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Mazur-Ulam theorem for probabilistic 2-normed spaces

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Abstract

In this paper we prove the Mazur-Ulam theorem for probabilistic 2-normed spaces. Our study is a natural continuation of that of Cobzas [S. Cobzas, Aequationes Math., 77 (2009) 197–205].

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1. Introduction

A mapping T from a metric space X into a metric space Y is called an isometry map if T satisfies $d_Y(T(x), T(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. The map T is called affine if T is linear up to translation.

Mazur and Ulam [11], proved that every isometry T from a real normed space X onto another real normed space Y is affine, while Baker [5] proved that an isometry map from a real normed linear space X into a strictly convex real normed linear space Y is affine.

For related works on this subject, we refer the reader to Aleksandrov [1], Cobzas [6], Chu *et al.* [7, 8, 9], and Rassias *et al.* [13, 17, 18].

Probabilistic metric spaces are spaces on which there is a distance function taking as values distribution functions, the distance between two points a and b is a distribution function in the sense of probability theory $\nu(a, b)$, whose values $\nu(p, q)(x)$ can be interpreted as the probability that the distance between a and b is less than x. The notion of probabilistic metric space was introduced by Menger [12]. The idea of Menger's was to use distribution functions instead of nonnegative real numbers as values of the metric.

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Probabilistic normed spaces were introduced by $\tilde{S}$ertnev in 1963 [19]. New definitions of probabilistic normed spaces were studied by Alsina *et al.* [2, 3, 4]. It is remarkable that the probabilistic generalization of metric spaces appears to be well adapted for the investigation of quantum particle physics, particularly in connections with both string and ε^∞ theory, which where given and studied by El Naschie [14, 15].

The notion of the probabilistic n-normed space was introduced by A. Poumoslemi and M. Salimi [16], while the notion of probabilistic 2-normed space was introduced by I. Golet [10]. In 2009, S. Cobzas studied the Mazur-Ulam theorem for probabilistic normed spaces [6].

In this paper, we study the Mazur-Ulam theorem for probabilistic 2-normed spaces.

2. Basic Concepts

Denote by $\triangle$ the set of distribution functions, meaning, nondecreasing, left continuous functions $\nu\colon \mathbb{R} \to [0,1]$, with $\nu(-\infty) = 0$ and $\nu(\infty) = 1$. Let D be the subclass of $\triangle$ formed by all functions $\nu \in \triangle$ such that

$$\lim_{x\to-\infty} \nu(x) = 0 \text{ and } \lim_{x\to\infty} \nu(x) = 1.$$

The set of distance functions are

$$\triangle^+ = \{\nu \in \triangle : \nu(0) = 0\} \quad \text{and} \quad D^+ = D \cap \triangle^+.$$

It follows that for $\nu \in D^+$, we have $\nu(x) = 0$ for all $x \le 0$. Two important distance functions are

$$\varepsilon_0(x) = \begin{cases} 0, & x \le 0; \\ 1, & x > 1 \end{cases}$$

and

$$\varepsilon_\infty(x) = \begin{cases} 0, & x < \infty; \\ 1, & x = \infty \end{cases}$$

A triangle function T is a binary operation on $\triangle^+$ that is commutative and associative, nondecreasing in each place and has ε_0 as identity, that is $T(\nu, \varepsilon_0) = \nu$. A t-norm is a continuous binary operation on $[0,1]$, that is commutative, associative, nondecreasing in each variable and has 1 as identity. The triangle function τ_T associated to a t-norm T is defined by

$$\tau_T(F,G)(x) = \sup\{T(F(s), G(t)) : s + t = x\}.$$

In this paper we are interested in the definition of probabilistic n-normed spaces, specially in the case of $n = 2$.

