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On Suzuki-Wardowski type fixed point theorems

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Abstract

Recently, Piri and Kumam [Fixed Point Theory and Applications 2014, 2014:210] improved concept of F-contraction and proved some Wardowski and Suzuki type fixed point results in metric spaces. The aim of this article is to define generalized $\alpha{-}GF$-contraction and establish Wardowski and Suzuki type fixed point results in metric and ordered metric spaces and derive main results of Piri *et al.* as corollaries. We also deduce certain fixed and periodic point results for orbitally continuous generalized F-contractions and certain fixed point results for integral inequalities are derived. Moreover, we discuss some illustrative examples to highlight the realized improvements.

Keywords: Fixed point, α-GF-contraction, $\alpha-\eta$-continuous function, orbitally continuous function.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

In 1922, Banach established the most famous fundamental fixed point theorem (the so-called the Banach contraction principle [4]) which has played an important role in various fields of applied mathematical analysis. It is known that the Banach contraction principle has been extended in many various directions by several authors (see [2]-[28]). One of the interesting results was given by Suzuki [27] which characterize the completeness of underlying metric spaces. He introduced a weaker notion of contraction and discussed the existence of some new fixed point theorems. Wardowski [29] introduced a new contraction called F-contraction and proved a fixed point result as a generalization of the Banach contraction principle. Abbas *et al.* [1] further generalized the concept of F-contraction and proved certain fixed and common fixed point results. Hussain *et al.* [14] introduced α-η-GF-contractions and obtained fixed point results in metric spaces and partially ordered metric spaces. They also established Suzuki type results for such GF-contractions.

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Recently Piri *et al.* [18] described a large class of functions by using condition $(F3^{'})$ instead of the condition $(F3)$ in the defintion of F-contraction introduced by Wardowski [29] . In this paper, we improve the results of Hussain *et al.* [14] by replacing general conditions $(F2^{'})$ and $(F3^{'})$ instead of the conditions $(F2)$ and $(F3)$. We begin with some basic definitions and results which will be used in the sequel.

In 2012, Samet *et al.* [22] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings defined on complete metric spaces. Afterwards Salimi *et al.* [21] and Hussain *et al.* [10, 11, 12] modified the notions of α-ψ-contractive and α-admissible mappings and established certain fixed point theorems.

Definition 1.1 ([22])**.** Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.2 ([21])**.** Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that if we take $\eta(x, y) = 1$ then this definition reduces to Definition 1.1. Also, if we take, $\alpha(x, y) = 1$ then we say that T is an η-subadmissible mapping.

Definition 1.3 ([12])**.** Let (X, d) be a metric space. Let $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$ be functions. We say T is an α-η-continuous mapping on (X, d), if, for given $x \in X$ and sequence $\{x_n\}$ with

$$x_n \to x \,\text{as}\, n \to \infty,\ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \,\text{for all}\, n \in \mathbb{N} \Longrightarrow\ Tx_n \to Tx.$$

Example 1.4 ([12])**.** Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ be a metric on X. Assume, $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be defined by

$$Tx = \begin{cases} x^5, & \text{if}\, x \in [0, 1] \\ \\ \sin \pi x + 2, & \text{if } (1, \infty) \end{cases}, \quad \alpha(x, y) = \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [0, 1] \\ \\ 0, & \text{otherwise} \end{cases}$$

and $\eta(x, y) = x^2$. Clearly, T is not continuous, but T is α-η-continuous on (X, d).

A mapping $T : X \to X$ is called orbitally continuous at $p \in X$ if $lim_{n\to\infty} T^n x = p$ implies that $lim_{n\to\infty} TT^n x = Tp$. The mapping T is orbitally continuous on X if T is orbitally continuous for all $p \in X$.

Remark 1.5. [12] Let $T : X \to X$ be a self-mapping on an orbitally T-complete metric space X. Define, $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 3, & \text{if } x, y \in O(w) \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \eta(x, y) = 1,$$

where $O(w)$ is an orbit of a point $w \in X$. If, $T : X \to X$ is an orbitally continuous map on (X, d), then T is α-η-continuous on (X, d).

2. Fixed point results for α-η-GF-contractions

Wardowski [29] introduced and studied a new contraction called F-contraction to prove a fixed point result as a generalization of the Banach contraction principle.

Definition 2.1. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying the following conditions:

(F_1) F is strictly increasing;

(F_2) for all sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F_3) there exists $0 < k < 1$ such that $\lim_{a\to 0^+} \alpha^k F(\alpha) = 0$.

Consistent with Wordowski [29], we denote by $\digamma$ the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying conditions F_1, F_2 and F_3.

Definition 2.2 ([29])**.** Let (X, d) be a metric space. A self-mapping T on X is called an $F-$contraction if there exists $\tau > 0$ such that for $x, y \in X$

$$d(Tx, Ty) > 0 \Longrightarrow \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big),$$

where $F \in \digamma$.

Hussain *et al.* [14] generalized the results of Wordowski [29] by introducing Θ_G set of functions $G : R^{+^4} \to R^+$ which satisfy

(G) for all $t_1, t_t, t_3, t_4 \in R^+$ with $t_1t_2t_3t_4 = 0$ there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

They also given some examples of such functions.

Example 2.3 ([14])**.** If $G(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4\} + \tau$ where $L \in \mathbb{R}^+$ and $\tau > 0$, then $G \in \Theta_G$.

Example 2.4 ([14])**.** If $G(t_1, t_2, t_3, t_4) = \tau e^{L \min\{t_1, t_2, t_3, t_4\}}$ where $L \in \mathbb{R}^+$ and $\tau > 0$, then $G \in \Theta_G$.

Example 2.5 ([14])**.** If $G(t_1, t_2, t_3, t_4) = L \ln(\min\{t_1, t_2, t_3, t_4\} + 1) + \tau$ where $L \in \mathbb{R}^+$ and $\tau > 0$, then $G \in \Theta_G$.

On the other hand Secelean [23] proved the following lemma and replaced condition (F_2) by an equivalent but a more simple condition $(F_{2'})$.

