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Fixed point theorems for multivalued G-contractions in Hausdorff b-Gauge spaces

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Abstract

In this paper, we extend gauge spaces in the setting of b metric spaces and prove fixed point theorems for multivalued mappings in this new setting endowed with a graph. An example is constructed to substantiate our result. We also discuss possible application of our result for solving integral equations. ©2015 All rights reserved.

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1. Introduction and Preliminaries

Czerwik [11] introduced the notion of a b-metric space. Let X be a nonempty set. A mapping $d\colon X \times X \to [0,\infty)$ is said to be a b-metric on X, if there exists $s \geq 1$ such that for each $x, y, z \in X$, we have

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

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The triplet (X, d, s) is said to be a b-metric space.

Note that every metric space is a b-metric but converse is not true.

Convergence of a sequence in a b-metric space is defined in a similar fashion as in a metric space. A sequence $\{x_n\} \subset X$ is a Cauchy sequence in (X, d, s), if for each $\epsilon > 0$ there exists a natural number $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for each $m, n \geq N(\epsilon)$. A b-metric space (X, d, s) is a complete if each Cauchy sequence in X converges to some point of X.

Czerwik [11] extended Banach contraction principle for self mappings on b metric spaces; for recent research in this direction, please see: Phiangsungnoen *et al.* [20], Shatanawi *et al* [23]. Czerwik [12] further extended the notion of a b-metric space (X, d, s) by defining Hausdorff metric for the space of all nonempty closed and bounded subsets of the b-metric space (X, d, s).

Let (X, d, s) be a b metric space. For $x \in X$ and $A \subset X$, $d(x, A) = \inf\{d(x, a) : a \in A\}$. Denote by $CB(X)$ the class of all nonempty closed and bounded subsets of X and by $CL(X)$ the class of all nonempty closed subsets of X. For $A, B \in CB(X)$, the function

$$H : CB(X) \times CB(X) \to [0, \infty), \quad H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$$

is said to be a Hausdorff b-metric induced by a b-metric space (X, d, s). A Hausdorff b metric space enjoys the same properties as a Hausdorff metric, expect for triangular inequality which in Hausdorff b metric spaces has the following form $H(A, B) \leq s[H(A, C) + H(C, B)]$. Czerwik [12] extended Nadler's fixed point theorem in the setting of Hausdorff b metric spaces.

Jachymaski [15] introduced the notion of Banach G-contraction to extend the notion of Banach contraction, where G is a graph in the metric space whose vertex set coincides with the metric space. He obtain some fixed point theorems for such mappings on complete metric space. Afterwards, many authors extended Banach G-contraction in single as well as multivalued case, see for examples: Tiammee and Suantai [24], Samreen and Kamran [21, 17, 22], Bojor [4, 5, 6], Nicolae *et al.* [19], Aleomraninejad *et al.* [2], Asl *et al.* [3].

One may characterize gauge spaces by the fact that the distance between two distinct points of the space may be zero. For details on gauge spaces, we refer the reader to [13]. Frigon [14] and Chis and Precup [10] generalized the Banach contraction principle on gauge spaces. Some interesting results are also been obtained by the authors: Agarwal *et al.* [1], Chifu and Petrusel [9], Cherichi *et al.* [8, 7], Lazara and Petrusel [18], Jleli *et al.* [16].

By using b metric spaces, in this paper, we first introduce the notion of b_s-gauge spaces. Then we extend this notion to define b_s-gauge structure on the space of nonempty closed subsets of the b metric space and prove some fixed point theorems for multivalued G contractions. To substantiate our main result we have constructed an example. Moreover, a possible application of our result in solving an integral equation is also been discussed.

2. Main results

We begin this section by introducing the notion of a b_s-pseudo metric space.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called b_s-pseudo metric on X if there exists $s \geq 1$ such that for each $x, y, z \in X$, we have

(i) $d(x, x) = 0$ for each $x \in X$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq s[d(x, y) + d(y, x)]$.

Remark 2.2. Every b-metric space (X, d, s) is a b_s-pseudo metric space, but the converse is not true.

Example 2.3. Let $X = C([0, \infty), \mathbb{R})$. Define a function

$$d : X \times X \to [0, \infty), \quad d(x(t), y(t)) = \max_{t \in [0,1]} (x(t) - y(t))^2.$$

Then

(i) It is clear that d is not a metric on X.

