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Multivalued f-weakly Picard mappings on partial metric spaces

Xianjiu Huang*, Yangyang Li, Chuanxi Zhu

Department of Mathematics, Nanchang University, Nanchang, 330031, Jiangxi, P. R. China.

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Abstract

In this paper, we introduce the notions of multivalued f-weak contraction and generalized multivalued f-weak contraction on partial metric spaces. We obtain some coincidence and fixed point theorems. Our results extend and generalize some well known fixed point theorems on partial metric spaces.

Keywords: Partial metric, common fixed point, hybrid maps, weakly Picard operators.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

In 1969, Nadler [24] extended Banach's contraction mapping principle [11] to a fundamental fixed point theorem for multivalued mappings on metric spaces. The study of fixed points for multivalued contractions using the Hausdorff metric was initiated by Markin [19]. Since then, an interesting and rich fixed point theory for such mappings was developed in many directions (see [14, 22, 23, 27, 28, 32, 35, 36, 37, 38, 39]). The theory of multi-valued mapping has applications in optimization problems, control theory, differential equations and economics. Berinde and Berinde [12] introduced the notion of multivalued (θ, L)-weak contraction and generalized multivalued (θ, L)-weak contraction and obtained some fixed point theorems. Kamran [17] further extended the notion of weak contraction mapping which is more general than the contraction mapping and introduced the notion of multi-valued (f, θ, L)-weak contraction mapping and generalized multi-valued (f, α, L)-weak contraction mapping. He established some coincidence and common fixed point theorems. We state the results of [17] for convenience as follows:

*Corresponding author
Email addresses: xjhuangxwen@163.com (Xianjiu Huang), ncuyyli@163.com (Yangyang Li), chuanxizhu@126.com (Chuanxi Zhu)

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Theorem 1.1. *Let (X, d) be a metric space, $f : X \to X$ and $T : X \to CB(X)$ be a multivalued (f, θ, L)-weak contraction such that $TX \subset fX$. Suppose fX is complete. Then the set of coincidence points of f and $T, C(f, T)$, is nonempty. Further, if f is T -weakly commuting at coincidence point u and $ffu = fu$, then f and T have a common fixed point.*

Theorem 1.2. *Let (X, d) be a metric space, $f : X \to X$ and $T : X \to CB(X)$ be a generalized multivalued (f, α, L)-weak contraction such that $TX \subset fX$. Suppose fX is a complete subspace of X. Then f and T have a coincidence point $u \in X$. Further, if f is T -weakly commuting at u and $ffu = fu$, then f and T have a common fixed point.*

The aim of this paper is to introduce the multivalued $f-$weak contractions and multivalued $f-$weakly Picard operators on partial metric space as the parallel manner on metric space. First, we recall the concept of partial metric space and some properties. In 1992, Matthews [20] introduced the notion of a partial metric space, which is a generalization of usual metric space in which the self distance for any point need not be equal to zero. The partial metric space has wide applications in many branches of mathematics as well as in the field of computer domain and semantics.

We recall that given a (nonempty) set X, a function $p : X \times X \to R^+$ is called a partial metric if and only if for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(p_2) $p(x, x) \le p(x, y)$;
(p_3) $p(x, y) = p(y, x)$;
(p_4) $p(x, z) \le p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair (R^+, p), where $p(x, y) = \max\{x, y\}$ for all $x, y \in R^+$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [26, 40, 41].

Each partial metric p on X generates a τ_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X; \varepsilon > 0\}$, where $\{B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

From this fact it immediately follows that a sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ with respect to τ_p if and only if $p(x, x) = \lim\limits_{n\to\infty} p(x, x_n)$. According to [20], a sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ with respect to τ_{p^s} if and only if

$$p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_m, x_n). \tag{1.1}$$

Following [20], a sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists $\lim\limits_{n,m\to\infty} p(x_n, x_m)$. A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to $\mathcal{T}(p)$, to a point $x \in X$ such that $p(x, x) = \lim\limits_{n,m\to\infty} p(x_n, x_m)$.

It is easy to see that, every closed subset of a complete partial metric space is complete.