Lemma 2.6. *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing map and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following assertions hold:*

(a) if $\lim_{n\to\infty} F(\alpha_n) = -\infty$ then $\lim_{n\to\infty} \alpha_n = 0$;

(b) if $\inf F = -\infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

He replaced the following condition.

$(F_{2'})$ $\inf F = -\infty$

or, also, by

$(F_{2''})$ there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Very recently Piri *et al.* [18] utilized the following condition $(F_{3'})$ instead of (F_3) in Definition (6).

$(F_{3'})$ F is continuous on $(0, \infty)$.

For $p \geq 1$, $F(\alpha) = -\frac{1}{\alpha^p}$ satisfies (F_1) and (F_2) but it does not satisfy (F_3) while it satisfies $(F_{3'})$. We denote by $\Delta_\digamma$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy conditions (F_1), $(F_{2'})$ and $(F_{3'})$.

Definition 2.7. Let (X, d) be a metric space and T be a self-mapping on X. Also suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say T is an $\alpha - \eta - GF$-contraction if for $x, y \in X$ with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have

$$G\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big), \tag{2.1}$$

where $G \in \Theta_G$ and $F \in \Delta_\digamma$.

First, we prove the main result of Hussain *et al.* [14] by replacing conditions (F_2) and (F_3) with $(F_{2'})$ and $(F_{3'})$.

Theorem 2.8. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is α-admissible mapping with respect to η;

(ii) T is α-η-GF-contraction;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) T is α-η-continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For such x_0, we define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$. Now since, T is α-admissible mapping with respect to η then, $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing this process we have,

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n), \tag{2.2}$$

for all $n \in \mathbb{N}$. If there exist $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is fixed point of T and we have nothing to prove. Hence, we assume, $x_n \neq x_{n+1}$ or $d(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$. Since, T is α-η-GF-contraction, so we have

$$G\big(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\big) + F\big(d(Tx_{n-1}, Tx_n)\big) \leq F\big(d(x_{n-1}, x_n)\big),$$

which implies,

$$G\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0\big) + F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big). \tag{2.3}$$

Now since, $d(x_{n-1}, x_n).d(x_n, x_{n+1}).d(x_{n-1}, x_{n+1}).0 = 0$, so from the definition of $G \in \Theta_G$, there exists $\tau > 0$ such that

$$G\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0\big) = \tau. \tag{2.4}$$

From (2.4), we deduce that

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big) - \tau. \tag{2.5}$$

Therefore

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big) - \tau \leq F\big(d(x_{n-2}, x_{n-1})\big) - 2\tau \leq \ldots \leq F(d(x_0, x_1)) - n\tau. \tag{2.6}$$

Since $F \in \Delta_F$, so by taking limit as $n \to \infty$ in (2.6) we have,

$$\lim_{n\to\infty} F\big(d(x_n, x_{n+1})\big) = -\infty \Longleftrightarrow \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

Now, we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. We suppose on the contrary that $\{x_n\}_{n=1}^{\infty}$ is not Cauchy then we assume there exists $\varepsilon > 0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that for $p(n) > q(n) > n$, we have

$$d(x_{p(n)}, x_{q(n)}) \geq \varepsilon. \tag{2.8}$$

Then

$$d(x_{p(n)-1}, x_{q(n)}) < \varepsilon$$

for all $n \in \mathbb{N}$. So, by triangle inequality and (2.8), we have

$$\begin{aligned}\varepsilon \leq d(x_{p(n)}, x_{q(n)}) &\leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \\ &\leq d(x_{p(n)}, x_{p(n)-1}) + \varepsilon.\end{aligned}$$

By taking the limit and using inequality (2.7), we get

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \varepsilon. \tag{2.9}$$

Also, from (2.7) there exists a natural number $n_0 \in \mathbb{N}$ such that

$$d(x_{p(n)}, x_{p(n)+1}) < \frac{\varepsilon}{4} \text{ and } d(x_{q(n)}, x_{q(n)+1}) < \frac{\varepsilon}{4} \tag{2.10}$$

for all $n \geq n_0$. Next, we claim that

$$d(Tx_{p(n)}, Tx_{q(n)}) = d(x_{p(n)+1}, x_{q(n)+1}) > 0 \tag{2.11}$$

for all $n \geq n_0$. We suppose on the contrary that there exists $m \geq n_0$ such that

$$d(x_{p(m)+1}, x_{q(m)+1}) = 0. \tag{2.12}$$

Then from (2.10), (2.11) and (2.12), we have

$$\begin{aligned}\varepsilon \leq d(x_{p(m)}, x_{q(m)}) &\leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)}) \\ &\leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)+1}, x_{q(m)}) \\ &< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},\end{aligned}$$

which is a contradiction , so (2.11) holds. Thus

$$\begin{aligned}G\big(d(x_{p(n)}, Tx_{p(n)}), d(x_{q(n)}, Tx_{q(n)}), d(x_{p(n)}, Tx_{q(n)}), d(x_{q(n)}, Tx_{p(n)})\big)& \\ + F\big(d(Tx_{p(n)}, Tx_{q(n)})\big) &\leq F\big(d(x_{p(n)}, x_{q(n)})\big)\end{aligned}$$

which implies,

$$\begin{aligned}G\big(d(x_{p(n)}, x_{p(n)+1}), d(x_{q(n)}, x_{q(n)+1}), d(x_{p(n)}, x_{q(n)+1}), d(x_{q(n)}, x_{p(n)+1}\big)& \\ + F\big(d(x_{p(n)+1}, x_{q(n)+1})\big) &\leq F\big(d(x_{p(n)}, x_{q(n)})\big).\end{aligned} \tag{2.13}$$

Since G is continuous, so from $(F_{3'})$, (2.9) and (2.13), we get

$$\tau + F(\varepsilon) \leq F(\varepsilon). \tag{2.14}$$

Which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exist $x^* \in X$ such that, $x_n \to x^*$ as $n \to \infty$. Now since, T is α-η-continuous and $\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)$, so

$$x_{n+1} = Tx_n \to Tx^* \text{ as } n \to \infty. \tag{2.15}$$

That is, $x^* = Tx^*$. Thus T has a fixed point. Let $x, y \in Fix(T)$ where $x \neq y$. Then from

$$G\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)$$

we get,

$$\tau + F\big(d(x, y)\big) \leq F\big(d(x, y)\big)$$

which is a contradiction. Hence, $x = y$. Therefore, T has a unique fixed point. □

Combining Theorem 2.8 and Example 2.3 we deduce the following Corollary.