(ii) d is not a pseudo metric on X. In this respect, consider $x, y, z \in C([0, \infty), \mathbb{R})$ be defined by

$$x(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ t - 1 & \text{if } t > 1, \end{cases}$$

$y(t) = 3$, for each $t \geq 0$, and $z(t) = -3$, for each $t \geq 0$. Then, we can see that $d(y, z) = 36 \nleq 18 = d(y, x) + d(x, z)$.

(iii) d is not a b-metric on X. Since, if $u, v \in C([0, \infty), \mathbb{R})$ are defined by

$$u(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ t - 1 & \text{if } t > 1, \end{cases}$$

and

$$v(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ 2t - 2 & \text{if } t > 1, \end{cases}$$

then $u \neq v$, but $d(u, v) = 0$.

(iv) d is b_2-pseudo metric on X with $s = 2$.

In order to define gauge spaces in the setting of b_s-pseudo metrics we need to define following.

Definition 2.4. Let X be a nonempty set endowed with the b_s-pseudo metric d. The d_s-ball of radius $\epsilon > 0$ centered at $x \in X$ is the set
$$B(x, d, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

Definition 2.5. A family $\mathcal{F} = \{d_\nu : \nu \in \mathcal{A}\}$ of b_s-pseudo metrics is said to be separating if for each pair (x, y) with $x \neq y$, there exists $d_\nu \in \mathcal{F}$ with $d_\nu(x, y) \neq 0$.

Definition 2.6. Let X be a nonempty set and $\mathcal{F} = \{d_\nu : \nu \in \mathcal{A}\}$ be a family of b_s-pseudo metrics on X. The topology $\mathcal{T}(\mathcal{F})$ having subbases the family

$$\mathcal{B}(\mathcal{F}) = \{B(x, d_\nu, \epsilon) : x \in X, d_\nu \in \mathcal{F} \text{ and } \epsilon > 0\}$$

of balls is called topology induced by the family $\mathcal{F}$ of b_s-pseudo metrics. The pair $(X, \mathcal{T}(\mathcal{F}))$ is called a b_s-gauge space. Note that $(X, \mathcal{T}(\mathcal{F}))$ is Hausdorff if $\mathcal{F}$ is separating.

Definition 2.7. Let $(X, \mathcal{T}(\mathcal{F}))$ be a b_s-gauge space with respect to the family $\mathcal{F} = \{d_\nu : \nu \in \mathcal{A}\}$ of b_s-pseudo metrics on X and $\{x_n\}$ is a sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ converges to x if for each $\nu \in \mathcal{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_\nu(x_n, x) < \epsilon$ for each $n \geq N_0$. We denote it as $x_n \to^{\mathcal{F}} x$;

(ii) The sequence $\{x_n\}$ is a Cauchy sequence if for each $\nu \in \mathcal{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_\nu(x_n, x_m) < \epsilon$ for each $n, m \geq N_0$;

(iii) $(X, \mathcal{T}(\mathcal{F}))$ is complete if each Cauchy sequence in $(X, \mathcal{T}(\mathcal{F}))$ is convergent in X;

(iv) A subset of X is said to be closed if it contains the limit of each convergent sequence of its elements.

Remark 2.8. When $s = 1$, then all above definitions reduce to the corresponding definitions in a gauge space.

Subsequently, in this paper, $\mathcal{A}$ is directed set and X is a nonempty set endowed with a separating complete b_s-gauge structure $\{d_\nu : \nu \in \mathcal{A}\}$. Further, $G = (V, E)$ is a directed graph in $X \times X$, where the set of its vertices V is equal to X and set of its edges E contains $\{(x, x) : x \in V\}$. Furthermore, G has no parallel edges. For each $d_\nu \in \mathcal{F}$, $CL_\nu(X)$ denote the set of all nonempty closed subsets of X with respect to d_v. For each $\nu \in \mathcal{A}$ and $A, B \in CL_\nu(X)$, the function $H_\nu : CL_\nu(X) \times CL_\nu(X) \to [0, \infty)$ defined by

$$H_\nu(A, B) = \begin{cases} \max\left\{ \sup_{x \in A} d_\nu(x, B), \sup_{y \in B} d_\nu(y, A) \right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

is a generalized Hausdorff b_s-pseudo metric on $CL_\nu(X)$. We denote by $CL(X)$ the set of all nonempty closed subsets in the b_s-gauge space $(X, \mathcal{T}(\mathcal{F}))$.

