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Boundedness and asymptotic behavior of positive solutions for difference equations of exponential form

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Abstract

In this paper we study the boundedness and the asymptotic behavior of the positive solutions of the difference equation

$$x_{n+1} = a + bx_n e^{-x_{n-1}},$$

where a, b are positive constants, and the initial values x_{-1}, x_0 are positive numbers.

Keywords: Difference equations, boundedness, asymptotic stability.
2010 MSC: 39A23, 39A10.

1. Introduction

In recent years, the behavior of positive solutions of the difference equations of exponential form has been one of the main topics in the theory of difference equations [1, 2, 3, 4, 5, 6]. In particular, In [2] the authors studied the existence of the equilibrium and the boundedness of solutions of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n},$$

where α, β are positive constants and the initial values x_{-1}, x_0 are positive numbers. Later in [3], Fotiades studied the existence, uniqueness and attractivity of prime period two solution of this equation. For similar research on difference equation, we refer the reader to [1, 4, 5, 6] and the references therein.

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In this paper, we investigate the global stability, the boundedness nature of the positive solutions of the difference equation

$$x_{n+1} = a + bx_n e^{-x_{n-1}}, \tag{1.1}$$

where the parameters a, b are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. Equation (1.1) could be also viewed as a model in mathematical biology, in which case we consider a to be the immigration rate and b to be the population growth rate of one species x_n.

2. Preliminaries

Let I be an interval of real numbers, and let $f : I \times I \longrightarrow I$ be a continuous function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), n = 0, 1, \cdots, \tag{2.1}$$

where the initial values $x_{-1}, x_0 \in I$.

The main tool we will use is the following lemma which is a minor modification of Theorem 5.2 in [4].

Lemma 2.1. *Support that f satisfies the following conditions:*

(a) There exist positive number a and b with $a < b$ such that $a \le f(x, y) \le b$ for all $x, y \in [a, b]$.

(b) $f(x, y)$ is increasing in $x \in [a, b]$ for each $y \in [a, b]$,and $f(x, y)$ is decreasing in $y \in [a, b]$ for each $x \in [a, b]$.

(c) (2.1) *has no solutions of prime period two in $[a, b]$.*

Then there exists exactly one equilibrium solution $\bar{x}$ (2.1) *which lies in $[a, b]$. Moreover, every solution of* (2.1) *with initial conditions $x_{-1}, x_0 \in [a, b]$ converges to $\bar{x}$.*

Before we give the main result of this paper, we establish the existence and uniqueness of equilibrium of (1.1).

Proposition 2.2. *Suppose that $b < e^a$. Then* (1.1) *has a unique positive equilibrium $\bar{x}$.*

Here we omit its proof since it is similar as in [2].

3. Boundedness and the asymptotic behavior of Solutions

The following theorem gives a sufficient condition for every positive solution of (1.1) to be bounded.

Theorem 3.1. *Every positive solution of* (1.1) *is bounded if*

$$b < e^a. \tag{3.1}$$

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary solution of Eq.(1.1). Observe that for all $n \ge 2$,

$$x_{n+1} = a + bx_n e^{-x_{n-1}} \le a + bx_n e^{-a}. \tag{3.2}$$

We will now consider the non-homogeneous difference equations

$$y_{n+1} = a + by_n e^{-a}, n = 2, 3, \cdots. \tag{3.3}$$

From (3.3), an arbitrary solution $\{y_n\}_{n=2}^{\infty}$ of (3.3) is given by

$$y_n = r(be^{-a})^n + \frac{a}{1 - be^{-a}}, \tag{3.4}$$

where r depends on the initial values y_2. Thus we see that relations (3.1) and (3.4) imply that y_n is a bounded sequence. Now we will consider the solution y_n of (3.3) such that

$$y_2 = x_2. \tag{3.5}$$

Thus from (3.2) and (3.5) we get

$$x_n \leq y_n, \quad n \geq 2.$$

Therefore it follows that x_n is bounded. □

Next theorem will study the existence of invariant intervals of (1.1).

Theorem 3.2. *Consider* (1.1) *where relations* (3.1) *hold. Then the following statements are true:*
(i) The set

$$[a, \frac{a}{1-be^{-a}}]$$

is an invariant set for (1.1).
(ii) Let ε *be an arbitrary positive number and* x_n *be an arbitrary solution of* (1.1). *We then consider the set*

$$I = [a, \frac{a+\varepsilon}{1-be^{-a}}]. \tag{3.6}$$

Then there exists an n_0 *such that for all* $n \geq n_0$

$$x_n \in I. \tag{3.7}$$

Proof. (i) Let x_n be a solution of (1.1) with initial values x_{-1}, x_0 such that

$$x_{-1}, x_0 \in [a, \frac{a}{1-be^{-a}}]. \tag{3.8}$$

Then from (1.1) and (3.8) we get

$$a \leq x_1 = a + bx_0e^{-x_{-1}} \leq a + b\frac{a}{1-be^{-a}}e^{-a} = \frac{a}{1-be^{-a}}.$$

Then it follows by induction that

$$a \leq x_n \leq \frac{a}{1-be^{-a}}, n = 1, 2, \cdots .$$

This completes the proof of statement (i).
(ii) Let x_n be an arbitrary solution of (1.1). Therefore, from Theorem 3.1 we get

$$0 < l = \liminf_{n\to\infty} x_n, \ L = \limsup_{n\to\infty} x_n < \infty. \tag{3.9}$$

It follows from (1.1) and (3.9) that

$$L \leq a + bLe^{-l}, \ l \geq a + ble^{-L},$$

which imply that

$$a \leq L \leq \frac{a}{1-be^{-a}}.$$

Thus from (1.1), we see that there exists an n_0 such that (3.7) holds true. This completes the proof of the proposition. □

In the following, we will study the asymptotic behavior of the positive solutions of (1.1).

