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Common fixed point theorems under strict contractive conditions in Menger probabilistic G-metric spaces

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Abstract

In this paper, a new concept of the property G^*-$(E.A)$ in Menger PGM-spaces is introduced. Based on this, some common fixed point theorems under strict contractive conditions for mappings satisfying the property G^*-$(E.A)$ in Menger PGM-spaces and the corresponding results in G-metric spaces are obtained. Finally, an example is given to exemplify our main results. ©2015 All rights reserved.

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1. Introduction

As a generalization of a metric space, the concept of a probabilistic metric space has been introduced by Menger [17, 23]. Fixed point theory in a probabilistic metric space is an important branch of probabilistic analysis and many results on the existence of fixed points or solutions of nonlinear equations under various types of conditions in Menger PM-spaces have been extensively studied by many scholars (see *e.g.* [27, 28]). In 2006, Mustafa and Sims [19] introduced the concept of a generalized metric space and other authors obtained many fixed point theorems in generalized metric spaces (see [4, 5, 6, 7, 10, 11, 12]). Moreover, Zhou *et al.* [26] defined the notion of a generalized probabilistic metric space or a PGM-space as a generalization of a PM-space and a G-metric space.

Jungck [13] introduced the concept of compatible mappings in metric spaces and proved some common fixed point theorems for such mappings. The concept of weakly compatible mappings was given by [14]. On

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the other hand, the concept of compatible mappings in Menger spaces was initiated by Mishra [18], and since then many fixed point results for compatible mappings and weakly compatible mappings have been studied [8, 16, 24, 25]. The concept of noncompatible mappings was introduced and studied by Pant [20, 21, 22]. In 2002, Aamri and Moutawakil [1] defined a new property for a pair of mappings, *i.e.*, the so-called property $(E.A)$, which is a generalization of the concept of noncompatibility. In 2009, Fang [9] defined the property $(E.A)$ for two mappings in Menger PM-spaces and studied the existence of common fixed points in such spaces.

The main purpose of this paper is to establish some common fixed point theorems under strict contractive conditions for a pair of weakly compatible mappings satisfying the property G^*-$(E.A)$ in Menger PGM-spaces. We also obtain the corresponding results in G-metric spaces. Finally, an example is given to illustrate our main results.

2. Preliminaries

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{Z}^+$ be the set of all positive integers.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing left-continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$.

We shall denote by $\mathcal{D}$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied:

(1) $\Delta(a, 1) = a$;
(2) $\Delta(a, b) = \Delta(b, a)$;
(3) $a \ge b, c \ge d \Rightarrow \Delta(a, c) \ge \Delta(b, d)$;
(4) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

A typical example of a t-norm is Δ_m, where $\Delta_m(a, b) = \min\{a, b\}$, for each $a, b \in [0, 1]$.

Definition 2.1 ([19]). Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions:

(1) $G(x, y, z) = 0$ if $x = y = z$ for all $x, y, z \in X$;
(2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \ne y$;
(3) $G(x, x, y) \le G(x, y, z)$ for all $x, y, z \in X$ with $z \ne y$;
(4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ for all $x, y, z \in X$;
(5) $G(x, y, z) \le G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a generalized metric or a G-metric on X and the pair (X, G) is a G-metric space.

Definition 2.2 ([26]). A Menger probabilistic G-metric space (shortly, a PGM-space) is a triple (X, G^*, Δ), where X is a nonempty set, Δ is a continuous t-norm and G^* is a mapping from $X \times X \times X$ into $\mathcal{D}$ ($G^*_{x,y,z}$ denote the value of G^* at the point (x, y, z)) satisfying the following conditions:

(1) $G^*_{x,y,z}(t) = 1$ for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;
(2) $G^*_{x,x,y}(t) \ge G^*_{x,y,z}(t)$ for all $x, y, z \in X$ with $z \ne y$ and $t > 0$;
(3) $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = ...$(symmetry in all three variables);
(4) $G^*_{x,y,z}(t + s) \ge \Delta(G^*_{x,a,a}(s), G^*_{a,y,z}(t))$ for all $x, y, z, a \in X$ and $s, t \ge 0$.

Example 2.3 ([26]). Let (X, G) be a G-metric space, where $G(x, y, z) = |x - y| + |y - z| + |z - x|$. Define $G^*_{x,y,z}(t) = \frac{t}{t+G(x,y,z)}$ for all $x, y, z \in X$. Then (X, G^*, Δ_m) is a Menger PGM-space.

Example 2.4. Let (X, G) be a G-metric space. Define a mapping $G^* : X \times X \times X \to \mathcal{D}$ by

$$G^*_{x,y,z}(t) = H(t - G(x, y, z)), \tag{2.1}$$

for $x, y, z \in X$ and $t > 0$. Then (X, G^*, Δ_m) is a Menger PGM-space, called the induced Menger PGM-space by (X, G).

Definition 2.5 ([26])**.** Let (X, G^*, Δ) be a PGM-space, and $\{x_n\}$ is a sequence in X.

(1) $\{x_n\}$ is said to be convergent to a point $x \in X$ (write $x_n \to x$), if for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $x_n \in N_{x_0}(\epsilon, \delta)$ whenever $n > M_{\epsilon,\delta}$;
(2) $\{x_n\}$ is called a *Cauchy* sequence, if for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $G^*_{x_n,x_m,x_l}(\epsilon) > 1 - \delta$ whenever $n, m, l > M_{\epsilon,\delta}$;
(3) (X, G^*, Δ) is said to be complete, if every *Cauchy* sequence in X converges to a point in X.

Remark 2.6. Let (X, G^*, Δ) be a Menger PGM-space, $\{x_n\}$ is a sequence in X. Then the following are equivalent:

(1) $\{x_n\}$ is convergent to a point $x \in X$;
(2) $G^*_{x_n,x_n,x}(t) \to 1$ as $n \to \infty$, for all $t > 0$;
(3) $G^*_{x_n,x,x}(t) \to 1$ as $n \to \infty$, for all $t > 0$.

Remark 2.7. If $G^*_{x_n,x_n,u}(t) \to 1$ and $G^*_{y_n,y_n,u}(t) \to 1$, or $G^*_{x_n,u,u}(t) \to 1$ and $G^*_{y_n,u,u}(t) \to 1$ as $n \to \infty$ for all $t > 0$, then it is easy to obtain from (PGM-4) that $G^*_{x_n,y_n,u}(t) \to 1$ as $n \to \infty$ for all $t > 0$.