Theorem 2.9. *Let $T : X \to CL(X)$ be a mapping such that for each $\nu \in \mathcal{A}$, we have*

$$H_\nu(Tx, Ty) \le a_\nu d_\nu(x, y) + b_\nu d_\nu(x, Tx) + c_\nu d_\nu(y, Ty) + e_\nu d_\nu(x, Ty) + L_\nu d_\nu(y, Tx), \tag{2.1}$$

for all $(x, y) \in E$, where $a_\nu, b_\nu, c_\nu, e_\nu, L_\nu \ge 0$, and $s^2 a_\nu + s^2 b_\nu + s^2 c_\nu + 2s^3 e_\nu < 1$.

Assume that following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) if $(x, y) \in E$, for $u \in Tx$ and $v \in Ty$ such that $d_\nu(u, v) \le d_\nu(x, y)$ for each $\nu \in \mathcal{A}$, then $(u, v) \in E$;

(iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$;

(iv) for each $\{q_\nu : q_\nu > 1\}_{\nu \in \mathcal{A}}$ and $x \in X$ there exists $y \in Tx$ such that

$$d_\nu(x, y) \le q_\nu d_\nu(x, Tx) \quad \forall \, \nu \in \mathcal{A}.$$

Then T has a fixed point.

Proof. By hypothesis (i), there exist $x_0, x_1 \in X$ such that $x_1 \in Tx_0$ and $(x_0, x_1) \in E$. Now, it follows form (2.1) that

$$H_\nu(Tx_0, Tx_1) \le a_\nu d_\nu(x_0, x_1) + b_\nu d_\nu(x_0, Tx_0) + c_\nu d_\nu(x_1, Tx_1) + e_\nu d_\nu(x_0, Tx_1) + L_\nu d_\nu(x_1, Tx_0), \tag{2.2}$$

for all $\nu \in \mathcal{A}$.

Since $d_\nu(x_1, Tx_1) \le H_\nu(Tx_0, Tx_1)$ and $d_\nu(x_0, Tx_1) \le d_\nu(x_0, x_1) + d_\nu(x_1, Tx_1)$, therefore from (2.2), we get

$$d_\nu(x_1, Tx_1) \le \frac{1}{\xi_\nu} d_\nu(x_0, x_1) \tag{2.3}$$

where, $\xi_\nu = \frac{1 - c_\nu - se_\nu}{a_\nu + b_\nu + se_\nu} > 1$. Using hypothesis (iv) there exists $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \le \sqrt{\xi_\nu} d_\nu(x_1, Tx_1). \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$d_\nu(x_1, x_2) \le \frac{1}{\sqrt{\xi_\nu}} d_\nu(x_0, x_1) \quad \forall \, \nu \in \mathcal{A}. \tag{2.5}$$

Hypothesis (ii) and (2.5), implies that $(x_1, x_2) \in E$. Continuing in the same way, we get a sequence $\{x_m\}$ in X such that $(x_m, x_{m+1}) \in E$ and

$$d_\nu(x_m, x_{m+1}) \leq \left(\frac{1}{\sqrt{\xi_\nu}}\right)^m d_\nu(x_0, x_1) \quad \forall \, \nu \in \mathcal{A}.$$

For convenience we assume that $\eta_\nu = \frac{1}{\sqrt{\xi_\nu}}$ for each $\nu \in \mathcal{A}$. Now we show that $\{x_m\}$ is a Cauchy sequence. For each $m, p \in \mathbb{N}$ and $\nu \in \mathcal{A}$, we have

$$\begin{aligned}
d_\nu(x_m, x_{m+p}) &\leq \sum_{i=m}^{m+p-1} s^i d_\nu(x_i, x_{i+1}) \\
&\leq \sum_{i=m}^{m+p-1} s^i (\eta_\nu)^i d_\nu(x_0, x_1) \\
&\leq \sum_{i=m}^{\infty} (s\eta_\nu)^i < \infty \quad (since \ s\eta_\nu < 1).
\end{aligned}$$

