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Fixed point results for probabilistic φ-contractions in generalized probabilistic metric spaces

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Abstract

In this paper, we present some new fixed point theorems for probabilistic contractions with a gauge function φ in generalized probabilistic metric spaces proposed by Zhou *et al.* Our theorems not only are generalizations of the corresponding results of Ćirić [L. Ćirić, Nonlinear Anal., 72 (2010), 2009–2018] and Jachymski [J. Jachymski, Nonlinear Anal., 73 (2010), 2199–2203], but also improve and extend the recent results given by Zhou *et al.* [C. Zhou, S. Wang, L. Ćirić, S. M. Alsulami, Fixed Point Theory Appl. 2014 (2014), 15 pages].

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1. Introduction and Preliminaries

Suppose that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, and let $\mathbb{Z}^+$ be the set of all positive integers. A function $F : \overline{\mathbb{R}} \to [0, 1]$ is called a distribution function if it is nondecreasing and left-continuous with $F(-\infty) = 0$ and $F(+\infty) = 1$. The set of all probability distribution functions is denoted by $\mathcal{D}_\infty$. Suppose that $\mathcal{D} = \{F \in \mathcal{D}_\infty : \inf_{t\in\mathbb{R}} F(t) = 0, \sup_{t\in\mathbb{R}} F(t) = 1\}$, $\mathcal{D}^+_\infty = \{F \in \mathcal{D}_\infty : F(0) = 0\}$, and $\mathcal{D}^+ = \mathcal{D} \cap \mathcal{D}^+_\infty$.

Definition 1.1 ([15])**.** A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if T satisfies the following conditions:

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(1) T is commutative and associative, i.e., $T(a,b) = T(b,a)$ and $T(a,T(b,c)) = T(T(a,b),c)$, for all $a,b,c \in [0,1]$;
(2) T is continuous;
(3) $T(a,1) = a$ for all $a \in [0,1]$;
(4) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

From the definition of T it follows that $T(a,b) \leq \min\{a,b\}$ for all $a,b \in [0,1]$.
Two typical examples of continuous t-norms are $T_M(a,b) = \min\{a,b\}$ and $T_p(a,b) = ab$ for all $a,b \in [0,1]$.

Definition 1.2 ([6])**.** A t-norm T is said to be of H-type (Hadžić type) if the family of functions $\{T^n(t)\}_{n=1}^{+\infty}$ is equicontinuous at $t = 1$, that is, for any $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that

$$t > 1 - \delta \Rightarrow T^n(t) > 1 - \varepsilon, \ \forall\ n \geq 1,$$

where $T^n : [0,1] \to [0,1]$ is defined as follows:

$$T^1(t) = T(t,t), T^2(t) = T(t,T^1(t)), \cdots, T^n(t) = T(t,T^{n-1}(t)), \cdots .$$

Obviously, $T^n(t) \leq t$ for all $n \in \mathbb{Z}^+$ and $t \in [0,1]$.

T_M is a trivial example of t-norm of Hadžić-type [7].

Definition 1.3. If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\varphi(0) = 0$, then φ is called a gauge function. If $t \in \mathbb{R}^+$, then $\varphi^n(t)$ denotes the nth iteration of $\varphi(t)$ and $\varphi^{-1}(\{0\}) = \{t \in \mathbb{R}^+ : \varphi(t) = 0\}$.

In 1942, Menger [11] introduced the concept of Menger probabilistic metric space (abbreviated, Menger PM-space) as follows.

Definition 1.4 ([11])**.** A Menger PM-space is a triple (X,F,T), where X is a nonempty set, T is a continuous t-norm and F is a mapping from $X \times X$ to $\mathcal{D}_\infty^+$($F_{x,y}$ denotes the value of F at the pair (x,y)) satisfying the following conditions:

(PM-1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;

(PM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x,y \in X$ and $t > 0$;

(PM-3) $F_{x,z}(t+s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x,y,z \in X$ and all $s,t > 0$.

It is well known that Menger PM-spaces are a very important generalization of metric spaces, and are considered to be of interest in the investigation of physical quantities and physiological thresholds. They are also of fundamental importance in probabilistic functional analysis [14]. Many results regarding generalizations of the notion of Menger PM-space or the existence and uniqueness of fixed points under various types of conditions in Menger PM-spaces have been obtained (see [1], [2], [3], [5], [9], [10], [14]).

In 2006, Mustafa and Sims [12] established the following interesting result.

Definition 1.5 ([12])**.** Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:
(G1) $G(x,y,z) = 0$ if and only if $x = y = z$,
(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$,
(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ for all $x,y,z \in X$ (symmetry in all three variables),
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).
Then the function G is called a generalized metric or, more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.

In 2014, Zhou et al. [16] presented a probabilistic version of G-metric spaces, called Menger probabilistic G-metric spaces (briefly, Menger PGM-spaces). The authors discussed the topological properties of these spaces and proved two important fixed point theorems under a probabilistic λ-contractive condition in this setting. Now we recall some definitions and results on Menger PGM-spaces which are used later on in the paper. For more details, we refer the reader to [16].

Definition 1.6 ([16])**.** A Menger PGM-space is a triple (X, G^*, T), where X is a nonempty set, T is a continuous t-norm and G^* is a mapping from $X \times X \times X$ into $\mathcal{D}^+_\infty$ ($G^*_{x,y,z}$ denotes the value of G^* at the point (x, y, z)) satisfying the following conditions:

(PGM-1) $G^*_{x,y,z}(t) = 1$ for all $t > 0$ if and only if $x = y = z$;
(PGM-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for all $x, y, z \in X$ with $z \neq y$ and $t > 0$;
(PGM-3) $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = \dots$ (symmetry in all three variables);
(PGM-4) $G^*_{x,y,z}(t + s) \geq T(G^*_{x,a,a}(s), G^*_{a,y,z}(t))$ for all $x, y, z, a \in X$ and all $s, t > 0$.

Definition 1.7 ([16])**.** Let (X, G^*, T) be a Menger PGM-space and x_0 be a point in X. For any $\varepsilon > 0$ and δ with $0 < \delta < 1$, an (ε, δ)-neighborhood of x_0 is the set of all points y in X for which $G^*_{x_0,y,y}(\varepsilon) > 1 - \delta$ and $G^*_{y,x_0,x_0}(\varepsilon) > 1 - \delta$. We write

$$N_{x_0}(\varepsilon, \delta) = \{y \in X : G^*_{x_0,y,y}(\varepsilon) > 1 - \delta, G^*_{y,x_0,x_0}(\varepsilon) > 1 - \delta\}.$$

Definition 1.8 ([16])**.** (1) A sequence $\{x_n\}$ in a Menger PGM-space (X, G^*, T) is said to be convergent to a point $x \in X$ (written $x_n \to x$) if, for any $\varepsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\varepsilon,\delta}$ such that $x_n \in N_x(\varepsilon, \delta)$ whenever $n > M_{\varepsilon,\delta}$.

