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Common fixed point theorems for mappings satisfying $(E.A)$ $-$property in $b-$metric spaces

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Abstract

In this paper, we give a fixed point theorem and some results for mappings satisfying $(E.A)$ $-$property in $b-$metric spaces.

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2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

The concept of $b-$metric space was introduced by Bakhtin [7] in 1989, who used it to prove a generalization of the Banach contraction principle in spaces endowed with such kind of metrics. Since then, this notion has been used by many authors to obtain various fixed point theorems. Aydi et al. in [4] proved common fixed point results for single valued and multi-valued mappings satisfying a weak $\varphi-$contraction in $b-$metric spaces. Roshan et al. in [25] used the notion of almost generalized contractive mappings in ordered complete $b-$metric spaces and established some fixed and common fixed point results. Păcurar [21] proved the existence and uniqueness of fixed points of $\varphi-$contractions on b-metric spaces. Hussain and Shah in [16] introduced the notion of a cone $b-$metric space, generalizing both notions of $b-$metric spaces and cone metric spaces. Fixed point theorems of contractive mappings in cone $b-$metric spaces without the assumption of the normality of a corresponding cone are proved by Huang and Xu in [17]. In [14], Hussain introduced partially ordered $b-$metric space. After that, several interesting results about the existence of a fixed point for single-valued and multi-valued operators in $b-$metric spaces have been obtained ([2],[9]–[13], [15], [19], [23], [22], [24]).

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Definition 1.1 ([7])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:
(**b1**) $d(x, y) = 0$ if and only if $x = y$,
(**b2**) $d(x, y) = d(y, x)$,
(**b3**) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
In this case, the pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, every metric is a b-metric with $s = 1$.

Example 1.2. Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a $b-$metric with $s = 2^{p-1}$.

However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space. For example, if $X = R$, where R is the set of real numbers, and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a $b-$metric on R with $s = 2$, but it is not a metric on R.

Definition 1.3 ([9])**.** Let $\{x_n\}$ be a sequence in a b-metric space (X, d).

a. $\{x_n\}$ is called $b-$convergent if and only if there is $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.
b. $\{x_n\}$ is a $b-$Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

A b-metric space is said to be complete if and only if each $b-$Cauchy sequence in this space is $b-$convergent.

Proposition 1.4 ([9])**.** *In a $b-$metric space (X, d), the following assertions hold:*

p1. A $b-$convergent sequence has a unique limit.
p2. Each $b-$convergent sequence is $b-$Cauchy.
p3. In general, a $b-$metric is not continuous.

Definition 1.5 ([10])**.** Let (X, d) be a $b-$metric space. A subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y which $b-$converges to an element x, we have $x \in Y$.

On the other hand, $(E.A)-$property was introduced in 2002 by Aamri and Moutaawakil [1]. Later, some authors employed this concept to obtain some new fixed point results ([3, 5, 6, 8, 20]). In this paper, we prove a common fixed point theorem for two pairs of mappings which satisfy the $b-(E.A)$ property in $b-$metric spaces.

We will give the definition of $b-(E.A)$ property in $b-$metric spaces.

Definition 1.6. Let (X, d) be a $b-$metric space and f and g be selfmappings on X.

(i) f and g are said to compatible if whenever a sequence $\{x_n\}$ in X is such that $\{fx_n\}$ and $\{gx_n\}$ are $b-$convergent to some $t \in X$, then $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.
(ii) f and g are said to noncompatible if there exists at least one sequence $\{x_n\}$ in X is such that $\{fx_n\}$ and $\{gx_n\}$ are $b-$convergent to some $t \in X$, but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either nonzero or does not exist.
(iii) f and g are said to satisfy the $b-(E.A)$ property if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$.

Remark 1.7. Noncompatibility implies property $(E.A)$.