This implies that $\{x_m\}$ is a Cauchy sequence in X. By completeness of X, we have $x^* \in X$ such that $x_m \to x^*$ as $m \to \infty$. By using hypothesis (iii), triangular inequality and (2.1), we have

$$\begin{aligned}
d_\nu(x^*, Tx^*) &\leq sd_\nu(x^*, x_{m-1}) + sd_\nu(x_{m-1}, Tx^*) \\
&\leq sd_\nu(x^*, x_{m-1}) + sH_\nu(Tx_m, Tx^*) \\
&\leq sd_\nu(x^*, x_{m-1}) + sa_\nu d_\nu(x_m, x^*) + sb_\nu d_\nu(x_m, Tx_m) \\
&\quad + sc_\nu d_\nu(x^*, Tx^*) + se_\nu d_\nu(x_m, Tx^*) + sL_\nu d_n(x^*, Tx_m) \\
&\leq sd_\nu(x^*, x_{m-1}) + sa_\nu d_\nu(x_m, x^*) + sb_\nu d_\nu(x_m, x_{m+1}) \\
&\quad + sc_\nu d_\nu(x^*, Tx^*) + se_\nu d_\nu(x_m, Tx^*) + sL_\nu d_\nu(x^*, x_{m+1}) \quad \forall \, \nu \in \mathcal{A}.
\end{aligned}$$

Letting $m \to \infty$, we get
$$d_\nu(x^*, Tx^*) \leq (sc_\nu + se_\nu) d_\nu(x^*, Tx^*) \quad \forall \, \nu \in \mathcal{A}.$$

Which is only possible if $d_\nu(x^*, Tx^*) = 0$. Since the structure $\{d_\nu : \nu \in \mathcal{A}\}$ on X is separating, we have $x^* \in Tx^*$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

In case of single valued mapping $T \colon X \to X$ we have the following result:

Theorem 2.10. *Let $T \colon X \to X$ be a mapping such that for each $\in \mathcal{A}$ we have*

$$d_\nu(Tx, Ty) \leq a_\nu d_\nu(x, y) + b_\nu d_\nu(x, Tx) + c_\nu d_\nu(y, Ty) + e_\nu d_\nu(x, Ty) + L_\nu d_\nu(y, Tx), \qquad (2.6)$$

for all $(x, y) \in E$, where, $a_\nu, b_\nu, c_\nu, e_\nu, L_\nu \geq 0$, and $sa_\nu + sb_\nu + sc_\nu + 2s^2 e_\nu < 1$.

Assume that following conditions hold:

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$;

(ii) for $(x, y) \in E$, we have $(Tx, Ty) \in E$, provided $d_\nu(Tx, Ty) \leq d_\nu(x, y)$ for each $\nu \in \mathcal{A}$;

(iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$;

Then T has a fixed point.

Example 2.11. Let $X = C([0, 10], \mathbb{R})$ endowed with the $d_n(x(t), y(t)) = \max_{t \in [0,n]} (x(t) - y(t))^2$ for each $n \in \{1, 2, 3, \ldots, 10\}$ and the graph $G = (V, E)$ as $V = X$ and

$$E = \{(x(t), y(t)) : x(t) \leq y(t)\} \cup \{(x(t), x(t)) : x \in X\}.$$

Define $T \colon X \to X$ by $Tx(t) = \frac{x(t)+1}{5}$, for each $x \in X$. It is easy to see that (2.5) holds with $a_n = 1/5$ and $b_n = c_n = e_n = L_n = 0$ for each $n \in \{1, 2, 3, \ldots, 10\}$. For $x_0 = 0$ and $x_1 = Tx_0 = 1/5$, we have $(x_0, Tx_0) \in E$. Since T is nondecreasing, for each $(x, y) \in E$, we have $(Tx, Ty) \in E(G)$. For each sequence $\{x_m\}$ in X such that $(x_m, x_{m+1}) \in E$ for each $m \in \mathbb{N}$ and $x_m \to x$ as $m \to \infty$, then $(x_m, x) \in E$ for each $m \in \mathbb{N}$. Therefore, all conditions of Theorem 2.10 are satisfied and T has a fixed point.

