold for only the balls $S(\theta, r)$ such that

$$r \geq \frac{1}{4}.$$

The reason of this is explained in the following discuss:

For the includings (3.19) and (3.20) hold, the set $S(\theta, r)$ must contain the elements

$$v_1 = \left(0, 0, 0, \cdots, 0, 0, \overset{k_n.\ term}{\frac{1}{4}}, 0, 0, \cdots, 0, 0, \overset{k_m.\ term}{-\frac{1}{4}}, 0, 0, \cdots\right) \tag{3.21}$$

and

$$v_2 = \left(0, 0, 0, \cdots, 0, 0, \overset{k_n.\ term}{-\frac{1}{4}}, 0, 0, \cdots, 0, 0, \overset{k_m.\ term}{\frac{1}{4}}, 0, 0, \cdots\right), \tag{3.22}$$

respectively, where it is considered $k_m > k_n$ without loss of generality. Then for containing the elements v_1 and v_2 of the ball $S(\theta, r)$, it must be

$$h_X(v_1,\theta) = \|v_1\|_{c_0} = \frac{1}{4} \leq r$$

and

$$h_X(v_2,\theta) = \|v_2\|_{c_0} = \frac{1}{4} \leq r.$$

So, we obtain

$$h_X\left(z + \frac{1}{4}\{e_{k_n}\}, z + \frac{1}{4}\{e_{k_m}\}\right) = \frac{1}{4}.$$

Clearly $\left(z + \frac{1}{4}\{e_{k_n}\}\right)$ can not be a Cauchy sequence. Consequently $S\left(z, \frac{1}{4}\right)$ is not compact. This result completes the proof of assertion in this example.

Corollary 3.38. *Let E be a finite dimensional normed linear space. Any subset M in proper qls $\Omega_C(E)$ is compact if and only if M is closed and bounded.*

Theorem 3.39. *Let E be a normed linear space. If closed unit ball of E is compact then $\Omega_C(E)$ is finite dimensional.*

Proof. Suppose that the closed unit ball $S(\theta,1) = \{x \in E : \ \|x\| \leq 1\}$ of normed linear space E is compact. Then E is finite dimensional. Since

$$\dim(\Omega_C(E)) = r - \dim(\Omega_C(E)) = \dim\left((\Omega_C(E))_r\right) = \dim(E),$$

we say that proper qls $\Omega_C(E)$ is finite dimensional. □

Theorem 3.40. *Let X be a proper normed qls. If the closed unit ball of X is compact then X is finite dimensional.*

Proof. We write $(S(\theta,1))_X$ and $(S(\theta,1))_{X_r}$to denote the closed balls of X and X_r, respectively. Firstly it is obvious that $(S(\theta,1))_{X_r} \subseteq (S(\theta,1))_X$ since $X_r \subseteq X$.

Now we suppose that closed unit ball $(S(\theta,1))_X$ of X is compact. Since $(S(\theta,1))_{X_r}$ is closed in $(S(\theta,1))_X$ (it can be easily prove this), $(S(\theta,1))_{X_r}$ is compact, too. Thus linear subspace X_r is finite dimensional. Since X is proper qls we say

$$\dim X = r - \dim X = s - \dim X$$

and because of

$$r - \dim X = \dim X_r$$

we obtain that X is finite dimensional. □

We will complete this section by deriving several significant results concerning the proper quasilinear space $\Omega_C(\mathbb{R}^n)$.

Let us consider any element y in proper normed qls $\Omega_C(\mathbb{R}^n)$. We have shown that the representation

$$y = \sup_{(\preceq)}\{x \in (\Omega_C(\mathbb{R}^n))_r : x \preceq y\}$$

is unique.

Now let us present the following lemma giving information about the norm of elements in $\Omega_C(\mathbb{R}^n)$. This lemma states that there exists always an element in floor of y such that its norm is equal to $\|y\|$. We note that this element is important since it is a regular element.