We can analogously prove the following lemma in Menger PM-spaces.

Lemma 2.8. *Let (X, G^*, Δ) be a Menger PGM-space with Δ a continuous t-norm, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in X and $x, y, z \in X$, if $\{x_n\} \to x$, $\{y_n\} \to x$ and $\{z_n\} \to x$ as $n \to \infty$. Then*

(1) $\liminf_{n\to\infty} G^*_{x_n,y_n,z_n}(t) \geq G^*_{x,y,z}(t)$ for all $t > 0$;
(2) $G^*_{x,y,z}(t + o) \geq \limsup_{n\to\infty} G^*_{x_n,y_n,z_n}(t)$ for all $t > 0$.

Particularly, if t_0 is a continuous point of $G_{x,y,z}(\cdot)$, then $\lim_{n\to\infty} G_{x_n,y_n,z_n}(t_0) = G_{x,y,z}(t_0)$.

Lemma 2.9 ([29])**.** *Let (X, G^*, Δ) be a Menger PGM-space. For each $\lambda \in (0, 1]$, define a function G^*_λ by*

$$G^*_\lambda(x, y, z) = \inf_t \{t \geq 0 : G^*_{x,y,z}(t) > 1 - \lambda\},$$

for $x, y, z \in X$, then

(1) $G^*_\lambda(x, y, z) < t$ if and only if $G^*_{x,y,z}(t) > 1 - \lambda$;
(2) $G^*_\lambda(x, y, z) = 0$ for all $\lambda \in (0, 1]$ if and only if $x = y = z$;
(3) $G^*_\lambda(x, y, z) = G^*_\lambda(y, x, z) = G^*_\lambda(y, z, x) = ...$;
(4) if $\Delta = \Delta_m$, then for every $\lambda \in (0, 1]$, $G^*_\lambda(x, y, z) \leq G^*_\lambda(x, a, a) + G^*_\lambda(a, y, z)$.

Lemma 2.10 ([22])**.** *Let (X, G^*, Δ) be a Menger PGM-space and Δ be a continuous t-norm. Then the following statements are equivalent:*

(i) the sequence $\{x_n\}$ is a Cauchy sequence;
(ii) for any $\epsilon > 0$ and $0 < \lambda < 1$, there exists $M \in \mathbb{Z}^+$ such that $G^*_{x_n,x_m,x_m}(\epsilon) > 1 - \lambda$, for all $n, m > M$.

Definition 2.11 ([14, 15])**.** A pair of self-mappings S and T on X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence point, i.e., if $Tu = Su$ for some $u \in X$ implies that $TSu = STu$.

Definition 2.12 ([2])**.** Let X be a G-metric space. The mappings $f, g : X \to X$ are called

(i) G-weakly commuting if for all $x \in X$

$$G(fgx, fgx, gfx) \leq G(fx, fx, gx);$$

(ii) G-R-weakly commuting if there exists a positive real number R, such that

$$G(fgx, fgx, gfx) \leq R \cdot G(fx, fx, gx),$$

holds for each $x \in X$;

(iii) G-compatible if, whenever a sequence $\{x_n\}$ in X is such that $\{fx_n\}$ and $\{gx_n\}$ are G-convergent to some $u \in X$, then $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 0$;

(iv) G-incompatible if there exists at least one sequence $\{x_n\}$ in X such that the sequences $\{fx_n\}$ and $\{gx_n\}$ are G-convergent to some $u \in X$, but $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n)$ is either nonzero or does not exist.

Definition 2.13 ([3])**.** Let (X, G) be a G-metric space. Self-mappings f and g on X are satisfy the G-$(E.A)$ property if there exists a sequence $\{x_n\}$ in X such that $\{fx_n\}$ and $\{gx_n\}$ are G-convergent to some $u \in X$.

Definition 2.14 ([9])**.** Let $F_1, F_2 \in \mathcal{D}$. The algebraic sum $F_1 \oplus F_2$ of F_1 and F_2 is defined by

$$(F_1 \oplus F_2)(t) = \sup_{t_1+t_2=t} \min\{F_1(t_1), F_2(t_2)\}$$

for all $t \in \mathbb{R}$.

Definition 2.15 ([9])**.** Let f and g be two functions defined on $\mathbb{R}$ with positive values. The notation $f > g$ means that $f \geq g$ for all $t \in \mathbb{R}$ and there exists at least one $t_0 \in \mathbb{R}$ such that $f(t_0) > g(t_0)$.

3. Main results

In this section, we will establish some new common fixed point theorems in Menger PGM-spaces. To this end, we first introduce the concepts of weakly compatible mappings and G^*-$(E.A)$ property in Menger PGM-spaces.

Definition 3.1. Let S and T be two self-mappings of a Menger PGM-space (X, G^*, Δ). S and T are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, *i.e.*, if $Tu = Su$ for some $u \in X$ implies that $TSu = STu$.

Definition 3.2. Let S and T be two self-mappings of a Menger PGM-space (X, G^*, Δ). S and T are said to satisfy the G^*-$(E.A)$ property, if there exists a sequence $\{x_n\}$ in X and $u \in X$, such that $G^*_{Tx_n,Tx_n,u}(t) \to 1$ and $G^*_{Sx_n,Sx_n,u}(t) \to 1$ for all $t > 0$.

We are now ready to give our main results.

Theorem 3.3. *Let (X, G^*, Δ) be a Menger PGM-space with a continuous t-norm Δ on $[0,1] \times [0,1]$, and S and T be two weakly compatible self-mappings on (X, G^*, Δ) satisfying the following conditions:*

(1) S and T satisfy the property G^*-$(E.A)$;

(2) for any $x, y \in X$, $x \neq y$, $t > 0$,

$$G^*_{Tx,Tx,Ty}(t) > \min\{G^*_{Sx,Sx,Sy}(t), [G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](\frac{2t}{k}), [G^*_{Ty,Ty,Sx} \oplus G^*_{Tx,Tx,Sy}](2t)\} \tag{3.1}$$

and

$$G^*_{Tx,Ty,Ty}(t) > \min\{G^*_{Sx,Sy,Sy}(t), [G^*_{Tx,Sx,Sx} \oplus G^*_{Ty,Sy,Sy}](\frac{2t}{k}), [G^*_{Ty,Sx,Sx} \oplus G^*_{Tx,Sy,Sy}](2t)\}, \tag{3.2}$$

for some k, $1 \leq k < 2$;

(3) $T(X) \subset S(X)$;
(4) $S(X)$ *or* $T(X)$ *is a closed subset of* X.