Theorem 3.3. *Consider* (1.1) *where the initial values* x_{-1}, x_0 *are positive constants and* a, b *are positive constants satisfying*

$$b < e^a. \tag{3.10}$$

Then (1.1) *has a unique positive equilibrium* $\bar{x}$ *such that*

$$\bar{x} \in [a, \frac{a}{1-be^{-a}}]. \tag{3.11}$$

Moreover, every positive solution of (1.1) *tends to the unique positive equilibrium* $\bar{x}$ *as* $n \to \infty$.

Proof. Following by Proposition 2.1 and Theorem 3.2, it suffices to show that any positive solution x_n converges to the unique positive equilibrium $\bar{x}$ of (1.1). So, by lemma2.1, we need to show that (1.1) has no positive solutions with prime period two.

Let $x, y \in (a, \infty)$ be such that

$$x = a + bye^{-x} \quad and \quad y = a + bxe^{-y}. \tag{3.12}$$

It suffices to show that $x = y, x > a, y > a$.

From (3.12) we get

$$x = \frac{y-a}{be^{-y}}, \quad y = \frac{x-a}{be^{-x}}$$

and so

$$x = \frac{(x-a-abe^{-x})e^{\frac{x-a+xbe^{-x}}{be^{-x}}}}{b^2}.$$

Set

$$F(x) = \frac{(x-a-abe^{-x})e^{\frac{x-a+xbe^{-x}}{be^{-x}}}}{b^2} - x, \quad x \in (a, \infty). \tag{3.13}$$

Since (1.1) has a unique positive equilibrium $\bar{x}$, the following relation holds

$$\bar{x} = \frac{a}{1-be^{-\bar{x}}}. \tag{3.14}$$

We claim that

$$F'(\bar{x}) > 0. \tag{3.15}$$

From (3.13) and (3.14), we have

$$F'(\bar{x}) = \frac{e^{2\bar{x}}(\bar{x}^4 + (2-2a)\bar{x}^3 + (a^2-2a)\bar{x}^2 + 2a\bar{x} - a^2)}{b^2\bar{x}^2}. \tag{3.16}$$

To prove (3.15) is true, it suffices to prove that for $u \geq a$,

$$\begin{aligned} &g(u) - h(u) > 0, \\ &g(u) = u^4 + 2u^3 + a^2u^2 + 2au, \\ &h(u) = 2au^3 + 2au^2 + a^2. \end{aligned} \tag{3.17}$$

From (3.17) we get

$$\begin{aligned} &g'(u) = 4u^3 + 6u^2 + 2a^2u + 2a, && h'(u) = 6au^2 + 4au, \\ &g''(u) = 12u^2 + 12u + 2a^2, && h''(u) = 12au + 4a, \\ &g'''(u) = 24u + 12, && h'''(u) = 12a, \\ &g^{(4)}(u) = 24, && h^{(4)}(u) = 0. \end{aligned} \tag{3.18}$$

Now from (3.10) and (3.18),we have

$$g^{(4)}(u) - h^{(4)}(u) = 24 > 0.$$

Since $u \geq a > 0$, we have

$$g'''(u) - h'''(u) > g'''(a) - h'''(a) = 12a + 12 > 0. \tag{3.19}$$

Using (3.19) and (3.10) we get

$$g''(u) - h''(u) > g''(a) - h''(a) = 2a^2 + 8a > 0. \tag{3.20}$$

Therefore from (3.20) and (3.10) it follows

$$g'(u) - h'(u) > g'(a) - h'(a) = 2a^2 + 2a > 0. \tag{3.21}$$

Hence from (3.21) and as $u \geq a$, we get

$$g(u) - h(u) > g(a) - h(a) = a^2 > 0,$$

which implies that (3.15) is true.