(2) A sequence $\{x_n\}$ in a Menger PGM-space (X, G^*, T) is called a Cauchy sequence if, for any $\varepsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\varepsilon,\delta}$ such that $G^*_{x_n,x_m,x_l}(\varepsilon) > 1 - \delta$ whenever $m, n, l > M_{\varepsilon,\delta}$.

(3) A Menger PGM-space (X, G^*, T) is said to be complete if every Cauchy sequence in X converges to a point in X.

Theorem 1.9 ([16])**.** *Let (X, G^*, T) be a Menger PGM-space. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences in X and $x, y, z \in X$. If $x_n \to x, y_n \to y$ and $z_n \to z$ as $n \to \infty$, then, for any $t > 0, G^*_{x_n,y_n,z_n}(t) \to G^*_{x,y,z}(t)$ as $n \to \infty$.*

Lemma 1.10 ([8])**.** *Suppose that $F \in \mathcal{D}^+$. For each $n \in \mathbb{Z}^+$, let $F_n : \mathbb{R} \to [0, 1]$ be nondecreasing, and $g_n : (0, +\infty) \to (0, +\infty)$ satisfy $\lim_{n\to\infty} g_n(t) = 0$ for any $t > 0$. If*

$$F_n(g_n(t)) \geq F(t)$$

for any $t > 0$, then $\lim_{n\to\infty} F_n(t) = 1$ for any $t > 0$.

Although probabilistic φ-contractions are a natural generalization of probabilistic λ-contractions, the techniques used in the proofs of fixed point results for probabilistic λ-contractions are no longer usable for probabilistic φ-contractions [4]. In 2009, Ćirić [4] presented a fixed point theorem for probabilistic φ-contractions. Jachymski [8] found a counterexample to the key lemma in [4], and established a corrected version of Ćirić's theorem. Inspired by the works in [4] and [8], in this paper, we try to obtain some new fixed point theorems under probabilistic φ-contractive conditions in Menger PGM-spaces. Our theorems not only are generalizations of the corresponding results of Ćirić [4], Jachymski [8] and other authors, but also improve and generalize the recent results given by Zhou et al. [16].

2. Fixed point results for probabilistic φ-contractions in generalized probabilistic metric spaces

Lemma 2.1. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. If*

$$G^*_{x,y,z}(\varphi(t)) = G^*_{x,y,z}(t) \tag{2.1}$$

for all $t > 0$, then $x = y = z$.

Proof. On the one hand, from $G^*_{x,y,z}(\varphi(t)) = G^*_{x,y,z}(t)$, we have

$$G^*_{x,y,z}(\varphi^n(t)) = G^*_{x,y,z}(t)$$

for all $n \in \mathbb{Z}^+$ and $t > 0$. Due to the fact that $\lim_{t\to+\infty} G^*_{x,y,z}(t) = 1$, for any $\varepsilon \in (0, 1)$, there exists $t_0 > 0$ such that $G^*_{x,y,z}(t_0) > 1 - \varepsilon$.

On the other hand, by $\lim_{n\to\infty} \varphi^n(t) = 0$, for any $\delta > 0$, there exists $N(\delta) \in \mathbb{Z}^+$ such that $\varphi^n(t_0) \leq \delta$ for all $n \geq N(\delta)$.

Thus

$$G^*_{x,y,z}(\delta) \geq G^*_{x,y,z}(\varphi^n(t_0)) = G^*_{x,y,z}(t_0) > 1 - \varepsilon,$$

which implies that $G^*_{x,y,z}(t) = 1$ for all $t > 0$.

Therefore, $x = y = z$. □

Following the proof of Lemma 2.1, we can similarly obtain the next result.

Lemma 2.2. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$. If*

$$G^*_{x,y,z}(t) = G^*_{x,y,z}(\varphi(t)) \tag{2.2}$$

for all $t > 0$, then $x = y = z$.

Lemma 2.3. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. If $g_1, g_2, \ldots, g_n : \mathbb{R} \to [0, 1]$, and*

$$G^*_{x,y,z}(\varphi(t)) \geq \min\{g_1(t), g_2(t), \ldots, g_n(t), G^*_{x,y,z}(t)\} \tag{2.3}$$

for all $t > 0$, then

$$G^*_{x,y,z}(\varphi(t)) \geq \min\{g_1(t), g_2(t), \ldots, g_n(t)\},$$

for all $t > 0$.

Proof. When $\min\{g_1(t), g_2(t), \ldots, g_n(t), G^*_{x,y,z}(t)\} < G^*_{x,y,z}(t)$, Lemma 2.3 obviously holds.

Suppose now that $\min\{g_1(t), g_2(t), \ldots, g_n(t), G^*_{x,y,z}(t)\} = G^*_{x,y,z}(t)$. From (2.3) we have

$$G^*_{x,y,z}(\varphi(t)) \geq G^*_{x,y,z}(t).$$

However, since and $\varphi(t) < t$,

$$G^*_{x,y,z}(t) \geq G^*_{x,y,z}(\varphi(t)).$$

Therefore $G^*_{x,y,z}(\varphi(t)) = G^*_{x,y,z}(t)$ for all $t > 0$. Then from Lemma 2.1, we obtain that

$$G^*_{x,y,z}(t) = 1$$

for all $t > 0$. Thus $g_1(t) = g_2(t) = \ldots = g_n(t) = 1$ for all $t > 0$. Consequently, $G^*_{x,y,z}(\varphi(t)) \geq \min\{g_1(t), g_2(t), \ldots, g_n(t)\}$ for all $t > 0$, and the proof of Lemma 2.3 is completed. □

In the same way as stated above, we can prove that the following lemma holds.

Lemma 2.4. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$. If $g_1, g_2, \ldots, g_n : \mathbb{R} \to [0, 1]$, and*

$$G^*_{x,y,z}(t) \geq \min\{g_1(\varphi(t)), g_2(\varphi(t)), \ldots, g_n(\varphi(t)), G^*_{x,y,z}(\varphi(t))\} \tag{2.4}$$

for all $t > 0$, then

$$G^*_{x,y,z}(t) \geq \min\{g_1(\varphi(t)), g_2(\varphi(t)), \ldots, g_n(\varphi(t))\}$$

for all $t > 0$.