Then S *and* T *have a unique common fixed point in* X.

Proof. Since S and T satisfy the property G^*-$(E.A)$, there exists a sequence $\{x_n\}$ in X and $u \in X$, such that $G^*_{Tx_n,Tx_n,u}(t) \to 1$ and $G^*_{Sx_n,Sx_n,u}(t) \to 1$, then we have $G^*_{Tx_n,Sx_n,u}(t) \to 1$ for all $t > 0$.
• Suppose that $S(X)$ is a closed subset of X. Since $\{Sx_n\} \subset S(X)$ and $Sx_n \to u$, we have $u \in S(X)$ and there exists $a \in X$ such that $Sa = u$. So, we obtain

$$\lim_{n\to\infty} G^*_{Tx_n,Sx_n,Sa}(t) = 1, \tag{3.3}$$

for all $t > 0$.
• Suppose that $T(X)$ is a subset of X. Since $\{Tx_n\} \subset T(X)$ and $Tx_n \to u$, we have $u \in T(X) \subset S(X)$, and so there exists $a \in X$ such that $Sa = u$. Therefore, (3.3) still holds.

Now we show that $Ta = Sa$. Suppose that $Ta \neq Sa$. It is not difficult to prove that there exists $t_0 > 0$ such that

$$G^*_{Ta,Ta,Sa}(\frac{2t_0}{k}) > G^*_{Ta,Ta,Sa}(t_0). \tag{3.4}$$

In fact, if not, then we have $G^*_{Ta,Ta,Sa}(t) = G^*_{Ta,Ta,Sa}(\frac{2t}{k})$ for all $t > 0$. Repeatedly using this equality, we obtain

$$G^*_{Ta,Ta,Sa}(t) = G^*_{Ta,Ta,Sa}(\frac{2t}{k}) = \cdots = G^*_{Ta,Ta,Sa}((\frac{2}{k})^n t) \to 1 \qquad (n \to \infty).$$

This shows that $G^*_{Ta,Ta,Sa}(t) = 1$ for all $t > 0$, which contradicts $Ta \neq Sa$, and so (3.4) is proved.

Without loss of generality, we assume that t_0 in (3.4) is a continuous point of $G_{Ta,Ta,Sa}(\cdot)$. By the left-continuity of distribution function, there exists $\delta > 0$ such that

$$G^*_{Ta,Ta,Sa}(\frac{2t}{k}) > G^*_{Ta,Ta,Sa}(t),$$

for all $t \in (t_0 - \delta, t_0]$. Since $G_{Ta,Ta,Sa}(\cdot)$ is nondecreasing, the set of all discontinuous points of $G_{Ta,Ta,Sa}(\cdot)$ is a countable set at most. Thus, when t_0 is a discontinuous point of $G_{Ta,Ta,Sa}(\cdot)$, we can choose a continuous point t_1 of $G_{Ta,Ta,Sa}(\cdot)$ in $(t_0 - \delta, t_0]$ to replace t_0.

Because of $Ta \neq Sa$ and $\lim_{n\to\infty} Tx_n = Sa$, there exists $n_0 \in \mathbb{Z}^+$ such that $Tx_n \neq Ta$ for all $n \geq n_0$. By (3.1), we have

$$G^*_{Tx_n,Tx_n,Ta}(t_0) > \min\{G^*_{Sx_n,Sx_n,Sa}(t_0), [G^*_{Tx_n,Tx_n,Sx_n} \oplus G^*_{Ta,Ta,Sa}](\frac{2t_0}{k}), [G^*_{Ta,Ta,Sx_n} \oplus G^*_{Tx_n,Tx_n,Sa}](2t_0)\} \tag{3.5}$$

for all $n \geq n_0$. In addition, it is easy to verify that

$$\liminf_{n\to\infty} [G^*_{Tx_n,Tx_n,Sx_n} \oplus G^*_{Ta,Ta,Sa}](\frac{2t_0}{k}) \geq G^*_{Ta,Ta,Sa}(\frac{2t_0}{k}). \tag{3.6}$$

In fact, for any $\delta \in (0, \frac{2t_0}{k})$, we have

$$[G^*_{Tx_n,Tx_n,Sx_n} \oplus G^*_{Ta,Ta,Sa}](\frac{2t_0}{k}) \geq \min\{G^*_{Tx_n,Tx_n,Sx_n}(\delta), G^*_{Ta,Ta,Sa}(\frac{2t_0}{k} - \delta)\}.$$

Since $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Sa$, by Lemma 2.8 the above inequality implies that

$$\liminf_{n\to\infty} [G^*_{Tx_n,Tx_n,Sx_n} \oplus G^*_{Ta,Ta,Sa}](\frac{2t_0}{k}) \geq G^*_{Ta,Ta,Sa}(\frac{2t_0}{k} - \delta).$$

Letting $\delta \to 0$, by the left-continuity of distribution function, (3.6) is proved. In the same way, we can prove that

$$\liminf_{n\to\infty}[G^*_{Ta,Ta,Sx_n} \oplus G^*_{Tx_n,Tx_n,Sa}](2t_0) \geq G^*_{Ta,Ta,Sa}(2t_0). \tag{3.7}$$

Notice that t_0 is a continuous point of $G_{Ta,Ta,Sa}(\cdot)$, by Lemma 2.8, we have $\lim\limits_{n\to\infty} G_{Tx_n,Tx_n,Ta}(t_0) = G_{Sa,Sa,Ta}(t_0)$. Letting $n \to \infty$ in (3.5) and using (3.6) and (3.7), we get

$$G^*_{Sa,Sa,Ta}(t_0) \geq \min\{1, G^*_{Ta,Ta,Sa}(\frac{2t_0}{k}), G^*_{Ta,Ta,Sa}(2t_0)\} = G^*_{Ta,Ta,Sa}(\frac{2t_0}{k}). \tag{3.8}$$