Corollary 2.9. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is α-admissible mapping with respect to η;

(ii) for $x,y \in X$ with $\eta(x,Tx) \leq \alpha(x,y)$ and $d(Tx,Ty) > 0$ we have,

$$\tau + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)$$

where $\tau > 0$ and $F \in \Delta_F$.

(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq \eta(x_0,Tx_0)$;

(iv) T is α-η-continuous function.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x,y) \geq \eta(x,y)$ for all $x,y \in Fix(T)$.

Theorem 2.10. *Let (X,d) be a complete metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is a α-admissible mapping with respect to η;

(ii) T is α-η-GF-contraction;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq \eta(x_0,Tx_0)$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either

$$\eta(Tx_n,T^2x_n) \leq \alpha(Tx_n,x) \text{ or } \eta(T^2x_n,T^3x_n) \leq \alpha(T^2x_n,x) \tag{2.16}$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(x,y) \geq \eta(x,x)$ for all $x,y \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq \eta(x_0,Tx_0)$. As in proof of Theorem 2.8 we can conclude that

$$\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1}) \text{ and } x_n \to x^* \text{ as } n \to \infty$$

where, $x_{n+1} = Tx_n$. So, from (iv), either

$$\eta(Tx_n,T^2x_n) \leq \alpha(Tx_n,x^*) \text{ or } \eta(T^2x_n,T^3x_n) \leq \alpha(T^2x_n,x^*)$$

holds for all $n \in \mathbb{N}$. This implies,

$$\eta(x_{n+1},x_{n+2}) \leq \alpha(x_{n+1},x) \text{ or } \eta(x_{n+2},x_{n+3}) \leq \alpha(x_{n+2},x)$$

holds for all $n \in \mathbb{N}$. Equivalently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k},Tx_{n_k}) = \eta(x_{n_k},x_{n_k+1}) \leq \alpha(x_{n_k},x^*) \tag{2.17}$$

and so from (2.17) we deduce that,

$$G\big(d(x_{n_k},Tx_{n_k}),d(x^*,Tx^*),d(x_{n_k},Tx^*),d(x^*,Tx_{n_k})\big) + F\big(d(Tx_{n_k},Tx^*)\big) \leq F\big(d(x_{n_k},x^*)\big)$$

which implies,

$$F\big(d(Tx_{n_k}, Tx^*)\big) \leq F\big(d(x_{n_k}, x^*)\big). \tag{2.18}$$

From (F_1) we have,

$$d(x_{n_k+1}, Tx^*) < d(x_{n_k}, x^*).$$

By taking limit as $k \to \infty$ in the above inequality we get, $d(x^*, Tx^*) = 0$, *i.e.*, $x^* = Tx^*$. Uniqueness follows similarly as in Theorem 2.8. □

Combining Theorem 2.10 and Example 2.3 we deduce the following Corollary.

Corollary 2.11. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is a α-admissible mapping with respect to η;

(ii) for $x, y \in X$ with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$ we have,

$$\tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)$$

where $\tau > 0$ and $F \in \Delta_F$.

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) if $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

If in Corollary 2.11 we take $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in X$, then we deduce main result of Piri *et al.*[18] as corollary.

Corollary 2.12 (Theorem 2.1 of [18])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. If for $x, y \in X$ with $d(Tx, Ty) > 0$ we have,*

$$\tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)$$

where $\tau > 0$ and $F \in \Delta_F$. Then T has a fixed point.

Example 2.13. Let $X = [0, +\infty)$. We endow X with usual metric. Define, $\alpha, \eta : X \times X \to [0, \infty)$, $T : X \to X$, $G : R^{+^4} \to R^+$ and $F : R^+ \to R$ by,

$$Tx = \begin{cases} \dfrac{1}{4} e^{-\tau} x^2, & \text{if } x \in [0, 1] \\ \\ 5x & \text{if } x \in (1, \infty) \end{cases}$$

$$\alpha(x, y) = \begin{cases} \dfrac{1}{4}, & \text{if } x, y \in [0, 1] \\ \\ \dfrac{1}{12}, & \text{otherwise} \end{cases}$$ and $\eta(x, y) = \frac{1}{6}$, $G(t_1, t_2, t_3, t_4) = \tau$ where $\tau > 0$ and $F(r) = \ln(r^2 + r)$.

Let, $\alpha(x, y) \geq \eta(x, y)$, then $x, y \in [0, 1]$. On the other hand, $Tu \in [0, 1]$ for all $u \in [0, 1]$. Then $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. That is, T is α-admissible mapping with respect to η. If $\{x_n\}$ is a sequence in X

such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$. Then, $Tx_n, T^2x_n, T^3x_n \in [0,1]$ for all $n \in \mathbb{N}$. That is,

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \text{ and } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

hold for all $n \in \mathbb{N}$. Clearly, $\alpha(0, T0) \geq \eta(0, T0)$. Let, $\alpha(x, y) \geq \eta(x, Tx)$. Now, if $x \notin [0,1]$ or $y \notin [0,1]$, then, $\frac{1}{12} \geq \frac{1}{6}$, which is a contradiction, so $x, y \in [0,1]$ and hence we obtain,

$$\begin{aligned} d(Tx,Ty)(d(Tx,Ty)+1) &= (\frac{1}{4}e^{-\tau}|x-y||x+y|)(\frac{1}{4}e^{-\tau}|x-y||x+y|+1) \\ &\leq e^{-\tau}(|x-y|)(e^{-\tau}(|x-y|)+1) \\ &\leq e^{-\tau}(|x-y|)((|x-y|)+1) \\ &= e^{-\tau}d(x,y)(d(x,y)+1) \end{aligned}$$

which implies,

$$\begin{aligned} \tau + F(d(Tx,Ty)) &= \tau + \ln(d(Tx,Ty)^2 + d(Tx,Ty)) \leq \tau + \ln(e^{-\tau}(d(x,y)^2 + d(x,y)) \\ &= \ln(d(x,y)^2 + d(x,y)) = F(d(x,y)). \end{aligned}$$

Hence, T is α-η-GF-contraction mapping. Thus all conditions of Corollary 2.11 (and Theorem 2.10) hold and T has a fixed point. Let $x = 0$, $y = 2$ and $\tau > 0$. Then,

$$\tau + F(d(T0,T2)) \geq F(d(T0,T2)) = \ln(10^2 + 10) > \ln(2^2 + 2) = F(d(0,2)).$$

That is Theorem 2.1 of [18] can not be applied for this example.