Lemma 3.41. *Let $y \in \Omega_C(\mathbb{R}^n)$. Then there exists an element $x_0 \in F_y$ such that*

$$\|x_0\| = \|y\|.$$

Proof. For every $y \in \Omega_C(\mathbb{R}^n)$, F_y is closed and bounded in $\mathbb{R}^n$ by Lemma 3.12. Since $\Omega_C(\mathbb{R}^n)$ is finite dimensional ($n-$dimensional), F_y is compact by Teorem 3.35. Because of the fact that the norm is continuous function which converts compact set F_y into $\mathbb{R}$, this mapping assumes a maximum at some points of F_y (see: [9, Corollary 2.5-7]). Thus we say that there exists the value of

$$\max\{\|x\| : x \in F_y\}.$$

Let $\max\{\|x\| : x \in F_y\} = z$. Then $\|x_0\| = z$ for an element x_0 of F_y by definition of maximum and we write

$$\|y\| = \sup\{\|x\|_{\mathbb{R}^n} : x \in y\} = \sup\{\|x\| : x \in F_y\} = \max\{\|x\| : x \in F_y\} = z = \|x_0\|$$

for $y \in \Omega_C(\mathbb{R}^n)$. □

The following considerable theorem is proved by Lemma 3.41.

Theorem 3.42.

$$\|y\| = \left\| \sup_{(\preceq)}\{x \in (\Omega_C(\mathbb{R}^n))_r : x \preceq y\} \right\| = \sup\{\|x\| : x \in (\Omega_C(\mathbb{R}^n))_r\,,\ x \preceq y\}$$

for $y \in \Omega_C(\mathbb{R}^n)$.

Proof. Let us recall that the element y in $\Omega_C(\mathbb{R}^n)$ has a unique representation such that $y = \sup_{(\preceq)}\{x \in (\Omega_C(\mathbb{R}^n))_r : x \preceq y\}$. Since $\|x\| \leq \|y\|$ for every x such that $x \preceq y$ by the normed qls axioms, we have

$$\sup\{\|x\| : x \in (\Omega_C(\mathbb{R}^n))_r\,,\ x \preceq y\} \leq \|y\|.$$

From Lemma 3.41 we know that there exists an element $x_0 \in (\Omega_C(\mathbb{R}^n))_r$ such that $x_0 \preceq y$ and $\|y\| = \|x_0\|$. Hence we obtain

$$\sup\{\|x\| \ : \ x \in (\Omega_C(\mathbb{R}^n))_r\,,\ x \preceq y\} = \|y\| = \left\| \sup_{(\preceq)}\{x \in (\Omega_C(\mathbb{R}^n))_r : x \preceq y\} \right\|.$$

□

Theorem 3.43. *Every nontrivial proper subspace Y of the normed qls $\Omega_C(\mathbb{R}^n)$ is complete.*

Proof. We consider an arbitrary Cauchy sequence (y_m) in Y and show that it is convergent in Y, the limit will be denoted by y. Since Y is a nontrivial proper subspace of $\Omega_C(\mathbb{R}^n)$ we have

$$r - \dim Y = s - \dim Y = \dim Y = k$$

with $k \leq n$. Let $\{b_1, b_2, ..., b_k\}$ any basis for Y_r. Then each $x_m \in Y_r$ such that $x_m \preceq y_m$ has a unique representation of the form

$$x_m = \alpha_1^{(m)} b_1 + \alpha_2^{(m)} b_2 + \cdots + \alpha_k^{(m)} b_k = \sum_{j=1}^{k} \alpha_j^{(m)} b_j$$

and each y_m has a unique representation shaped

$$y_m = \sup_{(\preceq)}\{x_m \in Y_r : x_m \preceq y_m\}$$

by aid of the elements $x_m \in X_r$.

Since (y_m) is Cauchy sequence, for every $\varepsilon > 0$ there exists N such that $\|y_m - y_p\| < \varepsilon$ when $m, p > N$. From this and Lemma 3.33 we have

$$\begin{aligned}
\varepsilon &> \|y_m - y_p\| \\
&= \left\| \sup_{(\preceq)} \{x_m \in Y_r : x_m \preceq y_m\} - \sup_{(\preceq)} \{x_p \in Y_r : x_p \preceq y_p\} \right\| \\
&\geq \|x_m - x_p\| \\
&= \left\| \sum_{j=1}^{k} \alpha_j^{(m)} b_j - \sum_{j=1}^{k} \alpha_j^{(p)} b_j \right\| \\
&\geq \left\| \sum_{j=1}^{k} \left(\alpha_j^{(m)} - \alpha_j^{(p)} \right) b_j \right\| \\
&\geq c \sum_{j=1}^{k} \left| \alpha_j^{(m)} - \alpha_j^{(p)} \right|
\end{aligned}$$

for some $c > 0$. Division by $c > 0$ gives

$$\left| \alpha_j^{(m)} - \alpha_j^{(p)} \right| < \frac{\varepsilon}{c}, \text{ where } m, n > N.$$