Again by (3.2) and the same way above, we can obtain

$$G^*_{Sa,Ta,Ta}(t_0) \geq \min\{1, G^*_{Ta,Sa,Sa}(\frac{2t_0}{k}), G^*_{Ta,Sa,Sa}(2t_0)\} = G^*_{Ta,Sa,Sa}(\frac{2t_0}{k}). \tag{3.9}$$

By (3.8) and (3.9), we have

$$G^*_{Sa,Ta,Ta}(t_0) \geq G^*_{Ta,Sa,Sa}(\frac{2t_0}{k}) \geq G^*_{Ta,Sa,Sa}(t_0) \geq G^*_{Ta,Ta,Sa}(\frac{2t_0}{k}),$$

which is in contradiction to (3.4). Therefore $Ta = Sa$. Since S and T are weakly compatible, we have $TTa = TSa = STa = SSa$. We now show that Ta is a common fixed point of S and T. Suppose that $Ta \neq TTa$, then $a \neq Ta$. From (3.1), there exists some $t_* > 0$ such that

$$\begin{aligned} G^*_{Ta,Ta,TTa}(t_*) &> \min\{G^*_{Sa,Sa,STa}(t_*), [G^*_{Ta,Ta,Sa} \oplus G^*_{TTa,TTa,STa}](\frac{2t_*}{k}), [G^*_{TTa,TTa,Sa} \oplus G^*_{Ta,Ta,STa}](2t_*)\} \\ &= \min\{G^*_{Ta,Ta,TTa}(t_*), [G^*_{TTa,TTa,Ta} \oplus G^*_{Ta,Ta,TTa}](2t_*)\}. \end{aligned} \tag{3.10}$$

If $G^*_{Ta,Ta,TTa}(t_*) \leq [G^*_{TTa,TTa,Ta} \oplus G^*_{Ta,Ta,TTa}](2t_*)]$, it follows from (3.10) that $G^*_{Ta,Ta,TTa}(t_*) > G^*_{Ta,Ta,TTa}(t_*)$, which is a contradiction. Then we have

$$G^*_{Ta,Ta,TTa}(t_*) > [G^*_{TTa,TTa,Ta} \oplus G^*_{Ta,Ta,TTa}](2t_*).$$

Similary, by (3.2), we can also obtain

$$G^*_{Ta,TTa,TTa}(t_*) > [G^*_{TTa,Ta,Ta} \oplus G^*_{Ta,TTa,TTa}](2t_*).$$

So, we have

$$\min\{G^*_{Ta,Ta,TTa}(t_*), G^*_{Ta,TTa,TTa}(t_*)\} > [G^*_{Ta,Ta,Ta} \oplus G^*_{Ta,TTa,TTa}](2t_*). \tag{3.11}$$

On the other hand, it follows from Definition 2.14 that

$$\begin{aligned} [G^*_{TTa,TTa,Ta} \oplus G^*_{Ta,Ta,TTa}](2t_*) &= \sup_{t_1+t_2=2t_*} \min\{G^*_{TTa,TTa,Ta}(t_1), G^*_{Ta,Ta,TTa}(t_2)\} \\ &\geq \min\{G^*_{TTa,TTa,Ta}(t_*), G^*_{Ta,Ta,TTa}(t_*)). \end{aligned} \tag{3.12}$$

Combining (3.11) with (3.12) yields

$$\min\{G^*_{Ta,Ta,TTa}(t_*), G^*_{Ta,TTa,TTa}(t_*)\} > \min\{G^*_{Ta,Ta,TTa}(t_*), G^*_{Ta,TTa,TTa}(t_*)\},$$

which is a contradiction. Therefore $Ta = TTa$, and so $STa = TTa = Ta$. This shows that Ta is a common fixed point of S and T.

Finally, we prove the uniqueness. Suppose that p and q are two common fixed points of S and T, *i.e.*, $Sp = Tp = p$ and $Sq = Tq = q$. If $p \neq q$, then by (3.1) and (3.2), there exists some $t_1 > 0$ such that

$$\begin{aligned}G^*_{Tp,Tp,Tq}(t_1) &> \min\{G^*_{Sp,Sp,Sq}(t_1), [G^*_{Tp,Tp,Sp} \oplus G^*_{Tq,Tq,Sq}](\frac{2t_1}{k}), [G^*_{Tq,Tq,Sp} \oplus G^*_{Tp,Tp,Sq}](2t_1)\}\\ &= \min\{G^*_{p,p,q}(t_1), 1, [G^*_{q,q,p} \oplus G^*_{p,p,q}](2t_1)\}\\ &\geq \min\{G^*_{p,p,q}(t_1), 1, G^*_{p,p,q}(t_1), G^*_{p,q,q}(t_1)\}\\ &= \min\{G^*_{p,p,q}(t_1), G^*_{p,q,q}(t_1)\},\end{aligned}$$

and

$$\begin{aligned}G^*_{Tp,Tq,Tq}(t_1) &> \min\{G^*_{Sp,Sq,Sq}(t_1), [G^*_{Tp,Sp,Sp} \oplus G^*_{Tq,Sq,Sq}](\frac{2t_1}{k}), [G^*_{Tq,Sp,Sp} \oplus G^*_{Tp,Sq,Sq}](2t_1)\}\\ &= \min\{G^*_{p,p,q}(t_1), G^*_{p,q,q}(t_1)\}.\end{aligned}$$

Then, we have

$$\min\{G^*_{p,p,q}(t_1), G^*_{p,q,q}(t_1)\} > \min\{G^*_{p,p,q}(t_1), G^*_{p,q,q}(t_1)\},$$

which is a contradiction. Therefore, the common fixed points of S and T is unique. □

Taking $k = 1$ in Theorem 3.3, we get the following result.

Corollary 3.4. *Let (X, G^*, Δ) be a Menger PGM-space with a continuous t-norm Δ on $[0, 1] \times [0, 1]$, and let S and T be two weakly compatible self-mappings on (X, G^*, Δ) satisfying the following conditions:*

(1) S and T satisfy the property G^*-$(E.A)$;

(2) ′ for any $x, y \in X$, $x \neq y$, $t > 0$,

$$G^*_{Tx,Tx,Ty}(t) > \min\{G^*_{Sx,Sx,Sy}(t), [G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](2t), [G^*_{Ty,Ty,Sx} \oplus G^*_{Tx,Tx,Sy}](2t)\}$$

and

$$G^*_{Tx,Ty,Ty}(t) > \min\{G^*_{Sx,Sy,Sy}(t), [G^*_{Tx,Sx,Sx} \oplus G^*_{Ty,Sy,Sy}](2t), [G^*_{Ty,Sx,Sx} \oplus G^*_{Tx,Sy,Sy}](2t)\};$$

(3) $T(X) \subset S(X)$;

(4) $S(X)$ or $T(X)$ is a closed subset of X.