Recall that a self-mapping T is said to have the property P if $Fix(T^n) = F(T)$ for every $n \in \mathbb{N}$.

Theorem 2.14. *Let (X,d) be a complete metric space and $T : X \to X$ be an α-continuous self-mapping. Assume that there exists $\tau > 0$ such that*

$$\tau + F\big(d(Tx,T^2x)\big) \leq F\big(d(x,Tx)\big) \tag{2.19}$$

holds for all $x \in X$ with $d(Tx,T^2x) > 0$ where $F \in \Delta_F$. If T is an α-admissible mapping and there exists $x_0 \in X$ such that, $\alpha(x_0, Tx_0) \geq 1$, then T has the property P.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. For such x_0, we define the sequence $\{x_n\}$ by $x_n = T^nx_0 = Tx_{n-1}$. Now since, T is α-admissible mapping, so $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$. By continuing this process, we have

$$\alpha(x_{n-1}, x_n) \geq 1$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is fixed point of T and we have nothing to prove. Hence, we assume, $x_n \neq x_{n+1}$ or $d(Tx_{n-1}, T^2x_{n-1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. From (2.19) we have,

$$\tau + F\big(d(Tx_{n-1}, T^2x_{n-1})\big) \leq F\big(d(x_{n-1}, Tx_{n-1})\big)$$

which implies,

$$\tau + F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big)$$

and so,

$$F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big) - \tau.$$

Therefore,

$$\begin{aligned} F\big(d(x_n, x_{n+1})\big) &\leq F\big(d(x_{n-1}, x_n)\big) - \tau \\ &\leq F\big(d(x_{n-2}, x_{n-1})\big) - 2\tau \\ &\leq \ldots \leq F(d(x_0, x_1)) - n\tau. \end{aligned}$$

By taking limit as $n \to \infty$ in above inequality, we have, $\lim_{n\to\infty} F\big(d(x_n, x_{n+1})\big) = -\infty$, and since, $F \in \Delta_F$ we obtain,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.20}$$

Now, we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. We suppose on the contrary that $\{x_n\}_{n=1}^{\infty}$ is not Cauchy then we assume there exists $\varepsilon > 0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that for $p(n) > q(n) > n$, we have

$$d(x_{p(n)}, Tx_{q(n)-1}) = d(x_{p(n)}, x_{q(n)}) \geq \varepsilon. \tag{2.21}$$

Then

$$d(x_{p(n)-1}, Tx_{q(n)-1}) < \varepsilon \tag{2.22}$$

for all $n \in \mathbb{N}$. So, by triangle inequality and (2.21), we have

$$\begin{aligned}\varepsilon &\leq d(x_{p(n)}, Tx_{q(n)-1}) \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, Tx_{q(n)-1})\\ &\leq d(x_{p(n)}, x_{p(n)-1}) + \varepsilon.\end{aligned}$$

By taking the limit and using inequality (2.20), we get

$$\lim_{n\to\infty} d(x_{p(n)}, Tx_{q(n)-1}) = \varepsilon. \tag{2.23}$$

On the other hand, from (2.20) there exists a natural number $n_0 \in \mathbb{N}$ such that

$$d(x_{p(n)}, x_{p(n)+1}) < \frac{\varepsilon}{4} \text{ and } d(x_{q(n)}, x_{q(n)+1}) < \frac{\varepsilon}{4} \tag{2.24}$$

for all $n \geq n_0$. Next, we claim that

$$d(Tx_{p(n)}, T^2x_{q(n)-1}) = d(x_{p(n)+1}, Tx_{q(n)}) > 0 \tag{2.25}$$

for all $n \geq n_0$. We suppose on the contrary that there exists $m \geq n_0$ such that

$$d(Tx_{p(m)}, T^2x_{q(m)-1}) = d(x_{p(m)+1}, Tx_{q(m)}) = 0. \tag{2.26}$$

Then from (2.24), (2.25) and (2.26), we have

$$\begin{aligned}\varepsilon &\leq d(x_{p(m)}, Tx_{q(m)-1}) \leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, Tx_{q(m)-1})\\ &\leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)+1}, Tx_{q(m)-1})\\ &= d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, Tx_{q(m)}) + d(x_{q(m)+1}, x_{q(m)})\\ &< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},\end{aligned}$$

which is a contradiction. Thus

$$d(Tx_{p(n)}, T^2x_{q(n)-1}) = d(x_{p(n)+1}, Tx_{q(n)}) > 0. \tag{2.27}$$

and

$$\tau + F\big(d(Tx_{p(n)}, T^2x_{q(n)-1})\big) \leq F\big(d(x_{p(n)}, Tx_{q(n)-1})\big) \tag{2.28}$$

established. Which further implies that

$$\tau + F\big(d(x_{p(n)+1}, x_{q(n)+1})\big) \leq F\big(d(x_{p(n)}, x_{q(n)})\big).$$