Definition 2.1 ([16])**.** Let X be a real linear space with $\dim X \ge n$, let T be a triangle function, and let ν be a mapping from X into D^+. If the following conditions are satisfied:

1. $\nu(x_1, \dots, x_n) = \varepsilon_0$ if $x_1, \dots, x_n$ are linearly dependent,
2. $\nu(x_1, \dots, x_n) \ne \varepsilon_0$ if $x_1, \dots, x_n$ are linearly independent,
3. $\nu(x_1, \dots, x_n) = \nu(x_{j1}, \dots, x_{jn})$ for any permutation $(j1, j2, \dots, jn)$ of $(1, 2, \dots, n)$
4. $\nu(\beta x_1, \dots, x_n) = \nu(x_1, \dots, x_n)\left(\frac{s}{|\beta|}\right)$, for every $s > 0$, and $\beta \ne 0$,
5. $\nu(x_1, \dots, x_{n-1}, x_n + y) \ge T(\nu(x_1, \dots, x_{n-1}, x_n), \nu(x_1, \dots, x_{n-1}, y))$

for $y, x_1, \dots, x_n \in X$, then ν is called a probabilistic 2-norm on X and the triple (X, ν, T) is called a probabilistic 2-normed space.

Definition 2.2. Let X be a real linear space and x, y, z mutually disjoint elements of X. Then x, y and z are said to be 2-collinear if

$$y - z = t(x - z),$$

for some real number t.

3. Main Results

We start our work by giving the definition of probabilistic 2-normed space.

Definition 3.1 ([10])**.** Let X be a real linear space with $\dim X \geq 2$, let T be a triangle function, and let ν be a mapping from X into D^+. If the following conditions are satisfied:

1. $\nu(x_1, x_2) = \varepsilon_0$ if x_1 and x_2 are linearly dependent,
2. $\nu(x_1, x_2) \neq \varepsilon_0$ if x_1 and x_2 are linearly independent,
3. $\nu(x_1, x_2) = \nu(x_2, x_1)$,
4. $\nu(\beta x_1, x_2) = \nu(x_1, x_2)\left(\frac{s}{|\beta|}\right)$, for every $s > 0$, and $\beta \neq 0$,
5. $\nu(x_1 + x_2, y) \geq T(\nu(x_1, y), \nu(x_2, y))$

for $y, x_1, x_2 \in X$, then ν is called a probabilistic 2-norm on X and the triple (X, ν, T) is called a probabilistic 2-normed space.

From now on, unless otherwise stated, we let (X, ν, T) and (Y, ν, T) be probabilistic 2-normed spaces.

In our work, we assume that: If x and y are linearly independent elements in X or in Y, then $\nu(x, y)$ is strictly increasing.

The following lemma due to A. Pourmoslemi and M. Salimi [16] is crucial in proving our next result.

Lemma 3.2 ([16])**.** For $x_1, x_2 \in X$ and $\alpha \in \mathbb{R}$, we have

$$\nu(x_1, \alpha x_1 + x_2) = \nu(x_1, x_2).$$

The following result is essential for proving our main result.

Lemma 3.3. *Let x_1 and x_2 be any two distinct elements in X, and let*

$$u = \frac{x_1 + x_2}{2}.$$

Then u is the unique element in X satisfying for all $s > 0$ the following equalities:

$$\nu(x_1 - u, x_1 - c)(s) = \nu(x_2 - c, x_2 - u)(s) = \nu(x_1 - c, x_2 - c)(2s)$$

for $c \in X$ where $x_1 - c$ and $x_2 - c$ are linearly independent and x_1, x_2, u are 2-collinear.

Proof. Choose $c \in X$ with $x_1 - c, x_2 - c$ being linearly independent. For $s > 0$ we have

$$\begin{aligned}
\nu(x_1 - u, x_1 - c)(s) &= \nu\left(x_1 - \frac{x_1 + x_2}{2}, x_1 - c\right)(s) \\
&= \nu\left(\frac{x_1 - x_2}{2}, x_1 - c\right)(s) \\
&= \nu(x_1 - x_2, x_1 - c)(2s) \\
&= \nu(x_1 - c + c - x_2, x_1 - c)(2s) \\
&= \nu(x_2 - c, x_1 - c)(2s) \\
&= \nu(x_1 - c, x_2 - c)(2s).
\end{aligned}$$