Then S and T have a unique common fixed point in X.

Theorem 3.5. *Let (X, G^*, Δ) be a Menger PGM-space with a continuous t-norm Δ on $[0, 1] \times [0, 1]$, and let S and T be two weakly compatible self-mappings on (X, G^*, Δ) satisfying the following conditions:*

(1)′ there exists a mapping $\phi : X \to \mathbb{R}^+$ such that

$$G^*_{Sx,Sy,Tz}(t) \geq H(t - (\phi(Sx) + \phi(Sy) - 2\phi(Tz))), \tag{3.13}$$

for all $x, y, z \in X$, $t \in \mathbb{R}$;

(2) for any $x, y \in X$, $x \neq y$, $t > 0$,

$$G^*_{Tx,Tx,Ty}(t) > \min\{G^*_{Sx,Sx,Sy}(t), [G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](\frac{2t}{k}), [G^*_{Ty,Ty,Sx} \oplus G^*_{Tx,Tx,Sy}](2t)\}$$

and

$$G^*_{Tx,Ty,Ty}(t) > \min\{G^*_{Sx,Sy,Sy}(t), [G^*_{Tx,Sx,Sx} \oplus G^*_{Ty,Sy,Sy}](\frac{2t}{k}), [G^*_{Ty,Sx,Sx} \oplus G^*_{Tx,Sy,Sy}](2t)\},$$

for some k, $1 \leq k < 2$;

(3) $T(X) \subset S(X)$;

(4)′ $S(X)$ or $T(X)$ is a complete subspace of X.

Then S and T have a unique common fixed point in X.

Proof. From Theorem 3.3, we only need to show that S and T satisfy the property G^*-$(E.A)$, *i.e.*, condition (1) in Theorem 3.3.

Taking $x_0 \in X$, by condition (3), we can choose $x_1 \in X$ such that $Tx_0 = Sx_1$. Choose $x_2 \in X$ such that $Tx_1 = Sx_2$. In general, choosing $x_n \in X$ such that $Tx_{n-1} = Sx_n$. Then, by (3.13) we get

$$G^*_{Sx_n,Sx_n,Sx_{n+1}}(t) = G^*_{Sx_n,Sx_n,Tx_n}(t) \geq H(t - 2(\phi(Sx_n) - \phi(Tx_n))) = H(t - 2(\phi(Sx_n) - \phi(Sx_{n+1}))).$$

Hence $G^*_{Sx_n,S_n,S_{n+1}}(t) = 1$, where $t > 2(\phi(Sx_n) - \phi(Sx_{n+1}))$. So, by Lemma 2.9 , we have $G^*_\lambda(Sx_n, Sx_n, Sx_{n+1}) < t$ for all $\lambda \in (0, 1]$. Letting $t \to 2(\phi(Sx_n) - \phi(Sx_{n+1}))$, we obtain

$$0 \leq G_\lambda(Sx_n, Sx_n, Sx_{n+1}) \leq 2(\phi(Sx_n) - \phi(Sx_{n+1})).$$

It is not difficult to see that the sequence $\{\phi(Sx_n)\}$ is nonincreasing and bounded below, hence it is a convergent sequence.

On the other hand, from (3.13) we can also obtain

$$G^*_{Sx_n,Sx_n,Sx_{n+m}}(t) \geq H(t - 2(\phi(Sx_n) - \phi(Sx_{n+m}))).$$

Let $n \to \infty$, then $G^*_{Sx_n,Sx_n,S_{n+m}}(t) \to 1$ for all $t > 0$ and $m \in \mathbb{Z}^+$. By Lemma 2.10, $\{Sx_n\}$ is a *Cauchy* sequence in $S(X)$. As $Sx_n = Tx_{n-1} \in T(X)$, $\{Sx_n\}$ is also a *Cauchy* sequence in $T(X)$. And then, by condition $(4)'$, there exist $u \in S(X)$ or $u \in T(X)$ such that $\lim_{n\to\infty} Sx_n = u$. Obviously, we also have $\lim_{n\to\infty} Tx_n = u$. This shows that S and T satisfy the property G^*-$(E.A)$. Moreover, it is evident that condition $(4)' \Rightarrow (4)$. Therefore the conclusion follows from Theorem 3.3 immediately. □

Remark 3.6. In Theorem 3.5, if we replace condition (2) by condition $(2)'$ of Corollary 3.4, then the conclusion of the theorem still holds.

Taking $S = I_X$ (the identity mapping on X) and $k = 1$ in Theorem 3.5, we get the following result.

Corollary 3.7. *Let (X, G^*, Δ) be a Menger PGM-space with a continuous t-norm Δ on $[0, 1] \times [0, 1]$, and let T be a weakly compatible self-mapping on (X, G^*, Δ) satisfying the following conditions:*

(1)′ there exists a mapping $\phi : X \to \mathbb{R}^+$ such that

$$G^*_{x,y,Tz}(t) \geq H(t - (\phi(x) + \phi(y) - 2\phi(Tz))),$$

for all $x, y, z \in X$, $t \in \mathbb{R}$;

(2) for any $x, y \in X$, $x \neq y$, $t > 0$,

$$G^*_{Tx,Tx,Ty}(t) > \min\{G^*_{x,x,y}(t), [G^*_{Tx,Tx,x} \oplus G^*_{Ty,Ty,y}](2t), [G^*_{Ty,Ty,x} \oplus G^*_{Tx,Tx,y}](2t)\}$$

and

$$G^*_{Tx,Ty,Ty}(t) > \min\{G^*_{x,y,y}(t), [G^*_{Tx,x,x} \oplus G^*_{Ty,y,y}](2t), [G^*_{Ty,x,x} \oplus G^*_{Tx,y,y}](2t)\}.$$

Then T has a unique fixed point in X.

Also, we have the following corollary.