Theorem 2.5. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(\varphi(t)) \geq \min\{G^*_{x,y,z}(t), G^*_{y,fy,fy}(t), G^*_{z,fz,fz}(t)\} \tag{2.5}$$

for all $x, y, z \in X$ and $t > 0$. Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\}$ in the following way:

$$x_{n+1} = fx_n, \quad n \in \mathbb{N}.$$

From the assumption (2.5), for any $t > 0$, we find that

$$\begin{aligned} G^*_{x_{n+1},x_{n+2},x_{n+2}}(\varphi(t)) &= G^*_{fx_n,fx_{n+1},fx_{n+1}}(\varphi(t)) \\ &\geq \min\{G^*_{x_n,x_{n+1},x_{n+1}}(t), G^*_{x_{n+1},fx_{n+1},fx_{n+1}}(t), G^*_{x_{n+1},fx_{n+1},fx_{n+1}}(t)\} \\ &= \min\{G^*_{x_n,x_{n+1},x_{n+1}}(t), G^*_{x_{n+1},x_{n+2},x_{n+2}}(t), G^*_{x_{n+1},x_{n+2},x_{n+2}}(t)\} \\ &= \min\{G^*_{x_n,x_{n+1},x_{n+1}}(t), G^*_{x_{n+1},x_{n+2},x_{n+2}}(t)\}. \end{aligned} \tag{2.6}$$

From Lemma 2.3, for any $t > 0$, we have

$$G^*_{x_{n+1},x_{n+2},x_{n+2}}(\varphi(t)) \geq G^*_{x_n,x_{n+1},x_{n+1}}(t). \tag{2.7}$$

Denote $P_n(t) = G^*_{x_n,x_{n+1},x_{n+1}}(t)$. From (2.7), we have

$$P_{n+1}(\varphi(t)) \geq P_n(t),$$

which implies that

$$P_{n+1}(\varphi^{n+1}(t)) \geq P_n(\varphi^n(t)) \geq \ldots \geq P_1(\varphi(t)) \geq P_0(t). \tag{2.8}$$

Since $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$, using Lemma 1.10 , we have

$$\lim_{n\to\infty} P_n(t) = 1,$$

that is

$$\lim_{n\to\infty} G^*_{x_n,x_{n+1},x_{n+1}}(t) = 1 \tag{2.9}$$

for any $t > 0$.

For any $k \in \mathbb{Z}^+$ and $t > 0$, we shall show the following inequality by mathematical induction:

$$G^*_{x_n,x_{n+k},x_{n+k}}(t) \geq T^k(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))). \tag{2.10}$$

If $k = 1$,

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(t) &\geq G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t)) \\ &= T\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t)), 1\Big) \\ &\geq T\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t)), G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))\Big) \\ &= T^1\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))\Big). \end{aligned}$$

Thus (2.10) holds in this case.

Now we assume (2.10) holds for $1 \le k \le p$. When $k = p+1$, by (PGM-4) we have

$$\begin{aligned} G^*_{x_n,x_{n+p+1},x_{n+p+1}}(t) &= G^*_{x_n,x_{n+p+1},x_{n+p+1}}(t-\varphi(t)+\varphi(t)) \\ &\ge T\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t)), G^*_{x_{n+1},x_{n+p+1},x_{n+p+1}}(\varphi(t))\Big). \end{aligned} \tag{2.11}$$

Following (2.5), it is easy to find that

$$G^*_{x_{n+1},x_{n+2},x_{n+2}}(t) \ge G^*_{x_n,x_{n+1},x_{n+1}}(t)$$

for all n. In fact, if we suppose

$$G^*_{x_{n+1},x_{n+2},x_{n+2}}(t) < G^*_{x_n,x_{n+1},x_{n+1}}(t),$$

then from $\varphi(t) < t$, we have

$$G^*_{x_{n+1},x_{n+2},x_{n+2}}(t) \ge G^*_{x_{n+1},x_{n+2},x_{n+2}}(\varphi(t)).$$

Therefore, by (2.7) we obtain that

$$G^*_{x_{n+1},x_{n+2},x_{n+2}}(t) \ge G^*_{x_n,x_{n+1},x_{n+1}}(t),$$

which is a contradiction. So, for all n we have

$$G^*_{x_{n+1},x_{n+2},x_{n+2}}(t) \ge G^*_{x_n,x_{n+1},x_{n+1}}(t).$$

Thus

$$G^*_{x_{n+p},x_{n+p+1},x_{n+p+1}}(t) \ge G^*_{x_n,x_{n+1},x_{n+1}}(t). \tag{2.12}$$

From (2.5), (2.12), the induction hypothesis and the monotony of G^*, we obtain that

$$\begin{aligned} G^*_{x_{n+1},x_{n+p+1},x_{n+p+1}}(\varphi(t)) &= G^*_{fx_n,fx_{n+p},fx_{n+p}}(\varphi(t)) \\ &\ge \min\{G^*_{x_n,x_{n+p},x_{n+p}}(t), G^*_{x_{n+p},fx_{n+p},fx_{n+p}}(t), G^*_{x_{n+p},fx_{n+p},fx_{n+p}}(t)\} \\ &= \min\{G^*_{x_n,x_{n+p},x_{n+p}}(t), G^*_{x_{n+p},x_{n+p+1},x_{n+p+1}}(t)\} \\ &\ge \min\{G^*_{x_n,x_{n+p},x_{n+p}}(t), G^*_{x_n,x_{n+1},x_{n+1}}(t)\} \\ &\ge \min\{T^p(G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t))), G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t))\} \\ &= T^p(G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t))). \end{aligned} \tag{2.13}$$

Then from (2.11) and (2.13), for $k = p+1$ we have

$$\begin{aligned} G^*_{x_n,x_{n+p+1},x_{n+p+1}}(t) &\ge T\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t)), T^p(G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t)))\Big) \\ &= T^{p+1}(G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t))). \end{aligned}$$

Thus

$$G^*_{x_n,x_{n+k},x_{n+k}}(t) \ge T^k(G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t)))$$

for all $k \ge 1$.

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence, i.e., $\lim_{m,n,l\to\infty} G^*_{x_n,x_m,x_l}(t) = 1$ for any $t > 0$. To this end, first we show that $\lim_{m,n,\to\infty} G^*_{x_n,x_m,x_m}(t) = 1$ for any $t > 0$. Suppose that $\varepsilon \in (0,1]$ is given. Since T is a t-norm of H-type, there exists $\delta > 0$, such that

$$T^n(s) > 1-\varepsilon, \forall\, n \in \mathbb{Z}^+, \tag{2.14}$$

when $1-\delta < s \le 1$.