From $(F_{3'})$, (2.23) and (2.28), we get

$$\tau + F(\varepsilon) \leq F(\varepsilon). \tag{2.29}$$

Which is a contradiction. Thus we proved that $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exists $x^* \in X$ such that, $x_n \to x^*$ as $n \to \infty$. Now since, T is α-continuous and $\alpha(x_{n-1}, x_n) \geq 1$ then, $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. That is, $x^* = Tx^*$. Thus T has a fixed point and $F(T^n) = F(T)$ for $n = 1$. Let $n > 1$. Assume contrarily that $w \in F(T^n)$ and $w \notin F(T)$. Then, $d(w, Tw) > 0$. Now we have,

$$\begin{aligned} F(d(w,Tw)) &= F(d(T(T^{n-1}w)), T^2(T^{n-1}w))) \\ &\leq F(d(T^{n-1}w), T^n w)) - \tau \\ &\leq F(d(T^{n-2}w), T^{n-1}w)) - 2\tau \\ &\leq \cdots \leq d(w,Tw) - n\tau. \end{aligned}$$

By taking limit as $n \to \infty$ in the above inequality we have, $F(d(w,Tw)) = -\infty$. Hence, by $(F_{2'})$ we get, $d(w,Tw) = 0$ which is a contradictions. Therefore, $F(T^n) = F(T)$ for all $n \in \mathbb{N}$. $\square$

Let $(X, d, \preceq)$ be a partially ordered metric space. Recall that $T : X \to X$ is nondecreasing if $\forall x, y \in X,\ x \preceq y \Rightarrow T(x) \preceq T(y)$ and ordered GF-contraction if for $x, y \in X$ with $x \preceq y$ and $d(Tx, Ty) > 0$, we have

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)$$

where $G \in \Theta_G$ and $F \in \Delta_F$. Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [2, 3, 7, 12, 15, 17] and references therein). From Theorems 2.8-2.14, we derive following new results in partially ordered metric spaces.

Theorem 2.15. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the following assertions hold true:*

(i) T is nondecreasing and ordered GF-contraction;

(ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(iii) either for a given $x \in X$ and sequence $\{x_n\}$

$$x_n \to x \text{ as } n \to \infty \text{ and } x_n \preceq x_{n+1} \text{ for all } n \in \mathbb{N} \text{ we have } Tx_n \to Tx$$

or if $\{x_n\}$ is a sequence such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either

$$Tx_n \preceq x, \text{ or } T^2 x_n \preceq x$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point.

Theorem 2.16. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the following assertions hold true:*

(i) T is nondecreasing and satisfies (2.19) for all $x \in X$ with $d(Tx, T^2x) > 0$ where $F \in \Delta_F$ and $\tau > 0$;

(ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(iii) for a given $x \in X$ and sequence $\{x_n\}$

$$x_n \to x \text{ as } n \to \infty \text{ and } x_n \preceq x_{n+1} \text{ for all } n \in \mathbb{N} \text{ we have } Tx_n \to Tx.$$

Then T has a property P.

As an application of our results proved above, we deduce certain Suzuki-Wardowski type fixed point theorems.

Theorem 2.17. *Let (X, d) be a complete metric space and T be a continuous self-mapping on X. If for $x, y \in X$ with $\frac{1}{2}d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$ we have,*

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big) \tag{2.30}$$

where $G \in \Theta_G$ and $F \in \Delta_F$. Then T has a unique fixed point.

Proof. Define, $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = d(x,y) \text{ and } \eta(x,y) = \frac{1}{2}d(x,y)$$

for all $x, y \in X$. Now, since $\frac{1}{2}d(x,y) \leq d(x,y)$ for all $x, y \in X$, so $\eta(x,y) \leq \alpha(x,y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 2.8 hold true. Since T is continuous, so T is α-η-continuous. Let, $\eta(x, Tx) \leq \alpha(x,y)$ with $d(Tx,Ty) > 0$. Equivalently, if $\frac{1}{2}d(x,Tx) \leq d(x,y)$ with $d(Tx,Ty) > 0$, then from (2.30) we have,

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big).$$

That is, T is α-η-GF-contraction mapping. Hence, all conditions of Theorem 2.8 hold and T has a unique fixed point. □

Combining above Corollary and Example 2.3 we deduce Theorem 2.2 of Piri et al. [18] as Corollary.

Corollary 2.18 (Theorem 2.2 [18])**.** *Let (X, d) be a complete metric space and T be a continuous self-mapping on X. If for $x, y \in X$ with $\frac{1}{2}d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$ we have*

$$\tau + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)$$

where $\tau > 0$ and $F \in \Delta_F$. Then T has a unique fixed point.

Corollary 2.19. *Let (X, d) be a complete metric space and T be a continuous self-mapping on X. If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$ we have,*

$$\tau e^{L \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}} + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)$$

where $\tau > 0$, $L \geq 0$ and $F \in \Delta_F$. Then T has a unique fixed point.

Theorem 2.20. *Let (X, d) be a complete metric space and T be a self-mapping on X. Assume that there exists $\tau > 0$ such that*

$$\frac{1}{2(1+\tau)}d(x,Tx) \leq d(x,y) \text{ implies } \tau + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big) \tag{2.31}$$

for $x, y \in X$ with $d(Tx, Ty) > 0$ where $F \in \Delta_F$. Then T has a unique fixed point.

Proof. Define, $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = d(x,y) \text{ and } \eta(x,y) = \frac{1}{2(1+\tau)}d(x,y)$$

for all $x, y \in X$ where $\tau > 0$. Now, since, $\frac{1}{2(1+\tau)}d(x,y) \leq d(x,y)$ for all $x, y \in X$, so $\eta(x,y) \leq \alpha(x,y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 2.10 hold true. Let, $\{x_n\}$ be a sequence with $x_n \to x$ as $n \to \infty$. Assume that $d(Tx_n, T^2x_n) = 0$ for some n. Then $Tx_n = T^2x_n$. That is Tx_n is a fixed point of T and we have nothing to prove. Hence we assume, $Tx_n \neq T^2x_n$ for all $n \in \mathbb{N}$. Since, $\frac{1}{2(1+\tau)}d(Tx_n, T^2x_n) \leq d(Tx_n, T^2x_n)$ for all $n \in \mathbb{N}$. Then from (2.31) we get,

$$F\big(d(T^2x_n, T^3x_n)\big) \leq \tau + F\big(d(T^2x_n, T^3x_n)\big) \leq F\big(d(Tx_n, T^2x_n)\big)$$