Corollary 3.8. *Let (X, G^*, Δ) be a compact Menger PGM-space with a continuous t-norm Δ on $[0, 1] \times [0, 1]$, T be a self-mapping on (X, G^*, Δ) satisfying the following conditions:*

(i) there exists an $x_0 \in X$, such that $G^*_{T^n x_0, T^{n+1} x_0, T^{n+1} x_0}(t) \to 1 \quad (n \to \infty)$ for all $t > 0$;

(ii) for any $x, y \in X$, $x \neq y$, $t > 0$,

$$G^*_{Tx,Tx,Ty}(t) > \min\{G^*_{x,x,y}(t), [G^*_{Tx,Tx,x} \oplus G^*_{Ty,Ty,y}](2t), [G^*_{Ty,Ty,x} \oplus G^*_{Tx,Tx,y}](2t)\}$$

and

$$G^*_{Tx,Ty,Ty}(t) > \min\{G^*_{x,y,y}(t), [G^*_{Tx,x,x} \oplus G^*_{Ty,y,y}](2t), [G^*_{Ty,x,x} \oplus G^*_{Tx,y,y}](2t)\}.$$

Then T has a unique fixed point in X.

Proof. Taking $S = I_X$, it is evident that S and T are weakly compatible mappings and satisfy condition (2)-(4). In the following, we need to show that S and T satisfy the property G^*-$(E.A)$. By (i), putting $x_n = T^n(x_0)$, we have $G^*_{x_n,Tx_n,Tx_n}(t) \to 1 \quad (n \to \infty)$ for all $t > 0$. Since X is compact, there exists subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and Tx_{n_k} of Tx_n such that $x_{n_k} \to x \in X$ and $Tx_{n_k} \to y \in X$. Thus, we have

$$G^*_{x_{n_k},y,y}(t) \geq \Delta(G^*_{x_{n_k},Tx_{n_k},Tx_{n_k}}(\frac{t}{2}), G^*_{Tx_{n_k},y,y}(\frac{t}{2})) \to 1 \quad (k \to \infty)$$

for all $t > 0$, which implies that $\lim_{k\to\infty} Sx_{n_k} = \lim_{k\to\infty} Tx_{n_k} = y$, *i.e.*, S and T satisfy the property G^*-$(E.A)$. This shows that all the conditions of Theorem 3.3 are satisfied, and so the conclusion follows from Theorem 3.3 immediately. □

4. Common fixed point theorems in *G*-metric spaces

In this section, we shall apply the results obtained in Section 3 to establish the corresponding common fixed point theorems under strict contractive conditions in G-metric spaces.

Theorem 4.1. *Let S and T be two weakly compatible self-mappings of a G-metric space (X, G). If the following conditions are satisfied:*

(i) S and T satisfy the property G-$(E.A)$;

(ii) for any $x, y \in X$, $x \neq y$,

$$G(Tx, Tx, Ty) \quad < \max\{G(Sx, Sx, Sy), \tfrac{k[G(Tx,Tx,Sx)+G(Ty,Ty,Sy)]}{2}, \tfrac{[G(Ty,Ty,Sx)+G(Tx,Tx,Sy)]}{2}\} \quad (4.1)$$

and

$$G(Tx, Ty, Ty) \quad < \max\{G(Sx, Sy, Sy), \tfrac{k[G(Tx,Sx,Sx)+G(Ty,Sy,Sy)]}{2}, \tfrac{[G(Ty,Sx,Sx)+G(Tx,Sy,Sy)]}{2}\}, \quad (4.2)$$

for some k, $1 \leq k < 2$;

(iii) $T(X) \subset S(X)$;

(iv) $S(X)$ or $T(X)$ is a closed subset of X.

Then S and T have a unique common fixed point in X.

Proof. Let (X, G^*, Δ_m) be the PGM-space induced by (X, G), where G^* is defined by (2.1). It is easy to see that condition (i), (iii), (iv) of Theorem 4.1 imply condition (1), (3), (4) of Theorem 3.3, respectively. It remains to be proved that condition (ii) of Theorem 4.1 implies condition (2) of Theorem 3.3.

For any $x, y \in X$, $x \neq y$ and $t > 0$, we first verify that

$$G^*_{Tx,Ty,Ty}(t) \geq \min\{G^*_{Sx,Sy,Sy}(t), [G^*_{Tx,Sx,Sx} \oplus G^*_{Ty,Sy,Sy}](\frac{2t}{k}), [G^*_{Ty,Sx,Sx} \oplus G^*_{Tx,Sy,Sy}](2t)\}. \quad (4.3)$$

If $t > G(Tx, Ty, Ty)$, then $G^*_{Tx,Ty,Ty}(t) = 1$. It is clear that (4.3) holds. If $t \leq G(Tx, Ty, Ty)$, we can consider the following three cases:

Case(a): $t < G(Sx, Sy, Sy)$, we have $G^*_{Sx,Sy,Sy}(t) = 0$, and so (4.3) holds.

Case(b): $t < \frac{k[G(Tx,Sx,Sx)+G(Ty,Sy,Sy)]}{2}$, *i.e.*, $\frac{2}{k} < [G(Tx,Sx,Sx)+G(Ty,Sy,Sy)]$. Thus, for any $t_1,t_2 > 0$ with $t_1+t_2 = \frac{2t}{k}$, we have $G(Tx,Sx,Sx) > t_1$ or $G(Ty,Sy,Sy) > t_2$, *i.e.*, at least one of $G^*_{Tx,Sx,Sx}(t_1) = 0$ and $G^*_{Ty,Sy,Sy}(t_2) = 0$ holds. Hence,

$$[G^*_{Tx,Sx,Sx} \oplus G^*_{Ty,Sy,Sy}](\frac{2t}{k}) = \sup_{t_1+t_2=\frac{2t}{k}} \min\{G^*_{Tx,Sx,Sx}(t_1), G^*_{Ty,Sy,Sy}(t_2)\} = 0,$$

and so (4.3) holds.