On the other hand, by (2.9), we have

$$\lim_{n\to\infty} G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t)) = 1,$$

which implies that there exists $n_0 \in \mathbb{N}$ such that $G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t)) > 1-\delta$ for all $n \geq n_0$. Hence, from (2.10) and (2.14), we get $G^*_{x_n,x_{n+k},x_{n+k}}(t) > 1-\varepsilon$ for $k \in \mathbb{Z}^+$ and $n \geq n_0$. This shows that $\lim_{m,n\to\infty} G^*_{x_n,x_m,x_m}(t) = 1$ for any $t > 0$.

From (PGM-4), it follows that, for all $t > 0$,

$$G^*_{x_n,x_m,x_l}(t) \geq T\left(G^*_{x_n,x_n,x_m}\left(\frac{t}{2}\right), G^*_{x_n,x_n,x_l}\left(\frac{t}{2}\right)\right),$$

$$G^*_{x_n,x_n,x_m}\left(\frac{t}{2}\right) \geq T\left(G^*_{x_n,x_m,x_m}\left(\frac{t}{4}\right), G^*_{x_n,x_m,x_m}\left(\frac{t}{4}\right)\right)$$

and

$$G^*_{x_n,x_n,x_l}\left(\frac{t}{2}\right) \geq T\left(G^*_{x_n,x_l,x_l}\left(\frac{t}{4}\right), G^*_{x_n,x_l,x_l}\left(\frac{t}{4}\right)\right).$$

Therefore, by the continuity of T, we have

$$\lim_{m,n,l\to\infty} G^*_{x_n,x_m,x_l}(t) = 1$$

for any $t > 0$. This implies that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists some $\overline{x} \in X$ such that $\lim_{n\to\infty} x_n = \overline{x}$.

Now we show that $\overline{x}$ is a fixed point of X. Since $\varphi(t) < t$, by the monotony of G^* and from (2.5) we have

$$\begin{aligned} G^*_{f\overline{x},fx_n,fx_n}(t) &\geq G^*_{f\overline{x},fx_n,fx_n}(\varphi(t)) \\ &\geq \min\{G^*_{\overline{x},x_n,x_n}(t), G^*_{x_n,fx_n,fx_n}(t), G^*_{x_n,fx_n,fx_n}(t)\} \\ &= \min\{G^*_{\overline{x},x_n,x_n}(t), G^*_{x_n,x_{n+1},x_{n+1}}(t)\}. \end{aligned} \tag{2.15}$$

Since $\{x_{n+1}\}$ is a subsequence of $\{x_n\}$, $fx_n = x_{n+1} \to \overline{x}$ as $n \to \infty$. Letting $n \to \infty$ on both sides of inequality (2.15), we get

$$G^*_{f\overline{x},\overline{x},\overline{x}}(t) \geq G^*_{\overline{x},\overline{x},\overline{x}}(t) = 1$$

for all $t > 0$, hence, by (PGM-1),

$$\overline{x} = f\overline{x}.$$

Thus we have proved that f has a fixed point. Now, we shall show that $\overline{x}$ is the unique fixed point of f. Suppose that y is another fixed point of f. We define a sequence $\{y_n\}$ in the following way:

$$y_n = y, \quad n \in \mathbb{N}.$$

From (2.5), we have

$$\begin{aligned} G^*_{\overline{x},\overline{x},y_n}(\varphi(t)) = G^*_{\overline{x},\overline{x},y}(\varphi(t)) &= G^*_{f\overline{x},f\overline{x},fy}(\varphi(t)) \\ &\geq \min\{G^*_{\overline{x},\overline{x},y}(t), G^*_{\overline{x},f\overline{x},f\overline{x}}(t), G^*_{y,fy,fy}(t)\} \\ &= G^*_{\overline{x},\overline{x},y}(t) = G^*_{\overline{x},\overline{x},y_{n-1}}(t). \end{aligned} \tag{2.16}$$

Denote $Q_n(t) = G^*_{\overline{x},\overline{x},y_n}(t)$ $(t > 0)$. By (2.16), we have $Q_n(\varphi(t)) \geq Q_{n-1}(t)$, and hence for all $t > 0$,

$$Q_n(\varphi^n(t)) \geq Q_{n-1}(\varphi^{n-1}(t)) \geq \ldots \geq Q_1(\varphi(t)) \geq Q_0(t).$$

Since $\lim_{n\to\infty} \varphi^n(t) = 0$, by Lemma 1.10 we have

$$\lim_{n\to\infty} Q_n(t) = 1,$$

that is

$$\lim_{n\to\infty} G^*_{\overline{x},\overline{x},y_n}(t) = 1.$$

It follows that $G^*_{\overline{x},\overline{x},y}(t) = 1$ for any $t > 0$, which implies that $\overline{x} = y$. Therefore, f has a unique fixed point in X. This completes the proof. □

If we take $\varphi(t) = \lambda t$, $\lambda \in (0, 1)$, then from Theorem 2.5 we obtain the following consequence.

Corollary 2.6. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type and $\lambda \in (0, 1)$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(\lambda t) \geq \min\{G^*_{x,y,z}(t), G^*_{y,fy,fy}(t), G^*_{z,fz,fz}(t)\} \tag{2.17}$$

for all $x, y, z \in X$, $t > 0$. Then f has a unique fixed point in X.

Theorem 2.7. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(\varphi(t)) \geq G^*_{x,y,z}(t) \tag{2.18}$$

for all $x, y, z \in X$ and $t > 0$. Then f has a unique fixed point in X.

Proof. Due to

$$G^*_{fx,fy,fz}(\varphi(t)) \geq G^*_{x,y,z}(t) \geq \min\{G^*_{x,y,z}(t), G^*_{y,fy,fy}(t), G^*_{z,fz,fz}(t)\},$$

we obtain the conclusion from Theorem 2.5. □

Taking $y = z$ in Theorem 2.7, we obtain the following result.

Corollary 2.8. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(0) = \{0\}$, $\varphi(t) < t$, and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fy}(\varphi(t)) \geq G^*_{x,y,y}(t) \tag{2.19}$$

for all $x, y \in X$ and $t > 0$. Then f has a unique fixed point in X.

Moreover, if we take $\varphi(t) = \lambda t$, $\lambda \in (0, 1)$, then from Theorem 2.7 we obtain the following corollary.

Corollary 2.9 ([16])**.** *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type and $\lambda \in (0, 1)$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(\lambda t) \geq G^*_{x,y,z}(t) \tag{2.20}$$

for all $x, y, z \in X$, $t > 0$. Then f has a unique fixed point in X.