and so from (F_1) we get,

$$d(T^2x_n, T^3x_n) \leq d(Tx_n, T^2x_n). \tag{2.32}$$

Assume there exists $n_0 \in \mathbb{N}$ such that,

$$\eta(Tx_{n_0}, T^2x_{n_0}) > \alpha(Tx_{n_0}, x) \text{ and } \eta(T^2x_{n_0}, T^3x_{n_0}) > \alpha(T^2x_{n_0}, x)$$

then,

$$\frac{1}{2(1+\tau)}d(Tx_{n_0}, T^2x_{n_0}) > d(Tx_{n_0}, x) \text{ and } \frac{1}{2(1+\tau)}d(T^2x_{n_0}, T^3x_{n_0}) > d(T^2x_{n_0}, x)$$

so by (2.32) we have,

$$\begin{aligned}
d(Tx_{n_0}, T^2x_{n_0}) &\leq d(Tx_{n_0}, x) + d(T^2x_{n_0}, x) \\
&< \frac{1}{2(1+\tau)}d(Tx_{n_0}, T^2x_{n_0}) + \frac{1}{2(1+\tau)}d(T^2x_{n_0}, T^3x_{n_0}) \\
&\leq \frac{1}{2(1+\tau)}d(Tx_{n_0}, T^2x_{n_0}) + \frac{1}{2(1+\tau)}d(Tx_{n_0}, T^2x_{n_0}) \\
&= \frac{2}{2(1+\tau)}d(Tx_{n_0}, T^2x_{n_0}) \\
&\leq d(Tx_{n_0}, T^2x_{n_0})
\end{aligned}$$

which is a contradiction. Hence, either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$. That is condition (iv) of Theorem 2.10 holds. Let, $\eta(x, Tx) \leq \alpha(x, y)$. So, $\frac{1}{2(1+\tau)}d(x, Tx) \leq d(x, y)$. Then from (2.31) we get, $\tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)$. Hence, all conditions of Theorem 2.10 hold and T has a unique fixed point. □

Example 2.21. Consider the sequence

$S_1 = 1 \times 2$
$S_2 = 1 \times 2 + 3 \times 4$
$S_3 = 1 \times 2 + 3 \times 4 + 5 \times 6$
$S_n = 1 \times 2 + 3 \times 4 + \ldots + (2n-1)(2n) = \frac{n(n+1)(4n-1)}{3}$.

Let $X = \{S_n : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define the mapping $T : X \to X$ by,

$$T(S_1) = S_1, \quad T(S_n) = S_{n-1}, \quad \text{for all } n \geq 2.$$

Let us consider the mapping $F(t) = \frac{-1}{t} + t$, we obtain that T is F-contraction, with $\tau = 12$. To see this, let us consider the following calculations. First observe that

$$\frac{1}{2(1+12)}d(S_n, T(S_n)) < d(S_n, S_m) \ \Leftrightarrow [(1 = n < m) \vee (1 = m < n) \vee (1 < n < m)].$$

For $1 = n < m$, we have

$$|T(S_m) - T(S_1)| = |S_{m-1} - S_1| = 3 \times 4 + 5 \times 6 + \ldots + (2m-3)(2m-2) \tag{2.33}$$

and

$$d(S_m, S_1) = |S_m - S_1| = 3 \times 4 + 5 \times 6 + \ldots + (2m-1)(2m). \tag{2.34}$$

Since $m > 1$, so we have

$$\frac{-1}{3 \times 4 + \ldots + (2m-3)(2m-2)} < \frac{-1}{3 \times 4 + \ldots + (2m-1)(2m)}. \tag{2.35}$$

From (2.35), we have

$$\begin{aligned}
&12 - \frac{1}{3 \times 4 + 5 \times 6 + \ldots + (2m-3)(2m-2)} + 3 \times 4 + +5 \times 6 + \ldots + (2m-3)(2m-2) \\
&< 12 - \frac{1}{3 \times 4 + 5 \times 6 + \ldots + (2m-1)(2m)} + [3 \times 4 + +5 \times 6 + \ldots + (2m-3)(2m-2)] \\
&\leq -\frac{1}{3 \times 4 + 5 \times 6 + \ldots + (2m-1)(2m)} + [3 \times 4 + +5 \times 6 + \ldots + (2m-3)(2m-2)] + (2m-1)(2m).
\end{aligned}$$

Thus from (2.33) and (2.34), we get

$$12 - \frac{1}{|T(S_m), T(S_1)|} + |T(S_m), T(S_1)| < -\frac{1}{|S_m - S_1|} + |S_m - S_1|. \tag{2.36}$$

For every $m, n \in N$ with $m > n > 1$, we have

$$|T(S_m) - T(S_n)| = (2n-1)(2n) + (2n+1)(2n+2) + ... + (2m-3)(2m-2) \tag{2.37}$$

and

$$|S_m - S_n| = (2n+1)(2n+2) + (2n+3)(2n+4) + ... + (2m-1)(2m). \tag{2.38}$$

Since $m > n > 1$, we have

$$(2m-1)(2m) \geq (2n+2)(2n+1) > (2n+2)(2n+2) = 2n(2n+2) + 2(2n+2) \geq 2n(2n+2) + 12.$$

We know that

$$\frac{-1}{(2n-1)(2n) + ... + (2m-3)(2m-2)} < \frac{-1}{(2n+1)(2n+2) + ... + (2m-1)(2m)}. \tag{2.39}$$

From (2.39), we get

$$\begin{aligned}
&12 - \frac{1}{(2n-1)(2n) + (2n+1)(2n+2) + ... + (2m-3)(2m-2)} \\
&\quad + (2n-1)(2n) + (2n+1)(2n+2) + ... + (2m-3)(2m-2) \\
&< 12 - \frac{1}{(2n+1)(2n+2) + (2n+3)(2n+4) + ... + (2m-1)(2m)} \\
&\quad + (2n-1)(2n) + (2n+1)(2n+2) + ... + (2m-3)(2m-2) \\
&< -\frac{1}{(2n+1)(2n+2) + (2n+3)(2n+4) + ... + (2m-1)(2m)} \\
&\quad + (2n-1)(2n) + (2n+1)(2n+2) + ... + (2m-3)(2m-2) + (2m-1)(2m) \\
&= -\frac{1}{(2n+1)(2n+2) + (2n+3)(2n+4) + ... + (2m-1)(2m)} \\
&\quad + (2n-1)(2n) + (2n+1)(2n+2) + ... + (2m-1)(2m)
\end{aligned}$$

So from (2.37) and (2.38), we get

$$12 - \frac{1}{|T(S_m) - T(S_1)|} + |T(S_m) - T(S_1)| < -\frac{1}{|S_m - S_1|} + |S_m - S_1|.$$

Hence all the conditions of Theorem (25) are satisfied and S_1 is a unique fixed point of T.