Example 1.8. $X = [0,2]$ and define $d : X \times X \to [0,\infty)$ as follows

$$d(x,y) = (x-y)^2.$$

Let $f, g : X \to X$ be defined by

$$f(x) = \begin{cases} 1, & x \in [0,1] \\ \frac{x+1}{8}, & x \in (1,2] \end{cases}, \quad g(x) = \begin{cases} \frac{3-x}{2}, & x \in [0,1] \\ \frac{x}{4}, & x \in (1,2]. \end{cases}$$

For a sequence $\{x_n\}$ in X such that $x_n = 1 + \frac{1}{n+2}$, $n = 0,1,2,...$, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \frac{1}{4}$. So f and g satisfy the $b-(E.A)$ property. But $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$. Thus f and g are noncompatible.

Definition 1.9 ([18])**.** f and g be given self-mappings on a set X. The pair (f,g) is said to be weakly compatible if f and g commute at their coincidence points (i.e. $fgx = gfx$ whenever $fx = gx$).

2. Main results

During this paper, we assume the control functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ are continuous, nondecreasing functions with $\psi(t) = 0$ if and only if $t = 0$.

Theorem 2.1. *Let (X,d) be a $b-$metric space and $f,g,S,T : X \to X$ be mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ such that for all $x, y \in X$,*

$$\psi(s^2 d(fx,gy)) \le \psi(M_s(x,y)) - \varphi(M_s(x,y)), \tag{2.1}$$

where,

$$M_s(x,y) = \max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(fx,Ty)+d(Sx,gy)}{2s}\right\},$$

Suppose that one of the pairs (f,S) and (g,T) satisfy the $(E.A)$-property and that one of the subspaces $f(X), g(X), S(X)$ and $T(X)$ is closed in X. Then the pairs (f,S) and (g,T) have a point of coincidence in X. Moreover, if the pairs (f,S) and (g,T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. If the pairs (f,S) satisfy the $(E.A)$-property, then there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = q,$$

for some $q \in X$. As $f(X) \subseteq T(X)$ there exists a sequence $\{y_n\}$ in X such that $fx_n = Ty_n$. Hence $\lim_{n\to\infty} Ty_n = q$. Let us show that $\lim_{n\to\infty} gy_n = q$. By (2.1),

$$\psi(d(fx_n,gy_n)) \le \psi\left(s^2 d(fx_n,gy_n)\right) \le \psi(M_s(x_n,y_n)) - \varphi(M_s(x_n,y_n)) \le \psi(M_s(x_n,y_n)) \tag{2.2}$$

where

$$\begin{aligned} M_s(x_n,y_n) &= \max\left\{d(Sx_n,Ty_n), d(fx_n,Sx_n), d(Ty_n,gy_n), \frac{d(Sx_n,gy_n)+d(fx_n,Ty_n)}{2s}\right\} \\ &= \max\left\{d(Sx_n,fx_n), d(fx_n,Sx_n), d(fx_n,gy_n), \frac{d(Sx_n,gy_n)+d(fx_n,fx_n)}{2s}\right\}, \\ &\le \max\left\{d(Sx_n,fx_n), d(fx_n,gy_n), \frac{s[d(Sx_n,fx_n), d(fx_n,gy_n)]}{2s}\right\}. \end{aligned}$$

In (2.2), taking the limit ,

$$\psi\left(s^2 \lim_{n\to\infty} d(q,gy_n)\right) \le \psi\left(\lim_{n\to\infty} d(q,gy_n)\right).$$

Using the definition of ψ,

$$s^2 \lim_{n\to\infty} d(q, gy_n) \leq \lim_{n\to\infty} d(q, gy_n).$$

Thus $\lim_{n\to\infty} d(q, gy_n) = 0$. Hence $\lim_{n\to\infty} gy_n = q$.