Theorem 2.10. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(t) \geq \min\{G^*_{x,y,z}(\varphi(t)), G^*_{y,fy,fy}(\varphi(t)), G^*_{z,fz,fz}(\varphi(t))\} \tag{2.21}$$

for all $x, y, z \in X$, $t > 0$. Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Put $x_{n+1} = fx_n,\ n \in \mathbb{N}$. From the assumption (2.21), for any $t > 0$, we have

$$\begin{aligned} G^*_{x_{n+1},x_{n+2},x_{n+2}}(t) &= G^*_{fx_n,fx_{n+1},fx_{n+1}}(t) \\ &\geq \min\{G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)), G^*_{x_{n+1},fx_{n+1},fx_{n+1}}(\varphi(t)), G^*_{x_{n+1},fx_{n+1},fx_{n+1}}(\varphi(t))\} \\ &= \min\{G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)), G^*_{x_{n+1},x_{n+2},x_{n+2}}(\varphi(t))\}. \end{aligned} \tag{2.22}$$

Hence from Lemma 2.4, for any $t > 0$, we obtain

$$G^*_{x_{n+1},x_{n+2},x_{n+2}}(t) \geq G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)). \tag{2.23}$$

Denote $E_n(t) = G^*_{x_n,x_{n+1},x_{n+1}}(t)$. From (2.23), we have

$$E_{n+1}(t) \geq E_n(\varphi(t)),$$

which implies that

$$E_{n+1}(t) \geq E_n(\varphi(t)) \geq E_{n-1}(\varphi^2(t)) \geq \ldots \geq E_1(\varphi^n(t)). \tag{2.24}$$

Since $\lim_{t\to+\infty} E_1(t) = \lim_{t\to+\infty} G^*_{x_1,x_2,x_2}(t) = 1$ and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for each $t > 0$, we have $\lim_{n\to\infty} E_1(\varphi^n(t)) = 1$. Moreover, by (2.24), we have $E_{n+1}(t) \geq E_1(\varphi^n(t))$. Hence,

$$\lim_{n\to\infty} E_n(t) = 1,$$

that is

$$\lim_{n\to\infty} G^*_{x_n,x_{n+1},x_{n+1}}(t) = 1, \quad t > 0. \tag{2.25}$$

In the next step we shall show by induction that for any $k \in \mathbb{Z}^+$,

$$G^*_{x_n,x_{n+k},x_{n+k}}(\varphi(t)) \geq T^k(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - t)). \tag{2.26}$$

For $k = 1$, from the monotony of G^* and the property (3) of T in Definition 1.1 we have

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) &\geq G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - t) \\ &= T\Big(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - t), 1\Big) \\ &\geq T\Big(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - t), G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - t)\Big) \\ &= T^1\Big(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - t)\Big). \end{aligned}$$

This means that (2.26) holds for $k = 1$.

Now we assume (2.26) holds for $k = p$ ($p \geq 1$). When $k = p + 1$, by (PGM-4) we have

$$\begin{aligned} G^*_{x_n,x_{n+p+1},x_{n+p+1}}(\varphi(t)) &= G^*_{x_n,x_{n+p+1},x_{n+p+1}}(\varphi(t) - t + t) \\ &\geq T\Big(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t) - t), G^*_{x_{n+1},x_{n+p+1},x_{n+p+1}}(t)\Big). \end{aligned} \tag{2.27}$$

Since $\varphi(t) > t$, by the monotony of G^* and from (2.23) we have

$$G^*_{x_{n+1},x_{n+2},x_{n+2}}(\varphi(t)) \geq G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t))$$

for all n. Thus

$$G^*_{x_{n+p},x_{n+p+1},x_{n+p+1}}(\varphi(t)) \geq G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)). \tag{2.28}$$

Hence, from (2.21), (2.28) and the induction hypothesis, we obtain

$$\begin{aligned}
G^*_{x_{n+1},x_{n+p+1},x_{n+p+1}}(t) &= G^*_{fx_n,fx_{n+p},fx_{n+p}}(t)\\
&\geq \min\{G^*_{x_n,x_{n+p},x_{n+p}}(\varphi(t)), G^*_{x_{n+p},fx_{n+p},fx_{n+p}}(\varphi(t)), G^*_{x_{n+p},fx_{n+p},fx_{n+p}}(\varphi(t))\}\\
&= \min\{G^*_{x_n,x_{n+p},x_{n+p}}(\varphi(t)), G^*_{x_{n+p},x_{n+p+1},x_{n+p+1}}(\varphi(t))\}\\
&\geq \min\{G^*_{x_n,x_{n+p},x_{n+p}}(\varphi(t)), G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t))\}\\
&\geq \min\{T^p(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)-t)), G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)-t)\}\\
&= T^p(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)-t)). \qquad (2.29)
\end{aligned}$$

From (2.27) and (2.29), we have

$$\begin{aligned}
G^*_{x_n,x_{n+p+1},x_{n+p+1}}(\varphi(t)) &\geq T\Big(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)-t), T^p(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)-t))\Big)\\
&= T^{p+1}(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)-t)).
\end{aligned}$$

Thus, by induction we obtain

$$G^*_{x_n,x_{n+k},x_{n+k}}(\varphi(t)) \geq T^k(G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)-t))$$

for all $k \in \mathbb{Z}^+$.

By the same method as in Theorem 2.5, we can infer that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $\overline{x} \in X$ such that $x_n \to \overline{x}$ as $n \to \infty$. By (2.21), it follows that

$$G^*_{f\overline{x},fx_n,fx_n}(t) \geq \min\{G^*_{\overline{x},x_n,x_n}(\varphi(t)), G^*_{x_n,fx_n,fx_n}(\varphi(t))\}. \qquad (2.30)$$

As $\{x_{n+1}\}$ is a subsequence of $\{x_n\}$, $fx_n = x_{n+1} \to \overline{x}$ as $n \to \infty$. Letting $n \to \infty$ on both sides of inequality (2.30), we obtain that

$$G^*_{f\overline{x},\overline{x},\overline{x}}(t) \geq G^*_{\overline{x},\overline{x},\overline{x}}(\varphi(t)) = 1$$

for any $t > 0$. Hence $\overline{x} = f\overline{x}$.