Similarly, we can show that

$$\nu(x_2 - c, x_2 - u)(s) = \nu(x_1 - c, x_2 - c)(2s).$$

To prove the uniqueness, assume that w is an element in X satisfying for all $s > 0$ the equalities:

$$\nu(x_1 - w, x_1 - c)(s) = \nu(x_2 - c, x_2 - w)(s) = \nu(x_1 - c, x_2 - c)(2s) \tag{3.1}$$

for $c \in X$ where $x_1 - c$ and $x_2 - c$ are linearly independent and x_1, x_2, w are 2-collinear. Since x_1, x_2, w are 2-collinear, there is a scalar t such that $w = (1-t)x_1 + tx_2$. Hence for $s > 0$, we have

$$\begin{aligned}\nu(x_1 - w, x_1 - c)(s) &= \nu(x_1 - (1-t)x_1 - tx_2), x_1 - c)(s)\\ &= \nu(tx_1 - tx_2 - ct + ct), x_1 - c)(s)\\ &= \nu(t(x_1 - c) - t(x_2 - c), x_1 - c)(s)\\ &= \nu(-t(x_2 - c), x_1 - c)(s)\\ &= \nu(x_1 - c, x_2 - c)\left(\frac{s}{|t|}\right)\end{aligned}$$

and

$$\begin{aligned}\nu(x_2 - c, x_2 - w)(s) &= \nu(x_2 - c, (1-t)x_2 - (1-t)x_1)(s)\\ &= \nu(x_2 - c, (1-t)x_2 - (1-t)x_1 - (1-t)c + (1-t)c)(s)\\ &= \nu(x_2 - c, (1-t)(x_2 - c) - (1-t)(x_1 - c))(s)\\ &= \nu(x_2 - c, -(1-t)(x_1 - c))(s)\\ &= \nu(x_2 - c, x_1 - c)\left(\frac{s}{|1-t|}\right)\\ &= \nu(x_1 - c, x_2 - c)\left(\frac{s}{|1-t|}\right).\end{aligned}$$

Since w satisfies Equation (3.1) and $\nu(x_1 - c, x_2 - c)$ is strictly increasing, we get that

$$2 = \frac{1}{|1-t|} = \frac{1}{|t|}.$$

So we conclude that $t = \frac{1}{2}$, and hence $w = u$. □

Using similar arguments as in the proof of Lemma 3.3, we can prove the following result.

Lemma 3.4. *Let x_1 and x_2 be any two distinct elements in X. Let*

$$u = \frac{x_1 + x_2}{2}.$$

Then u is the unique element in X satisfying for all $s > 0$ the following equalities:

$$\nu(u - x_1, x_2 - c)(s) = \nu(x_1 - c, u - x_2)(s) = \nu(x_1 - c, x_2 - c)(2s),$$

for $c \in X$ where $x_1 - c$ and $x_2 - c$ are linearly independent and x_1, x_2, u are 2-collinear.

To achieve our main result we introduce the following definition.

Definition 3.5. Let (X, ν, T) and (Y, ν, T) be probabilistic 2-normed spaces. We call the map $f\colon X \to Y$ probabilistic 2-isometry if

$$\nu(f(x) - f(c), f(y) - f(c))(s) = \nu(x - c, y - c)(s)$$

holds, for all $x, y, c \in X$ and all $s > 0$.

Lemma 3.6. *Let $f\colon X \to Y$ be probabilistic 2-isometry from probabilistic 2-normed space (X, ν, T) into probabilistic 2-normed space (Y, ν, T). Define the map f from (X, ν, T) into (Y, ν, T) by the rule $g(x) = f(x) - f(0)$. Then f is probabilistic 2-isometry iff g is probabilistic 2-isometry.*

Proof. Assume that f is probabilistic 2-isometry, then for $a, b, c \in X$ and $s > 0$ we have

$$\begin{aligned}\nu(g(a)-g(c), g(b)-g(c))(s) &= \nu(f(a)-f(0)-(f(c)-f(0)), f(b)-f(0)-(f(c)-f(0)))(s)\\ &= \nu(f(a)-f(c), f(b)-f(c))(s)\\ &= \nu(a-c, b-c)(s).\end{aligned}$$

So g is probabilistic 2-isometry.

Similarly we may show that if g is probabilistic 2-isometry, then f is probabilistic 2-isometry. □

We have furnished all necessary background to introduce and prove our main result.