If $T(X)$ is closed subspace of X, then there exists a $r \in X$, such that $Tr = q$. By (2.1),

$$\psi\left(s^2 d(fx_n, gr)\right) \leq \psi(M_s(x_n, r)) - \varphi(M_s(x_n, r)) \tag{2.3}$$

where

$$\begin{aligned} M_s(x_n, r) &= \max\left\{d(Sx_n, Tr), d(fx_n, Sx_n), d(Tr, gr), \frac{d(fx_n, Tr) + d(Sx_n, gr)}{2s}\right\} \\ &= \max\left\{d(Sx_n, q), d(fx_n, Sx_n), d(q, gr), \frac{d(fx_n, q) + d(Sx_n, gr)}{2s}\right\}. \end{aligned}$$

Letting $n \to \infty$,

$$\begin{aligned} \lim_{n\to\infty} M_s(x_n, r) &= \max\left\{d(q, q), d(q, q), d(q, gr), \frac{d(q, q) + d(q, gr)}{2s}\right\} \\ &= d(q, gr). \end{aligned}$$

Now,by (2.3) and the definitions of ψ and φ, as $n \to \infty$,

$$\psi(d(q, gr) \leq \psi(s^2 d(q, gr)) \leq \psi(d(q, gr)) - \varphi(d(q, gr))$$

which implies $\varphi(d(q, gr)) \leq 0$ giving $gr = q$. Thus r is a coincidence point of the pair (g, T). As $g(X) \subseteq S(X)$, there exists a point $z \in X$ such that $q = Sz$. We claim that $Sz = fz$. By (2.1), we have

$$\psi(s^2 d(fz, gr)) \leq \psi(M_s(z, r)) - \varphi(M_s(z, r)) \tag{2.4}$$

where

$$\begin{aligned} M_s(z, r) &= \max\left\{d(Sz, Tr), d(fz, Sz), d(Tr, gr), \frac{d(fz, Tr) + d(Sz, gr)}{2s}\right\} \\ &= \max\left\{d(q, q), d(fz, q), d(q, q), \frac{d(fz, q) + d(q, q)}{2s}\right\} \\ &\leq \max\left\{d(fz, q), \frac{d(fz, q)}{2s}\right\} \\ &= d(fz, q). \end{aligned}$$

Thus from (2.4),

$$\psi(s^2 d(fz, gr)) = \psi(s^2 d(fz, q)) \leq \psi(d(fz, q)) - \varphi(d(fz, q))$$

implies that $\varphi(d(fz, q)) \leq 0$. Therefore $Sz = fz = q$. Hence z is a coincidence point of the pair (f, S). Thus $fz = Sz = gr = Tr = q$. By the weak compatibility of the pairs (f, S) and (g, T), $fq = Sq$ and $gq = Tq$.

We will show that q is a common fixed point of f, g, S and T. From (2.1) ,

$$\psi(d(fq, q)) \leq \psi(s^2 d(fq, q)) = \psi(s^2 d(fq, gr)) \leq \psi(M_s(q, r)) - \varphi(M_s(q, r)) \tag{2.5}$$

where,

$$\begin{aligned} M_s(q, r) &= \max\left\{d(Sq, Tr), d(fq, Sq), d_\lambda(Tr, gr), \frac{d(fq, Tr) + d(Sq, gr)}{2s}\right\} \\ &= \max\left\{d(fq, q), d(fq, fq), d(q, q), \frac{d(fq, q) + d(fq, q)}{2s}\right\} \\ &= d(fq, q). \end{aligned}$$

By (2.5)

$$\psi\left(d\left(fq,q\right)\right) \leq \psi(d\left(fq,q\right)) - \varphi\left(d\left(fq,q\right)\right).$$

So $fq = Sq = q$. Similarly, it can be shown $gq = Tq = q$.

To prove the uniqueness of fixed point , suppose p is another fixed point of f, g, S and T. By (2.1),

$$\psi\left(d\left(q,p\right)\right) = \psi(d\left(fq,gp\right)) \leq \psi\left(s^2 d\left(fq,gp\right)\right) \leq \psi\left(M_s\left(q,p\right)\right) - \varphi\left(M_s\left(q,p\right)\right)$$

and

$$\begin{aligned} M_s\left(q,p\right) &= \max\left\{d\left(Sq,Tp\right), d\left(fq,Sq\right), d\left(Tp,gp\right), \frac{d\left(fq,Tp\right)+d\left(Sq,gp\right)}{2s}\right\} \\ &= \max\left\{d\left(q,p\right), d\left(q,q\right), d\left(p,p\right), \frac{d\left(q,p\right)+d\left(q,p\right)}{2s}\right\} \\ &= d\left(q,p\right). \end{aligned}$$