Case(c): $t < [G^*_{Ty,Sx,Sx} \oplus G^*_{Tx,Sy,Sy}](2t)$. By using the same way as (b), it is not difficult to show that $[G^*_{Ty,Sx,Sx} \oplus G^*_{Tx,Sy,Sy}](2t) = 0$, and so (4.3) holds. In view of the above discussions, we conclude that (4.3) is true. Next, by (4.2), we can choose a $t_0 > 0$ such that

$$\begin{aligned} G(Tx,Ty,Ty) \quad &< \quad t_0 < \max\{G(Sx,Sy,Sy), \\ &\frac{k[G(Tx,Sx,Sx)+G(Ty,Sy,Sy)]}{2}, \frac{[G(Ty,Sx,Sx)+G(Tx,Sy,Sy)]}{2}\}, \end{aligned}$$

which implies that $G^*_{Tx,Ty,Ty}(t_0) = 1$ and

$$\min\{G^*_{Sx,Sy,Sy}(t), [G^*_{Tx,Sx,Sx} \oplus G^*_{Ty,Sy,Sy}](\frac{2t}{k}), [G^*_{Ty,Sx,Sx} \oplus G^*_{Tx,Sy,Sy}](2t)\} = 0.$$

Hence,

$$G^*_{Tx,Ty,Ty}(t_0) > \min\{G^*_{Sx,Sy,Sy}(t_0), [G^*_{Tx,Sx,Sx} \oplus G^*_{Ty,Sy,Sy}](\frac{2t_0}{k}), [G^*_{Ty,Sx,Sx} \oplus G^*_{Tx,Sy,Sy}](2t_0)\}. \tag{4.4}$$

Similarly, we can verify that,

$$G^*_{Tx,Tx,Ty}(t) \geq \min\{G^*_{Sx,Sx,Sy}(t), [G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](\frac{2t}{k}), [G^*_{Ty,Ty,Sx} \oplus G^*_{Tx,Tx,Sy}](2t)\} \tag{4.5}$$

for any $x,y \in X$, $x \neq y$ and $t > 0$, and there exists a $t_1 > 0$ such that

$$G^*_{Tx,Tx,Ty}(t_1) > \min\{G^*_{Sx,Sx,Sy}(t_1), [G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](\frac{2t_1}{k}), [G^*_{Ty,Ty,Sx} \oplus G^*_{Tx,Tx,Sy}](2t_1)\}. \tag{4.6}$$

By (4.3)-(4.6) and Definition 2.15, we know that (3.1) and (3.2) hold. Therefore, all the conditions of Theorem 3.3 are satisfied and the conclusion follows from it immediately. □

Similarly, from Corollary 3.4 we get the following corollary.

Corollary 4.2. *Let S and T be two weakly compatible self-mappings of a G-metric space (X,G). If conditions (i), (iii), (iv) of Theorem 4.1 and the following condition (ii)$'$ are satisfied:*

(ii)$'$ for any $x,y \in X$, $x \neq y$

$$G(Tx,Tx,Ty) \quad < \max\{G(Sx,Sx,Sy), \tfrac{[G(Tx,Tx,Sx)+G(Ty,Ty,Sy)]}{2}, \tfrac{[G(Ty,Ty,Sx)+G(Tx,Tx,Sy)]}{2}\}$$

and

$$G(Tx,Ty,Ty) \quad < \max\{G(Sx,Sy,Sy), \tfrac{[G(Tx,Sx,Sx)+G(Ty,Sy,Sy)]}{2}, \tfrac{[G(Ty,Sx,Sx)+G(Tx,Sy,Sy)]}{2}\},$$

for some k, $1 \leq k < 2$.

Then S and T have a unique common fixed point in X.

From Theorem 3.5 we can also obtain the following corollary.

Corollary 4.3. *Let S and T be two weakly compatible self-mappings of a G-metric space (X,G) satisfy the following conditions:*

(i)′ there exist a mapping $\phi : X \to \mathbb{R}^+$ such that

$$G(Sx, Sy, Tz) \leq \phi(Sx) + \phi(Sy) - 2\phi(Tz), \tag{4.7}$$

for all $x, y, z \in X$;

(ii) for any $x, y \in X$, $x \neq y$

$$G(Tx, Tx, Ty) < \max\{G(Sx, Sx, Sy), \tfrac{k[G(Tx,Tx,Sx)+G(Ty,Ty,Sy)]}{2}, \tfrac{[G(Ty,Ty,Sx)+G(Tx,Tx,Sy)]}{2}\} \tag{4.8}$$

and

$$G(Tx, Ty, Ty) < \max\{G(Sx, Sy, Sy), \tfrac{k[G(Tx,Sx,Sx)+G(Ty,Sy,Sy)]}{2}, \tfrac{[G(Ty,Sx,Sx)+G(Tx,Sy,Sy)]}{2}\}, \tag{4.9}$$

for some k, $1 \leq k < 2$;

(iii) $T(X) \subset S(X)$;

(iv)′ $S(X)$ or $T(X)$ is a complete subspace of X.

Then S and T have a unique common fixed point.

Proof. Let (X, G^*, Δ_m) be PGM-space induced by (X, G). Obviously, conditions (ii), (iii) and (iv)′ imply conditions (2), (3) and (4)′ of Theorem 3.5 respectively. In addition, it is not difficult to prove that (4.7) implies (3.13).

In fact, if $t > G(Sx, Sx, Tx)$, then $G^*_{Sx,Sx,Tx}(t) = 1$, and so (3.13) holds. If $t \leq G(Sx, Sx, Tx)$, then it follows from (4.7) that $t \leq \phi(Sx) + \phi(Sy) - 2\phi(Tz)$ for all $x \in X$. Hence

$$G^*_{Sx,Sx,Tx}(t) = 0 = H(t - (\phi(Sx) + \phi(Sy) - 2\phi(Tz))),$$

and so (3.13) holds. Therefore, all conditions of Theorem 3.5 are satisfied and the conclusion follows from Theorem 3.5 immediately. □

Remark 4.4. If we replace condition (ii) in Corollary 4.3 by condition (ii)′ of Corollary 4.2, then the conclusion of Corollary 4.3 still holds.

Taking $S = I_X$ and $k = 1$ in Corollary 4.3, we get the following result.

Corollary 4.5. *Let T be a weakly compatible self-mapping on a G-metric space (X, G) satisfying the following conditions:*

(i) there exists a mapping $\phi : X \to \mathbb{R}^+$ such that

$$G(x, y, Tz) \leq \phi(x) + \phi(y) - 2\phi(Tz) \tag{4.10}$$

for all $x, y, z \in X$;

(ii) for any $x, y \in X$, $x \neq y$,

$$G(Tx, Tx, Ty) < \max\{G(x, x, y), \tfrac{[G(Tx,Tx,x)+G(Ty,Ty,y)]}{2}, \tfrac{[G(Ty,Ty,x)+G(Tx,Tx,y)]}{2}\}$$

and

$$G(Tx, Ty, Ty) < \max\{G(x, y, y), \tfrac{[G(Tx,x,x)+G(Ty,y,y)]}{2}, \tfrac{[G(Ty,x,x)+G(Tx,y,y)]}{2}\}.$$

Then T has a unique fixed point in X.