Before going towards our next theorem, we have to define Ψ_{s^2} family of mappings. Let $\psi \colon [0, \infty) \to [0, \infty)$ be a nondecreasing mappings such that it satisfies following conditions:

(ψ_1) $\psi(0) = 0$;

(ψ_2) $\psi(\rho t) = \rho \psi(t) < \rho t$ for each $\rho, t > 0$;

(ψ_3) $\sum_{i=1}^{\infty} s^{2i} \psi^i(t) < \infty$;

where $s \geq 1$.

Theorem 2.12. *Let $T \colon X \to CL(X)$ be a mapping such that for each $\nu \in \mathcal{A}$ we have*

$$H_\nu(Tx, Ty) \leq \psi_\nu(d_\nu(x, y)), \quad \forall\, (x, y) \in E, \tag{2.7}$$

where $\psi_\nu \in \Psi_{s^2}$. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) if $(x, y) \in E$, for $u \in Tx$ and $v \in Ty$ such that $\frac{1}{s} d_\nu(u, v) < d_\nu(x, y)$ for each $\nu \in \mathcal{A}$, then $(u, v) \in E$;

(iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$;

(iv) for each $x \in X$, we have $y \in Tx$ such that

$$d_\nu(x, y) \leq s d_\nu(x, Tx) \quad \forall\, \nu \in \mathcal{A}.$$

Then T has a fixed point.

Proof. By hypothesis we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. From (2.7), we get

$$d_\nu(x_1, Tx_1) \leq H_\nu(Tx_0, Tx_1) \leq \psi_\nu(d_\nu(x_0, x_1)) \quad \forall\, \nu \in \mathcal{A}. \tag{2.8}$$

By hypothesis (iv), for $x_1 \in X$, we have $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \leq s d_\nu(x_1, Tx_1) \leq s \psi_\nu(d_\nu(x_0, x_1)) \quad \forall\, \nu \in \mathcal{A}. \tag{2.9}$$

Applying ψ_ν, we have

$$\psi_\nu(d_\nu(x_1, x_2)) \leq \psi_\nu(s \psi_\nu(d_\nu(x_0, x_1))) = s \psi_\nu^2(d_\nu(x_0, x_1)) \quad \forall\, \nu \in \mathcal{A}.$$

From (2.9), it is clear that $(x_1, x_2) \in E$. Again from (2.7), we have

$$d_\nu(x_2, Tx_2) \leq H_\nu(Tx_1, Tx_2) \leq \psi_\nu(d_\nu(x_1, x_2)) \quad \forall\, \nu \in \mathcal{A}. \tag{2.10}$$

By hypothesis (iv), for $x_2 \in X$, we have $x_3 \in Tx_2$ such that

$$d_\nu(x_2, x_3) \leq s d_\nu(x_2, Tx_2) \leq s\psi_\nu(d_\nu(x_1, x_2)) \leq s^2\psi_\nu^2(d_\nu(x_0, x_1)) \ \ \forall \ \nu \in \mathcal{A}. \tag{2.11}$$

Clearly, $(x_2, x_3) \in E$. Continuing in the same way, we get a sequence $\{x_m\}$ in X such that $(x_m, x_{m+1}) \in E$ and

$$d_\nu(x_m, x_{m+1}) \leq s^m \psi_\nu^m(d_\nu(x_0, x_1)) \ \ \forall \ \nu \in \mathcal{A}.$$

Now, we show that $\{x_m\}$ is a Cauchy sequence. For $m, p \in \mathbb{N}$, we have

$$d_\nu(x_m, x_{m+p}) \leq \sum_{i=m}^{m+p-1} s^i d_\nu(x_i, x_{i+1})$$

$$\leq \sum_{i=m}^{m+p-1} s^{2i} \psi_\nu^i(d_\nu(x_0, x_1)) < \infty$$

This implies that $\{x_m\}$ is a Cauchy sequence in X. By completeness of X, we have $x^* \in X$ such that $x_m \to x^*$ as $m \to \infty$. Using hypothesis (iv), triangular inequality and (2.7), we have

$$d_\nu(x^*, Tx^*) \leq s d_\nu(x^*, x_{m-1}) + s d_\nu(x_{m-1}, Tx^*)$$

$$\leq s d_\nu(x^*, x_{m-1}) + s H_\nu(Tx_m, Tx^*)$$

$$\leq s d_\nu(x^*, x_{m-1}) + s\psi_\nu(d_n(x_m, x^*)) \ \ \forall \ \nu \in \mathcal{A}.$$

Letting $m \to \infty$, we get $d_\nu(x^*, Tx^*) = 0$ for each $\nu \in \mathcal{A}$. Since the structure $\{d_\nu : \nu \in \mathcal{A}\}$ on X is separating, we have $x^* \in Tx^*$. $\qquad \square$

By considering $T \colon X \to X$ in above theorem we get the following one.