Hence we have

$$\psi\left(d\left(q,p\right)\right) \leq \psi\left(d\left(q,p\right)\right) - \varphi\left(d\left(q,p\right)\right),$$

which implies that $\varphi\left(d\left(q,p\right)\right) = 0$. So $q = p$. □

Corollary 2.2. *Let (X,d) be a $b-$metric space and $f, T : X \to X$ be mappings with $f\left(X\right) \subseteq T\left(X\right)$ such that for all $x, y \in X$,*

$$\psi(s^2 d\left(fx,fy\right)) \leq \psi(M_s\left(x,y\right)) - \varphi(M_s\left(x,y\right)),$$

where

$$M_s\left(x,y\right) = \max\left\{d\left(Tx,Ty\right), d\left(fx,Tx\right), d\left(fy,Ty\right), \frac{d\left(fx,Ty\right)+d\left(Tx,fy\right)}{2s}\right\},$$

Suppose that the pair (f,T) satisfies the $(E.A)$-property and $T\left(X\right)$ is closed in X. Then the pair (f,T) has a common point of coincidence in X. Moreover, if the pair (f,T) is weakly compatible, then f and T have a unique common fixed point.

Corollary 2.3. *Let (X,d) be a $b-$metric space and $f, g, S, T : X \to X$ be mappings with $f\left(X\right) \subseteq T\left(X\right)$ and $g\left(X\right) \subseteq S\left(X\right)$ such that for all $x, y \in X$*

$$d\left(fx,gy\right) \leq M_s\left(x,y\right) - \varphi(M_s\left(x,y\right)),$$

where,

$$M_s\left(x,y\right) = \max\left\{d\left(Sx,Ty\right), d\left(fx,Sx\right), d\left(gy,Ty\right), \frac{d\left(fx,Ty\right)+d\left(Sx,gy\right)}{2s}\right\},$$

Suppose that one of the pairs (f,S) and (g,T) satisfy the $(E.A)$-property and that one of the subspaces $f\left(X\right), g\left(X\right), S\left(X\right)$ and $T\left(X\right)$ is closed in X. Then the pairs (f,S) and (g,T) have a point of coincidence in X. Moreover, if the pairs (f,S) and (g,T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Example 2.4. Let $X = \{0,1,2,3,4\}$ and define $d : X \times X \to [0,\infty)$ as follows

$$d\left(x,y\right) = \begin{cases} 0, & x = y \\ \left(x+y\right)^2, & x \neq y. \end{cases}$$

Then (X,d) is a $b-$metric space with constant $s = \frac{49}{25}$. Let $f, g, S, T : X \to X$ be defined by

$$f\left(x\right) = \left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 & 0 \end{array}\right), \quad S\left(x\right) = \left(\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 1 & 1 \end{array}\right), \; g\left(x\right) = 0 \text{ and } T\left(x\right) = x.$$

Clearly, $g(X)$ is closed and $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$. The sequence $\{x_n\}$, $x_n = 1$, is in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = 2$. So the pair (f, S) satisfies the $(E.A)$ −property But the pair (f, S) is noncompatible for $\lim_{n\to\infty} d(fSx_n, Sfx_n) \neq 0$. The control functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ are defined by $\psi(t) = \sqrt{t}$ and $\varphi(t) = \frac{t}{10^3}$. To check the contractive condition (2.1), for all $x, y \in X$,

if $x \neq 1$, then (2.1) is satisfied.

İf $x = 1$, then

$$\begin{aligned}\psi\left(s^2 d(fx, gy)\right) &= \frac{49}{25}2 \leq \frac{3984}{10^3} d(fx, Sx) \\ &\leq \frac{3984}{10^3} M_s(x, y) = \psi(M_s(x, y)) - \varphi(M_s(x, y)).\end{aligned}$$

Then (2.1) is satisfied for all $x, y \in X$. The pairs (f, S) and (g, T) are weakly compatible. Hence, all of the conditions of Theorem 2.1 are satisfied. Moreover 0 is the unique common fixed point of f, g, S and T.

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