Now we shall prove that $\overline{x}$ is the unique fixed point of f. Suppose that y is another fixed point of f. We define a sequence $\{y_n\}$ in the following way:

$$y_n = y, \quad n \in \mathbb{N}.$$

From (2.21), we have

$$\begin{aligned}
G^*_{\overline{x},\overline{x},y_n}(t) &= G^*_{\overline{x},\overline{x},y}(t) = G^*_{f\overline{x},f\overline{x},fy}(t)\\
&\geq \min\{G^*_{\overline{x},\overline{x},y}(\varphi(t)), G^*_{\overline{x},f\overline{x},f\overline{x}}(\varphi(t)), G^*_{y,fy,fy}(\varphi(t))\}\\
&= \min\{G^*_{\overline{x},\overline{x},y}(\varphi(t)), G^*_{\overline{x},\overline{x},\overline{x}}(\varphi(t)), G^*_{y,y,y}(\varphi(t))\}\\
&= G^*_{\overline{x},\overline{x},y}(\varphi(t))\\
&= G^*_{\overline{x},\overline{x},y_{n-1}}(\varphi(t)). \qquad (2.31)
\end{aligned}$$

Suppose that $Q_n(t) = G^*_{\overline{x},\overline{x},y_n}(t)$ $(t > 0)$. By (2.31), we have $Q_n(t) \geq Q_{n-1}(\varphi(t))$, and then

$$Q_n(t) \geq Q_{n-1}(\varphi(t)) \geq \ldots \geq Q_0(\varphi^n(t)). \qquad (2.32)$$

Since $\lim_{n\to\infty} \varphi^n(t) = +\infty$, we have

$$\lim_{n\to\infty} Q_0(\varphi^n(t)) = \lim_{n\to\infty} G^*_{\overline{x},\overline{x},y_0}(\varphi^n(t)) = 1. \qquad (2.33)$$

From (2.33) and (2.32), we obtain

$$\lim_{n\to\infty} Q_n(t) = \lim_{n\to\infty} G^*_{\overline{x},\overline{x},y_n}(t) \geq 1,$$

which implies

$$G^*_{\overline{x},\overline{x},y}(t) = 1$$

for any $t > 0$. Hence we conclude that $\overline{x} = y$. Therefore, f has a unique fixed point in X. The proof of Theorem 2.10 is completed. □

Theorem 2.11. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(t) \geq G^*_{x,y,z}(\varphi(t)) \tag{2.34}$$

for all $x, y, z \in X$ and $t > 0$. Then the operator f has a unique fixed point in X.

Proof. Similarly as in the proof of Theorem 2.7, but using Theorem 2.10 in place of Theorem 2.5, we immediately obtain that Theorem 2.11 holds. □

Theorem 2.12. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$\begin{aligned} G^*_{fx,fy,fz}(\varphi(t)) \geq\ & a_1 G^*_{x,y,z}(t) + a_2 G^*_{x,fx,fx}(t) + a_3 G^*_{y,fy,fy}(t) \\ & + a_4 G^*_{z,fz,fz}(t) + a_5 G^*_{y,fz,fz}(t) + a_6 G^*_{z,fy,fy}(t) \end{aligned} \tag{2.35}$$

for all $x, y, z \in X$ and $t > 0$, where $a_i \geq 0$ $(i = 1, 2, \ldots, 6)$, $a_1 + a_2 > 0$ and $\sum_{i=1}^{6} a_i = 1$. Then f has a unique fixed point in X.

Proof. Let x_0 in X be an arbitrary point. We define a sequence $\{x_n\}$ in the following way:

$$x_{n+1} = fx_n, \quad n \in \mathbb{N}.$$

Due to (2.35), for any $t > 0$, we have

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) &= G^*_{fx_{n-1},fx_n,fx_n}(\varphi(t)) \\ &\geq a_1 G^*_{x_{n-1},x_n,x_n}(t) + a_2 G^*_{x_{n-1},fx_{n-1},fx_{n-1}}(t) + a_3 G^*_{x_n,fx_n,fx_n}(t) \\ &\quad + a_4 G^*_{x_n,fx_n,fx_n}(t) + a_5 G^*_{x_n,fx_n,fx_n}(t) + a_6 G^*_{x_n,fx_n,fx_n}(t) \\ &= (a_1 + a_2) G^*_{x_{n-1},x_n,x_n}(t) + (a_3 + a_4 + a_5 + a_6) G^*_{x_n,fx_n,fx_n}(t) \\ &\geq (a_1 + a_2) G^*_{x_{n-1},x_n,x_n}(t) + (a_3 + a_4 + a_5 + a_6) G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)), \end{aligned} \tag{2.36}$$

which implies

$$G^*_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq G^*_{x_{n-1},x_n,x_n}(t) \tag{2.37}$$

for all n. Thus, for any $k \in \mathbb{Z}^+$, we have

$$G^*_{x_{n+k},x_{n+k+1},x_{n+k+1}}(\varphi(t)) \geq G^*_{x_n,x_{n+1},x_{n+1}}(t). \tag{2.38}$$

Denote $P_n(t) = G^*_{x_n,x_{n+1},x_{n+1}}(t)$. From the inequality (2.37), we have

$$P_n(\varphi(t)) \geq P_{n-1}(t),$$

which implies that

$$P_n(\varphi^n(t)) \geq P_{n-1}(\varphi^{n-1}(t)) \geq \ldots \geq P_1(\varphi(t)) \geq P_0(t). \tag{2.39}$$

Since $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, we obtain using Lemma 1.10 that

$$\lim_{n\to\infty} P_n(t) = 1,$$

that is

$$\lim_{n\to\infty} G^*_{x_n,x_{n+1},x_{n+1}}(t) = 1, \tag{2.40}$$

for all $t > 0$. Next we shall prove by induction that for all $k \in \mathbb{Z}^+$ and $t > 0$,

$$G^*_{x_n,x_{n+k},x_{n+k}}(t) \geq T^k(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))). \tag{2.41}$$

In fact, as $k = 1$, from the monotony of G^* and the property (3) of T in Definition 1.1 we have

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(t) &\geq G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t)) \\ &= T\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t)), 1\Big) \\ &\geq T\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t)), G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))\Big) \\ &= T^1\Big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))\Big). \end{aligned}$$

Therefore, (2.41) holds for $k = 1$.