Since $\bar{x}$ is a solution of the equation $F(x) = 0$ and (3.15) is satisfied there is $\epsilon > 0$ such that

$$F(\bar{x} - \epsilon) < 0 \;\; and \;\; F(\bar{x} + \epsilon) > 0. \tag{3.22}$$

Moreover from (3.13) we have

$$F(a) = -\frac{a + ab}{b} < 0, \quad \lim_{x \to \infty} F(x) = +\infty. \tag{3.23}$$

From (3.22) and (3.23) it follows that there exists exactly one root of $F(x) = 0$ in $(a, +\infty)$. So, (1.1) has no positive solutions of prime period two. The proof is complete. □

Theorem 3.4. *Suppose that $a \geq 2$,*

$$b < \frac{2}{a + \sqrt{a^2 - 4a}} e^{1+a}. \tag{3.24}$$

Then the equilibrium $\bar{x}$ of (1.1) *is locally asymptotically stable.*

Proof. The linearized equation of (1.1) at the equilibrium $\bar{x}$ is

$$y_{n+1} = py_n + qy_{n-1}, \quad n = 0, 1, \cdots, \tag{3.25}$$

where

$$p = be^{-\bar{x}} = \frac{\bar{x} - a}{x} > 0, \quad q = -b\bar{x}e^{-\bar{x}} = a - \bar{x} < 0. \tag{3.26}$$

From [2], a necessary and sufficient condition for absolute value of both eigenvalues of (3.25) less than one is

$$|p| < 1 - q < 2. \tag{3.27}$$

that is, by direct computation,

$$b < \frac{2}{a + \sqrt{a^2 - 4a}} e^{1+a}, \tag{3.28}$$

which imply $\bar{x}$ is locally asymptotically stable. The proof is complete. □

Remark 3.5. If $a \geq 2$, $b > \dfrac{2}{a + \sqrt{a^2 - 4a}} e^{1+a}$, the equilibrium $\bar{x}$ is unstable.

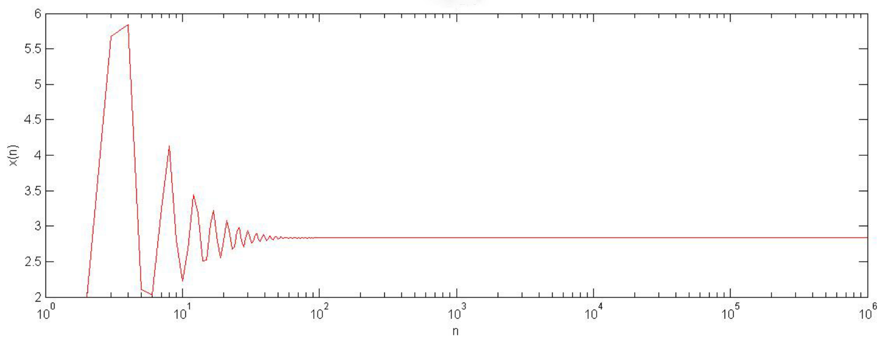
Theorem 3.6. *If* $a \geq 2$,

$$b < \min\{e^a, \frac{2}{a+\sqrt{a^2-4a}}e^{1+a}\},$$

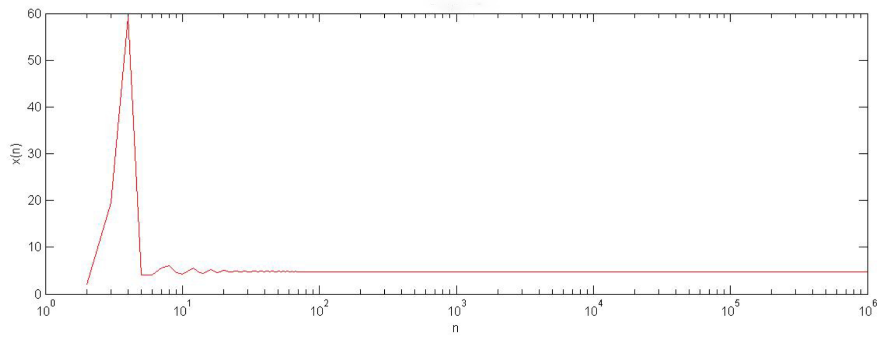
the equilibrium $\bar{x}$ *of* (1.1) *is globally asymptotically stable.*

Proof. It follows immediately from Theorem 3.3 and Theorem 3.4. □

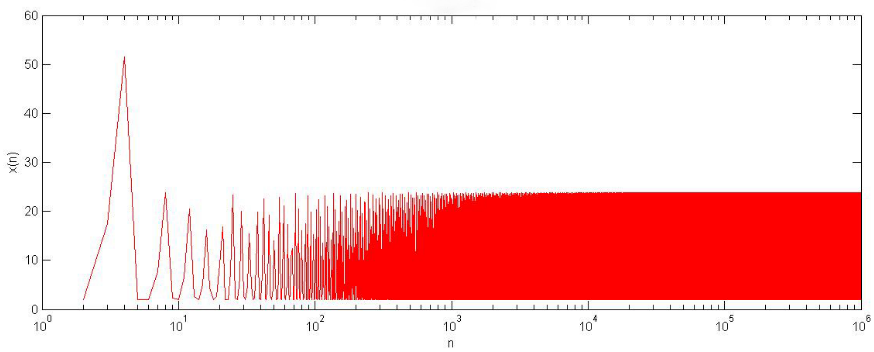
Example 3.7. See Figure1, (a) and (b) show the stability of equilibrium of (1.1) and (c) shows the unstable case whenever (3.24) is not satisfied.



(a) $a = 2, b = 5$.



(b) $a = 4, b = 21$.



(c) $a = 2, b = 21$.

Figure 1

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