3. Applications to orbitally continuous mappings

Theorem 3.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) for $x, y \in O(w)$ with $d(Tx, Ty) > 0$ we have,

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)$$

where $G \in \Theta_G$ and $F \in \Delta_F$;

(ii) T is an orbitally continuous function.

Then T has a fixed point. Moreover, T has a unique fixed point when $Fix(T) \subseteq O(w)$.

Proof. Define, $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 3, & \text{if } x, y \in O(w) \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \eta(x,y) = 1$$

where $O(w)$ is an orbit of a point $w \in X$. From Remark 1.5 we know that T is an α-η-continuous mapping. Let, $\alpha(x, y) \geq \eta(x, y)$, then $x, y \in O(w)$. So $Tx, Ty \in O(w)$. That is, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. Therefore, T is an α-admissible mapping with respect to η. Since $w, Tw \in O(w)$, then $\alpha(w, Tw) \geq \eta(w, Tw)$. Let, $\alpha(x, y) \geq \eta(x, Tx)$ and $d(Tx, Ty) > 0$. Then, $x, y \in O(w)$ and $d(Tx, Ty) > 0$. Therefore from (i) we have,

$$G\big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)$$

which implies, T is α-η-GF-contraction mapping. Hence, all conditions of Theorem 2.8 hold true and T has a fixed point. If $Fix(T) \subseteq O(w)$, then, $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in Fix(T)$ and so from Theorem 2.8, T has a unique fixed point. □

Corollary 3.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) for $x, y \in O(w)$ with $d(Tx, Ty) > 0$ we have,

$$\tau + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)$$

where $\tau > 0$ and $F \in \Delta_F$;

(ii) T is orbitally continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $Fix(T) \subseteq O(w)$.

Corollary 3.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) for $x, y \in O(w)$ with $d(Tx, Ty) > 0$ we have,

$$\tau e^{L \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}} + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)$$

where $\tau > 0$, $L \geq 0$ and $F \in \Delta_F$;

(ii) T is orbitally continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $Fix(T) \subseteq O(w)$.

In our next result, we prove improved version of Theorem 4 of [1].

Theorem 3.4. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) for $x \in X$ with $d(Tx,T^2x) > 0$ we have,

$$\tau + F\big(d(Tx,T^2x)\big) \leq F\big(d(x,Tx)\big)$$

where $\tau > 0$ and $F \in \Delta_F$;

(ii) T is an orbitally continuous function.

Then T has the property P.

Proof. Define, $\alpha : X \times X \to [0,+\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \in O(w) \\ 0, & \text{otherwise} \end{cases}$$

where $w \in X$. Let, $\alpha(x,y) \geq 1$, then $x,y \in O(w)$. So $Tx,Ty \in O(w)$. That is, $\alpha(Tx,Ty) \geq 1$. Therefore, T is α-admissible mapping. Since $w,Tw \in O(w)$, so $\alpha(w,Tw) \geq 1$. By Remark 1.5 we conclude that T is α-continuous mapping. If, $x \in X$ with $d(Tx,T^2x) > 0$, then, from (i) we have,

$$\tau + F\big(d(Tx,T^2x)\big) \leq F\big(d(x,Tx)\big).$$

Thus all conditions of Theorem 2.14 hold true and T has the property P. □

We can easily deduce following results involving integral inequalities.

Theorem 3.5. *Let (X,d) be a complete metric space and T be a continuous self-mapping on X. If for $x,y \in X$ with*

$$\int_0^{d(x,Tx)} \rho(t)dt \leq \int_0^{d(x,y)} \rho(t)dt \text{ and } \int_0^{d(Tx,Ty)} \rho(t)dt > 0$$

we have,

$$G\Big(\int_0^{d(x,Tx)} \rho(t)dt, \int_0^{d(y,Ty)} \rho(t)dt, \int_0^{d(x,Ty)} \rho(t)dt, \int_0^{d(y,Tx)} \rho(t)dt\Big) + F\big(\int_0^{d(Tx,Ty)} \rho(t)dt\big) \leq F\big(\int_0^{d(x,y)} \rho(t)dt\big)$$

where $G \in \Theta_G$, $F \in \Delta_F$ and $\rho : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)dt > 0$ for $\varepsilon > 0$. Then T has a unique fixed point.

Theorem 3.6. *Let (X,d) be a complete metric space and T be a self-mapping on X. Assume that there exists $\tau > 0$ such that*

$$\frac{1}{2(1+\tau)} \int_0^{d(x,Tx)} \rho(t)dt \leq \int_0^{d(x,y)} \rho(t)dt \Rightarrow$$

$$\tau + F\big(\int_0^{d(Tx,Ty)} \rho(t)dt\big) \leq F\big(\int_0^{d(x,y)} \rho(t)dt\big)$$

for $x,y \in X$ with $\int_0^{d(Tx,Ty)} \rho(t)dt > 0$ where $F \in \Delta_F$ and $\rho : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)dt > 0$ for $\varepsilon > 0$. Then T has a unique fixed point.

Theorem 3.7. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) for $x, y \in O(w)$ with $\int_0^{d(Tx,Ty)} \rho(t)dt > 0$ we have,

$$G\Big(\int_0^{d(x,Tx)} \rho(t)dt, \int_0^{d(y,Ty)} \rho(t)dt, \int_0^{d(x,Ty)} \rho(t)dt, \int_0^{d(y,Tx)} \rho(t)dt \Big) + F\Big(\int_0^{d(Tx,Ty)} \rho(t)dt \Big) \le F\Big(\int_0^{d(x,y)} \rho(t)dt \Big)$$

where $G \in \Theta_G$, $F \in \Delta_F$ and $\rho : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)dt > 0$ for $\varepsilon > 0$.