Suppose now that $G^*_{x_n,x_{n+k},x_{n+k}}(t) \geq T^k(G^*_{x_n,x_{n+1},x_{n+1}}(t-\varphi(t)))$ holds for some fixed $k \geq 1$. From (2.35), the monotony of G^*, (2.38) and the induction hypothesis we have

$$\begin{aligned} &G^*_{x_{n+1},x_{n+k+1},x_{n+k+1}}(\varphi(t)) = G^*_{fx_n,fx_{n+k},fx_{n+k}}(\varphi(t)) \\ \geq\ & a_1G^*_{x_n,x_{n+k},x_{n+k}}(t) + a_2G^*_{x_n,fx_n,fx_n}(t) + a_3G^*_{x_{n+k},fx_{n+k},fx_{n+k}}(t) + a_4G^*_{x_{n+k},fx_{n+k},fx_{n+k}}(t) \\ &+a_5G^*_{x_{n+k},fx_{n+k},fx_{n+k}}(t) + a_6G^*_{x_{n+k},fx_{n+k},fx_{n+k}}(t) \\ =\ & a_1G^*_{x_n,x_{n+k},x_{n+k}}(t) + a_2G^*_{x_n,x_{n+1},x_{n+1}}(t) + (a_3 + a_4 + a_5 + a_6)G^*_{x_{n+k},x_{n+k+1},x_{n+k+1}}(t) \\ \geq\ & a_1G^*_{x_n,x_{n+k},x_{n+k}}(t) + a_2G^*_{x_n,x_{n+1},x_{n+1}}(t) + (a_3 + a_4 + a_5 + a_6)G^*_{x_{n+k},x_{n+k+1},x_{n+k+1}}(\varphi(t)) \\ \geq\ & a_1G^*_{x_n,x_{n+k},x_{n+k}}(t) + (a_2 + a_3 + \ldots + a_6)G^*_{x_n,x_{n+1},x_{n+1}}(t) \\ \geq\ & a_1T^k(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))) + (a_2 + a_3 + \ldots + a_6)G^*_{x_n,x_{n+1},x_{n+1}}(t) \\ \geq\ & a_1T^k(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))) + (a_2 + a_3 + \ldots + a_6)G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t)) \\ \geq\ & a_1T^k(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))) + (a_2 + a_3 + \ldots + a_6)T^k(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))) \\ =\ & T^k(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))). \end{aligned} \tag{2.42}$$

Hence, by (PGM-4) and (2.42), we obtain

$$\begin{aligned} G^*_{x_n,x_{n+k+1},x_{n+k+1}}(t) &= G^*_{x_n,x_{n+k+1},x_{n+k+1}}(t - \varphi(t) + \varphi(t)) \\ &\geq T(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t)), G^*_{x_{n+1},x_{n+k+1},x_{n+k+1}}(\varphi(t))) \\ &\geq T\big(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t)), T^k(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t)))\big) \\ &= T^{k+1}(G^*_{x_n,x_{n+1},x_{n+1}}(t - \varphi(t))). \end{aligned}$$

Thus we have proved that if inequality (2.41) holds for some $k \geq 1$, then it must also hold for $k + 1$. By mathematical induction we conclude that inequality (2.41) holds for all $k \in \mathbb{Z}^+$ and $t > 0$.

As in the proof of the Theorem 2.5, it follows that the sequence $\{x_n\}$ is Cauchy.

Since X is complete, there exists $\overline{x} \in X$ such that $x_n \to \overline{x}$ as $n \to \infty$. Next we show that $\overline{x}$ is a fixed point of f.

By (2.35) we have

$$
\begin{aligned}
G^*_{x_{n+1},f\overline{x},f\overline{x}}(\varphi(t)) &= G^*_{fx_n,f\overline{x},f\overline{x}}(\varphi(t)) \\
&\geq a_1G^*_{x_n,\overline{x},\overline{x}}(t) + a_2G^*_{x_n,fx_n,fx_n}(t) + a_3G^*_{\overline{x},f\overline{x},f\overline{x}}(t) + a_4G^*_{\overline{x},f\overline{x},f\overline{x}}(t) + a_5G^*_{\overline{x},f\overline{x},f\overline{x}}(t) + a_6G^*_{\overline{x},f\overline{x},f\overline{x}}(t) \\
&= a_1G^*_{x_n,\overline{x},\overline{x}}(t) + a_2G^*_{x_n,x_{n+1},x_{n+1}}(t) + (a_3+a_4+a_5+a_6)G^*_{\overline{x},f\overline{x},f\overline{x}}(t) \\
&\geq a_1G^*_{x_n,\overline{x},\overline{x}}(t) + a_2G^*_{x_n,x_{n+1},x_{n+1}}(t) + (a_3+a_4+a_5+a_6)G^*_{\overline{x},f\overline{x},f\overline{x}}(\varphi(t)). \qquad (2.43)
\end{aligned}
$$

Now, since $x_n \to \overline{x}$ and $fx_n = x_{n+1} \to \overline{x}$ as $n \to \infty$, letting $n \to \infty$ on both sides of inequality (2.43), we get, for any $t > 0$,

$$G^*_{\overline{x},f\overline{x},f\overline{x}}(\varphi(t)) \geq (a_1+a_2)G^*_{\overline{x},\overline{x},\overline{x}}(t) + (a_3+a_4+a_5+a_6)G^*_{\overline{x},f\overline{x},f\overline{x}}(\varphi(t)),$$

which implies

$$G^*_{\overline{x},f\overline{x},f\overline{x}}(\varphi(t)) \geq G^*_{\overline{x},\overline{x},\overline{x}}(t) = 1.$$

Therefore $\overline{x} = f\overline{x}$.

Finally, we shall show that $\overline{x}$ is the unique fixed point of f. Suppose that, contrary to our claim, there exists another fixed point $y \in X$. From (2.35), we have, for any $t > 0$,

$$
\begin{aligned}
G^*_{\overline{x},y,y}(\varphi(t)) &= G^*_{f\overline{x},fy,fy}(\varphi(t)) \\
&\geq a_1G^*_{\overline{x},y,y}(t) + a_2G^*_{\overline{x},f\overline{x},f\overline{x}}(t) + a_3G^*_{y,fy,fy}(t) + a_4G^*_{y,fy,fy}(t) + a_5G^*_{y,fy,fy}(t) + a_6G^*_{y,fy,fy}(t) \\
&\geq a_1G^*_{\overline{x},y,y}(\varphi(t)) + a_2G^*_{\overline{x},\overline{x},\overline{x}}(t) + (a_3+\ldots+a_6)G^*_{y,y,y}(t) \\
&= a_1G^*_{\overline{x},y,y}(\varphi(t)) + a_2 + \ldots + a_6.
\end{aligned}
$$

This implies that

$$G^*_{\overline{x},y,y}(\varphi(t)) \geq 1$$

for all $t > 0$, so $\overline{x} = y$. Therefore, f has a unique fixed point in X. The proof of Theorem 2.12 is completed. □

Taking $a_5 = a_6 = 0$ in Theorem 2.12, we obtain the following result.

Corollary 2.13. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(0) = \{0\}$, $\varphi(t) < t$, and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(t) \geq a_1G^*_{x,y,z}(\varphi(t)) + a_2G^*_{x,fx,fx}(\varphi(t)) + a_3G^*_{y,fy,fy}(\varphi(t)) + a_4G^*_{z,fz,fz}(\varphi(t)) \qquad (2.44)$$

for all $x, y, z \in X$ and $t > 0$, where $a_i \geq 0$ $(i = 1, 2, 3, 4)$, $a_1 + a_2 > 0$ and $\sum_{i=1}^4 a_i = 1$. Then f has a unique fixed point in X.