5. An application

In this section, we will provide an example to show the validity of Theorem 3.3 of this paper.

Example 5.1. Consider $X = (-1, 1)$ and define $G^*_{x,y,z} = \frac{t}{t+|x-y|+|y-z|+|z-x|}$ for all $x, y \in X$ with $t > 0$. Then, by Example 2.1, (X, G^*, Δ_m) is a PGM-space . Define $T, S : X \to X$ as follows:

$$T(x) = \begin{cases} \frac{1}{5}, & x \in (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1), \\ \frac{1}{3}x, & x \in [-\frac{1}{2}, \frac{1}{2}]. \end{cases}$$

$$S(x) = \begin{cases} \frac{1}{3}, & x \in (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1), \\ \frac{1}{2}x, & x \in [-\frac{1}{2}, \frac{1}{2}]. \end{cases}$$

Consider the sequences $\{x_n = \frac{1}{n+1}\}$ and $\{y_n = -\frac{1}{n+1}\}$ in X, then

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0,$$

which shows that T and S are two weakly compatible self-mappings and also satisfy the common property G^*-$(E.A)$. Also T and S are closed subsets of X. By a routine calculation, one can verify that (3.1) holds for all $x, y \in X$, $x \neq y$, $t > 0$ and some $1 \leq k < 2$.

In fact, if $x, y \in (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$, $x \neq y$, then for any $t > 0$, $G^*_{Tx,Tx,Ty}(t) = G^*_{\frac{1}{5},\frac{1}{5},\frac{1}{5}}(t) = 1$,

$$\begin{aligned} [G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](\frac{2t}{k}) &= \sup_{t_1+t_2=\frac{2t}{k}} \min\{G^*_{Tx,Tx,Sx}(t_1), G^*_{Ty,Ty,Sy}(t_2)\} \\ &= \sup_{t_1+t_2=\frac{2t}{k}} \min\{G^*_{\frac{1}{5},\frac{1}{5},\frac{1}{3}}(t_1), G^*_{\frac{1}{5},\frac{1}{5},\frac{1}{3}}(t_2)\} \\ &= \sup_{t_1+t_2=\frac{2t}{k}} \min\{\frac{t_1}{t_1 + 2|\frac{1}{3} - \frac{1}{5}|}, \frac{t_2}{t_2 + 2|\frac{1}{3} - \frac{1}{5}|}\} < 1. \end{aligned}$$

So we have

$$G^*_{Tx,Tx,Ty}(t) > \min\{G^*_{Sx,Sx,Sy}(t), [G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](\frac{2t}{k}), [G^*_{Ty,Ty,Sx} \oplus G^*_{Tx,Tx,Sy}](2t)\}.$$

If $x, y \in [-\frac{1}{2}, \frac{1}{2}]$, $x \neq y$, then for any $t > 0$,

$$G^*_{Tx,Tx,Ty}(t) = 1,$$

and for any $t_1 + t_2 = \frac{2t}{k}$, at least one of $G^*_{Tx,Tx,Sx}(t_1) = \frac{t_1}{t_1+\frac{|x|}{3}} < 1$ and $G^*_{Ty,Ty,Sy}(t_2) = \frac{t_2}{t_2+\frac{|y|}{3}} < 1$ holds. Hence, $[G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](\frac{2t}{k}) < 1$. Then

$$\begin{aligned} G^*_{Tx,Tx,Ty}(t) &> [G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](\frac{2t}{k}) \\ &\geq \min\{G^*_{Sx,Sy,Sy}(t), [G^*_{Tx,Sx,Sx} \oplus G^*_{Ty,Sy,Sy}](\frac{2t}{k}), [G^*_{Ty,Sx,Sx} \oplus G^*_{Tx,Sy,Sy}](2t)\}. \end{aligned} \tag{5.1}$$

If $x \in (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$, $y \in [-\frac{1}{2}, \frac{1}{2}]$, $x \neq y$, then for any $t > 0$,

$$\begin{aligned} G^*_{Tx,Tx,Ty}(t) &= \frac{t}{t + 2|\frac{1}{5} - \frac{y}{3}|} > \frac{t}{t + 2|\frac{1}{3} - \frac{y}{2}|} = G^*_{Sx,Sx,Sy}(t) \\ &\geq \min\{G^*_{Sx,Sx,Sy}(t), [G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](\frac{2t}{k}), [G^*_{Ty,Ty,Sx} \oplus G^*_{Tx,Tx,Sy}](2t)\}. \end{aligned}$$

If $y \in (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$, $x \in [-\frac{1}{2}, \frac{1}{2}]$, $x \neq y$, then for any $t > 0$,

$$\begin{aligned} G^*_{Tx,Tx,Ty}(t) &= \frac{t}{t + 2|\frac{1}{5} - \frac{x}{3}|} > \frac{t}{t + 2|\frac{1}{3} - \frac{x}{2}|} = G^*_{Sx,Sx,Sy}(t) \\ &\geq \min\{G^*_{Sx,Sx,Sy}(t), [G^*_{Tx,Tx,Sx} \oplus G^*_{Ty,Ty,Sy}](\frac{2t}{k}), [G^*_{Ty,Ty,Sx} \oplus G^*_{Tx,Tx,Sy}](2t)\}. \end{aligned}$$

In view of the above discussions, we conclude that (3.1) is satisfied.

Similarly, we can verify that

$$G^*_{Tx,Ty,Ty}(t) > \min\{G^*_{Sx,Sy,Sy}(t), [G^*_{Tx,Sx,Sx} \oplus G^*_{Ty,Sy,Sy}](\frac{2t}{k}), [G^*_{Ty,Sx,Sx} \oplus G^*_{Tx,Sy,Sy}](2t)\},$$

for any $x, y \in X$, $x \neq y$, $t > 0$.

Thus, all the conditions of Theorem 3.3 are satisfied, so T and S have a unique common fixed point in X. In fact, 0 is the unique common fixed points of T and S.

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