This shows that each of the k sequences $\left(\alpha_j^{(m)}\right)$ is Cauchy in $\mathbb{R}$. Thus it converges; let α_j denote the limit. Using these k limits $\alpha_1, \alpha_2, ..., \alpha_k$, we define

$$z = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_k b_k.$$

It is obvious that $z \in Y_r$. We recall that each element y such that $z \preceq y$ can be represented shaped

$$y = \sup_{(\preceq)} \{z \in Y_r : z \preceq y\}.$$

Also, taking into account Theorem 3.42 we can write

$$\begin{aligned}
\|y_m - y\| &= \left\| \sup_{(\preceq)} \{x_m \in Y_r : x_m \preceq y_m\} - \sup_{(\preceq)} \{z \in Y_r : z \preceq y\} \right\| \\
&= \left\| \sup_{(\preceq)} \{x_m - z \in Y_r : x_m - z \preceq y_m - y\} \right\| \\
&= \left\| \sup_{(\preceq)} \left\{ \sum_{j=1}^{k} \left(\alpha_j^{(m)} - \alpha_j \right) b_j : \sum_{j=1}^{k} \left(\alpha_j^{(m)} - \alpha_j \right) b_j \preceq y_m - y \right\} \right\| \\
&= \sup \left\{ \left\| \sum_{j=1}^{k} \left(\alpha_j^{(m)} - \alpha_j \right) b_j \right\| : \sum_{j=1}^{k} \left(\alpha_j^{(m)} - \alpha_j \right) b_j \preceq y_m - y \right\} \\
&\leq \sup \left\{ \sum_{j=1}^{k} \left\| \left(\alpha_j^{(m)} - \alpha_j \right) b_j \right\| : \sum_{j=1}^{k} \left(\alpha_j^{(m)} - \alpha_j \right) b_j \preceq y_m - y \right\} \\
&= \sup \left\{ \sum_{j=1}^{k} \left| \alpha_j^{(m)} - \alpha_j \right| \|b_k\| : \sum_{j=1}^{k} \left(\alpha_j^{(m)} - \alpha_j \right) b_j \preceq y_m - y \right\}.
\end{aligned}$$

On the right

$$\alpha_j^{(m)} \to \alpha_j \text{ when } m \to \infty.$$

Hence $\|y_m - y\| \to 0$ when $m \to \infty$ that is $y_m \to y$. Since $h_X(y_m, y) \leq \|y_m - y\|$ in a qls, we write $h_X(y_m, y) \to 0$. This shows that (y_m) is convergent in Y. Since (y_m) is an arbitrary Cauchy sequence in Y, this proves that Y is complete. $\square$

Remark 3.44. In Theorem 3.43, the subspace Y of $\Omega_C(\mathbb{R}^n)$ must be proper. Note that improper subspaces of $\Omega_C(\mathbb{R}^n)$ may not be complete. Now it shows as an example, let's examine the following example:

Example 3.45. We consider the qls $\Omega_C(\mathbb{R})$. $(\Omega_C(\mathbb{R}))_s \cup \{0\}$ is improper subspace of $\Omega_C(\mathbb{R})$. The singular subspace $(\Omega_C(\mathbb{R}))_s \cup \{0\}$ is not complete. In fact, an example of a Cauchy sequence without limit in $(\Omega_C(\mathbb{R}))_s \cup \{0\}$ is given by

$$u_n = \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$$

converging to the singleton $\{1\}$ which is not a singular element.

As a result of Theorem 3.35 and Theorem 3.43, we have:

Theorem 3.46. *Every proper subspace Y of normed quasilinear space $\Omega_C(\mathbb{R}^n)$ is closed in $\Omega_C(\mathbb{R}^n)$.*

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