(ii) T is an orbitally continuous function;

Then T has a fixed point. Moreover, T has a unique fixed point when $Fix(T) \subseteq O(w)$.

Theorem 3.8. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) for $x \in X$ with $\int_0^{d(Tx,T^2x)} \rho(t)dt > 0$ we have,

$$\tau + F\Big(\int_0^{d(Tx,T^2x)} \rho(t)dt \Big) \le F\Big(\int_0^{d(x,Tx)} \rho(t)dt \Big)$$

where $\tau > 0$ and $F \in \Delta_F$ and $\rho : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)dt > 0$ for $\varepsilon > 0$.

(ii) T is an orbitally continuous function.

Then T has the property P.

References

[1] M. Abbas, B. Ali, S. Romaguera, Fixed and periodic points of generalized contractions in metric spaces, Fixed Point Theory Appl., **2013** (2013), 11 pages. 1, 3

[2] R. P. Agarwal, N. Hussain, M. A. Taoudi, *Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations*, Abstr. Appl. Anal., **2012** (2012), 15 pages. 1, 2

[3] Q. H. Ansari, A. Idzik, J. C. Yao, Coincidence and fixed point theorems with applications, Topol. Methods Nonlinear Anal., **15** (2000), 191–202. 2

[4] S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, Fundam. Math., **3** (1922), 133–181. 1

[5] D. Dorić, Z. Kadelburg, S. Radenović, *Edelstein-Suzuki-type fixed point results in metric and abstract metric spaces*, Nonlinear Anal., **75** (2012), 1927–1932.

[6] S. Fathollahi, N. Hussain, L. A. Khan, *Fixed point results for modified weak and rational $\alpha - \psi$-contractions in ordered 2-metric spaces*, Fixed Point Theory Appl., **2014** (2014), 25 pages.

[7] N. Hussain, S. Al-Mezel, P. Salimi, Fixed points for ψ-graphic contractions with application to integral equations, Abstr. Appl Anal., **2013** (2013), 11 pages. 2

[8] N. Hussain, D. Dorić, Z. Kadelburg, S. Radenović, Suzuki-type fixed point results in metric type spaces, Fixed Point Theory Appl., **2012** (2012), 12 pages.

[9] N. Hussain, E. Karapinar, P. Salimi, F. Akbar, α-admissible mappings and related fixed point theorems, J. Inequal. Appl., **2013** (2013), 11 pages.

[10] N. Hussain, M. A. Kutbi, S. Khaleghizadeh, P. Salimi, Discussions on recent results for α-ψ-contractive mappings, Abst. Appl. Anal., **2014** (2014), 13 pages. 1

[11] N. Hussain, M. A. Kutbi, P. Salimi, Best proximity point results for modified α-ψ-proximal rational contractions, Abstr. Appl. Anal., **2013** (2013), 14 pages. 1

[12] N. Hussain, M. A. Kutbi, P. Salimi, Fixed point theory in α-complete metric spaces with applications, Abstr. Appl. Anal., **2014** (2014), 11 pages. 1, 1.3, 1.4, 1.5, 2

[13] N. Hussain, A. Latif, P. Salimi, Best proximity point results for modified Suzuki α-ψ-proximal contractions, Fixed Point Theory Appl., **2014** (2014), 16 pages.

[14] N. Hussain, P. Salimi, Suzuki-Wardowski type fixed point theorems for α-GF-contractions, Taiwanese J. Math., **18** (2014), 1879–1895. 1, 2, 2.3, 2.4, 2.5, 2

[15] N. Hussain, M. A. Taoudi, *Krasnosel'skii-type fixed point theorems with applications to Volterra integral equations*, Fixed Point Theory Appli., **2013** (2013), 16 pages. 2

[16] S. K. Malhotra, S. Radenović, S. Shukla, *Some fixed point results without monotone property in partially ordered metric spaces*, J. Egyptian Math. Soc., **22** (2014), 83–89.

[17] J. J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223–229. 2

[18] H. Piri, P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, Fixed Point Theory Appl., **2014** (2014), 16 pages. 1, 2, 2, 2.12, 2.13, 2, 2.18

[19] S. Radenović, Z. Kadelburg, D. Jandrlić, A. Jandrlić, Some results on weakly contractive maps, Bull. Iranian Math. Soc., **38** (2012), 625–645.

[20] V. C. Rajić, S. Radenović, W. Shatanawi, N. Tahat, *Common fixed point results for weakly isotone increasing mappings in partially ordered partial metric spaces*, The Matematiche, **68** (2013), 191–204.

[21] P. Salimi, A. Latif, N. Hussain, Modified α-ψ-contractive mappings with applications, Fixed Point Theory Appl., **2013** (2013), 19 pages. 1, 1.2

[22] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal., **75** (2012), 2154–2165. 1, 1.1

[23] N. A. Secelean, Iterated function systems consisting of F-contractions, Fixed Point Theory Appl., **2013** (2013), 13 pages. 2

[24] M. Sgroi, C. Vetro, Multi-valued F-contractions and the solution of certain functional and integral equations, Filomat, **27** (2013), 1259–1268.

[25] S. Shukla, S. Radenović, *Some common fixed point theorems for F-contraction type mappings in 0-complete partial metric spaces*, J. Math., **2013** (2013), 7 pages.

[26] T. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Anal., **71** (2009), 5313–5317.

[27] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc., **136** (2008), 1861–1869. 1

[28] W. Takahashi, L. J. Lin, S.Y. Wang, *Fixed point theorems for contractively generalized hybrid mappings in complete metric spaces*, J. Nonlinear Convex Anal., **13** (2012), 195–206. 1

[29] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl., **2012** (2012), 6 pages. 1, 2, 2, 2.2, 2

[30] D. Wardowski, N. Van Dung, Fixed points of F-weak contractions on complete metric space, Demonstr. Mathe., **47** (2014), 10 pages.