If we set $a_1 = 0$ in Corollary 2.13, then we obtain:

Corollary 2.14. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(0) = \{0\}$, $\varphi(t) < t$, and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(t) \geq a_1G^*_{x,fx,fx}(\varphi(t)) + a_2G^*_{y,fy,fy}(\varphi(t)) + a_3G^*_{z,fz,fz}(\varphi(t)) \qquad (2.45)$$

for all $x, y, z \in X$ and $t > 0$, where $a_1 > 0, a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 = 1$. Then f has a unique fixed point in X.

In particular, if we set $\varphi(t) = \lambda t, \lambda \in (0, 1)$, and $a_1 = a_2 = a_3 = \frac{1}{3}$, then Corollary 2.14 becomes:

Corollary 2.15 ([16])**.** *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(\lambda t) \geq \frac{1}{3}[G^*_{x,fx,fx}(t) + G^*_{y,fy,fy}(t) + G^*_{z,fz,fz}(t)] \tag{2.46}$$

for all $x, y, z \in X$ and $t > 0$, where $\lambda \in (0, 1)$. Then f has a unique fixed point in X.

Following the proof of Theorem 2.12, we can show that the next result also holds.

Theorem 2.16. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$\begin{aligned} G^*_{fx,fy,fz}(t) \geq\ & a_1 G^*_{x,y,z}(\varphi(t)) + a_2 G^*_{x,fx,fx}(\varphi(t)) + a_3 G^*_{y,fy,fy}(\varphi(t)) \\ & + a_4 G^*_{z,fz,fz}(\varphi(t)) + a_5 G^*_{y,fz,fz}(\varphi(t)) + a_6 G^*_{z,fy,fy}(\varphi(t)) \end{aligned} \tag{2.47}$$

for all $x, y, z \in X$ and $t > 0$, where $a_i \geq 0$ $(i = 1, 2, \ldots, 6)$, $a_1 + a_2 > 0$ and $\sum_{i=1}^6 a_i = 1$. Then f has a unique fixed point in X.

Taking $a_5 = a_6 = 0$ in Theorem 2.16, we obtain

Corollary 2.17. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(t) \geq a_1 G^*_{x,y,z}(\varphi(t)) + a_2 G^*_{x,fx,fx}(\varphi(t)) + a_3 G^*_{y,fy,fy}(\varphi(t)) + a_4 G^*_{z,fz,fz}(\varphi(t)) \tag{2.48}$$

for all $x, y, z \in X$ and $t > 0$, where $a_i \geq 0$ $(i = 1, 2, 3, 4)$, $a_1 + a_2 > 0$ and $\sum_{i=1}^4 a_i = 1$. Then f has a unique fixed point in X.

In particular, if we set $a_1 = 0$ in Corollary 2.17, then we have

Corollary 2.18. *Let (X, G^*, T) be a complete Menger PGM-space with T of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$. Let $f : X \to X$ be a given mapping satisfying*

$$G^*_{fx,fy,fz}(t) \geq a_1 G^*_{x,fx,fx}(\varphi(t)) + a_2 G^*_{y,fy,fy}(\varphi(t)) + a_3 G^*_{z,fz,fz}(\varphi(t)) \tag{2.49}$$

for all $x, y, z \in X$, $t > 0$, where $a_1 > 0$, $a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 = 1$. Then f has a unique fixed point in X.

Finally, we give the following example to illustrate Theorem 2.12.

Example 2.19. Let $X = [0, \infty)$, $T(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and define the mappings $H : [0, \infty) \to [0, \infty)$ and $G^* : X^3 \times [0, \infty) \to [0, \infty)$ by

$$H(t) = \begin{cases} 0, & t = 0, \\ 1, & t > 0. \end{cases}$$

and

$$G^*_{x,y,z}(t) = \begin{cases} H(t), & x = y = z, \\ \frac{\alpha t}{\alpha t + G(x,y,z)}, & \text{otherwise.} \end{cases} \tag{2.50}$$

for all $x, y, z \in X$, where $\alpha > 0$, $G(x, y, z) = |x - y| + |y - z| + |z - x|$. Then G is a G-metric (see [13]). It is easy to check that G^* satisfies (PGM-1)-(PGM-3). Next we show $G^*(x, y, z)(s + t) \geq T(G^*(x, a, a)(s), G^*(a, y, y)(t))$ for all $x, y, z, a \in X$ and all $s, t > 0$. When $x = y = z$, it is easy to see that G^* satisfies (PGM-4). When at least one of x, y, z is not equal to the other two, since $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, we have

$$\begin{aligned}\frac{\alpha t + \alpha s}{\alpha s + \alpha t + G(x, y, z)} &\geq \frac{\alpha t + \alpha s}{\alpha s + \alpha t + G(x, a, a) + G(a, y, z)} \\ &\geq \min\left\{\frac{\alpha s}{\alpha s + G(x, a, a)}, \frac{\alpha t}{\alpha t + G(a, y, z)}\right\}.\end{aligned}$$

This shows that G^* satisfies (PGM-4). Hence (X, G^*, T_M) is a Menger PGM-space. Let $\varphi(t) = \lambda t$, $\lambda \in (0, 1)$. Define a mapping $f : X \to X$ by $f(x) = 1$ for all $x \in X$, and let $a_i \geq 0$ $(i = 1, 2, \ldots, 6)$ be such that $a_1 + a_2 > 0$ and $\sum_{i=1}^{6} a_i = 1$. For all $x, y, z \in X$ and $t > 0$, since

$$G^*_{fx,fy,fz}(\varphi(t)) = G^*_{1,1,1}(\lambda t) = 1$$

and

$$a_1 G^*_{x,y,z}(t) + a_2 G^*_{x,fx,fx}(t) + a_3 G^*_{y,fy,fy}(t) + a_4 G^*_{z,fz,fz}(t) + a_5 G^*_{y,fz,fz}(t) + a_6 G^*_{z,fy,fy}(t) \leq \sum_{i=1}^{6} a_i = 1,$$

we obtain that

$$G^*_{fx,fy,fz}(\varphi(t)) \geq a_1 G^*_{x,y,z}(t) + a_2 G^*_{x,fx,fx}(t) + a_3 G^*_{y,fy,fy}(t) + a_4 G^*_{z,fz,fz}(t) + a_5 G^*_{y,fz,fz}(t) + a_6 G^*_{z,fy,fy}(t).$$

Thus all conditions of Theorem 2.12 are satisfied. Therefore, we conclude that f has a fixed point in X. In fact, the fixed point is $x = 1$.

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