Theorem 2.13. *Let $T \colon X \to X$ be a mapping such that for each $\nu \in \mathcal{A}$ we have*

$$d_\nu(Tx, Ty) \leq \psi_\nu(d_\nu(x, y)), \ \ \forall \ (x, y) \in E, \tag{2.12}$$

where $\psi_\nu \in \Psi_{s^2}$. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) for $(x, y) \in E$, we have $(Tx, Ty) \in E$ provided $\frac{1}{s}d_\nu(Tx, Ty) < d_\nu(x, y)$ for each $\nu \in \mathcal{A}$, then $(u, v) \in E$;

(iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

3. Application

Consider the Volterra integral equation of the form:

$$x(t) = f(t) + \int_0^t K(t, s, x(s))ds, \ t \in I \tag{3.1}$$

where $f \colon I \to \mathbb{R}$ is a continuous function, and $K \colon I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing function.

Let $X = (C[0, \infty), \mathbb{R})$. Define the family of b_2-pseudo norms by $\|x\|_n = \max_{t \in [0,n]}(x(t))^2$, $n \in \mathbb{N}$. By using this family of b_2-pseudonorms we get a family of b_2-pseudo metrics as $d_n(x, y) = \|x - y\|_n$. Clearly, $\mathcal{F} = \{d_n : n \in \mathbb{N}\}$ defines b_2-gauge structure on X, which is complete and separating. Define graph $G = (V, E)$ such that $V = X$ and $E = \{(x, y) : x(t) \leq y(t), \forall t \geq 0\}$.

Theorem 3.1. *Let* $X = (C[0, \infty), \mathbb{R})$ *and let the operator*

$$T: X \to X \quad Tx(t) = f(t) + \int_0^t K(t, s, x(s))ds, \quad t \in I = [0, \infty),$$

where $f: I \to \mathbb{R}$ *is a continuous function, and* $K: I \times I \times \mathbb{R} \to \mathbb{R}$ *is a continuous and nondecreasing function. Assume that the following conditions hold:*

(i) for each $t, s \in [0, n]$ *and* $x, y \in X$ *with* $(x, y) \in E(G)$, *there exists a continuous mapping* $p: I \times I \to I$ *such that*
$$|K(t, s, x(s)) - K(t, s, y(s))| \leq \sqrt{p(t, s)d_n(x, y)} \text{ for each } n \in \mathbb{N};$$

(ii) $\sup_{t \geq 0} \int_0^t \sqrt{p(t, s)}ds = a < \frac{1}{\sqrt{2}};$

(iii) there exists $x_0 \in X$ *such that* $(x_0, Tx_0) \in E(G)$.

Then the integral equation (3.1) *has at least one solution.*

Proof. First we show that for each $(x, y) \in E(G)$, the inequality (2.1) holds. For any $(x, y) \in E(G)$ and $t \in [0, n]$ for each $n \geq 1$, we have

$$\begin{aligned}
(Tx(t) - Ty(t))^2 &\leq \left(\int_0^t |K(t, s, x(s)) - K(t, s, y(s))|ds \right)^2 \\
&\leq \left(\int_0^t \sqrt{p(t, s)d_n(x, y)}ds \right)^2 \\
&= \left(\int_0^t \sqrt{p(t, s)}ds \right)^2 d_n(x, y) \\
&= a^2 d_n(x, y).
\end{aligned}$$

Thus, we get $d_n(Tx, Ty) \leq a^2 d_n(x, y)$ for each $(x, y) \in E$ and $n \in \mathbb{N}$ with $a^2 < 1/2$. This implies that (2.1) holds with $a_n = a^2$, and $b_n = c_n = e_n = L_n = 0$ for each $n \in \mathbb{N}$. As K is nondecreasing, for each $(x, y) \in E(G)$, we have $(Tx, Ty) \in E(G)$. Therefore, by Theorem 2.10, there exists a fixed point of the operator T, that is, integral equation (3.1) has at least one solution. $\square$

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