Theorem 3.7. *Let $f\colon X \to Y$ be probabilistic 2-isometry from probabilistic 2-normed space (X, ν, T) into probabilistic 2-normed space (Y, ν, T) with the property that if a, b, and c are 2-collinear in X, then $f(a), f(b)$, and $f(c)$ are 2-collinear in Y. Then f is affine.*

Proof. By Lemma 3.6, we may assume that $f(0) = 0$. So it suffices to prove that f is linear. Let x and y be two distinct elements in X, and $u = \frac{x+y}{2}$. Since $\dim X \geq 2$, there is $c \in X$ such that $x - c$ and $y - c$ are linearly dependent. Now for $s > 0$, we have

$$\begin{aligned}\nu(f(x)-f(u), f(x)-f(c))(s) &= \nu(x-u, x-c)(s)\\ &= \nu\left(x - \frac{x+y}{2}, x-c\right)\\ &= \nu\left(\frac{x-y}{2}, x-c\right)(s)\\ &= \nu(x-c-(y-c), x-c)(2s)\\ &= \nu(y-c, x-c)(2s)\\ &= \nu(f(y)-f(c), f(x)-f(c))(2s)\\ &= \nu(f(x)-f(c), f(y)-f(c))(2s).\end{aligned}$$

Similarly, we may prove that

$$\nu(f(y)-f(u), f(y)-f(c))(s) = \nu(f(x)-f(c), f(y)-f(c))(2s).$$

By Lemma 3.3, we conclude that

$$f(u) = f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}. \tag{3.2}$$

For $x \in X$, $s > 0$, and $\alpha \in \mathbb{R}^+ \setminus \{0\}$, we have

$$\varepsilon_0(s) = \nu(\alpha x, x)(s) = \nu(\alpha x - 0, x - 0)(s) = \nu(f(\alpha x) - f(0), f(x) - f(0))(s) = \nu(f(\alpha x), f(x))(s).$$

So $f(\alpha x)$ and $f(x)$ are linearly dependent. Hence there is $k \in \mathbb{R}$ such that $f(\alpha x) = kf(x)$. Choose $y \in X$ such that x and y are linearly independent. Then for $s > 0$, we have

$$\begin{aligned}\nu(x, y)\left(\frac{s}{\alpha}\right) &= \nu(\alpha x, y)(s) = \nu(f(\alpha x), f(y))(s)\\ &= \nu(kf(x), f(y))(s) = \nu(f(x), f(y))\left(\frac{s}{|k|}\right)\\ &= \nu(x, y)\left(\frac{s}{|k|}\right),\end{aligned}$$

and hence $\alpha = |k|$.

Claim: $k = \alpha$.

If $k = -\alpha$, then for $s > 0$, we have

$$\begin{aligned}\nu(x,y)\left(\frac{s}{|\alpha-1|}\right) &= \nu((\alpha-1)x,y)(s) = \nu(\alpha x - x, y - x)(s)\\ &= \nu(f(\alpha x) - f(x), f(y) - f(x))(s) = \nu(-\alpha f(x) - f(x), f(y) - f(x))(s)\\ &= \nu(f(x), f(y) - f(x))\left(\frac{s}{\alpha+1}\right) = \nu(f(x), f(y))\left(\frac{s}{\alpha+1}\right)\\ &= \nu(x,y)\left(\frac{s}{\alpha+1}\right).\end{aligned}$$

So $|\alpha - 1| = \alpha + 1$, and hence $\alpha = 0$ which is a contradiction. Therefore $k = \alpha$ and so that $f(\alpha x) = \alpha f(x)$, for all $\alpha \in \mathbb{R}^+ \setminus \{0\}$.

Similarly, we can show that $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}^- \setminus \{0\}$. Given two distinct elements x and y in X. Since

$$f(x+y) = f\left(\frac{2x+2y}{2}\right)$$

by Equation (3.2), we get that

$$f(x+y) = \frac{f(2x)+f(2y)}{2} = \frac{2f(x)+2f(y)}{2} = f(x) + f(y).$$

If $x = y$, then $f(x+y) = f(2x) = 2f(x) = f(x) + f(x) = f(x) + f(y)$. So f is affine. □

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