If p is a partial metric on X, then the function $p^s, p^w : X \times X \to R^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

and

$$p^w(x, y) = p(x, y) - \min\{p(x, x), p(y, y)\} \tag{1.2}$$

are equivalent metric on X.

Lemma 1.3 ([20]). *Let (X, p) be a partial metric space.*

(1) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(2) *A partial metric space* (X, p) *is complete if and only if the metric space* (X, p^s) *is complete. Furthermore* $\lim_{n\to\infty} p^s(a, x_n) = 0$ *if and only if* $p(a, a) = \lim_{n\to\infty} p(a, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$.

In [20] Matthews obtained a partial metric version of the Banach fixed point theorem. Afterward, Acar *et al.* [1, 2], Altun *et al.* [4, 5, 7, 8], Karapinar and Erhan [18], Oltra and Valero [25], Romaguera [29, 30] and Valero [40] gave some generalizations of the result of Matthews. Also, Ciric *et al.* [13], Samet *et al.* [33] and Shatanawi *et al.* [34] proved some common fixed point results in partial metric spaces. But, so far all of the fixed point theorems have been given for single valued mappings. To prove Nadler's fixed point theorem for multi- valued maps on partial metric spaces, Aydi *et al.* [9] introduced the concept of partial Hausdorff distance a parallel manner to that in the Hausdorff metric in their nice paper [9]. Then, they give some properties of partial Hausdorff distance, some important lemmas and a fundamental fixed point theorem for multivalued mappings. We can find some nice fixed point results for single and multivalued maps on partial metric space in [3, 16, 21, 31].

Now we recall the concept of partial Hausdorff distance and some properties: Let (X, p) be partial metric space and $A \subseteq X$, then A is said to be bounded if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(a, a) + M$. A is closed if and only if $A = \overline{A}$, where $\overline{A}$ is the closure of A with respect to τ_p (τ_p is the topology induced by p). Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of (X, p). For $A, B \in CB^p(X)$ and $x \in X$, define

$$P(x, A) = \inf\{p(x, a) : a \in A\}, \delta_p(A, B) = \sup\{P(a, B) : a \in A\}$$

and

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

Lemma 1.4 ([9]). *Let* (X, p) *be a partial metric space,* $A \subseteq X$ *and* $x \in X$*. Then* $x \in \overline{A}$ *if and only if* $P(x, A) = p(x, x)$.

Proposition 1.5 ([9]). *Let* (X, p) *be a partial metric space. For any* $A, B, C \in CB_p(X)$*, we have the following:*

(1) $\delta_p(A, A) = \sup_{a\in A} p(a, a)$;
(2) $\delta_p(A, A) \leq \delta_p(A, B)$;
(3) $\delta_p(A, B) = 0$ *implies* $A \subseteq B$;
(4) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c\in C} p(c, c)$.

Proposition 1.6 ([9]). *Let* (X, p) *be a partial metric space. For any* $A, B, C \in CB_p(X)$*, we have the following:*

(1) $H_p(A, A) \leq H_p(A, B)$;
(2) $H_p(A, B) = H_p(B, A)$;
(3) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c\in C} p(c, c)$.

Remark 1.7. An example is given by Minak and Altun in [21] that $H_p(A, A) = H_p(A, B) = H_p(B, A)$, but $A \neq B$. That is H_p is not a partial metric on $CB_p(X)$. Nevertheless, as shown in [9] we have the following property:

$$H_p(A, B) = 0 \text{ implies } A = B.$$

Also, it is easy to see that, for all $A, B \in CB_P(X)$ and $a \in A$,

$$P(a, B) \leq \delta_p(A, B) \leq H_p(A, B).$$

The following lemma is very important to give fixed point results for multivalued maps on a partial metric space.

Lemma 1.8 ([9]). *Let (X,p) be a partial metric space, $A,B \in CB_p(X)$ and $h > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that $p(a,b) \leq hH_p(A,B)$.*

Lemma 1.8 can be expressed with the following version.

Lemma 1.9 ([10]). *Let (X,p) be a partial metric space, $A,B \in CB_p(X)$ and $\varepsilon > 0$. For any $a \in A$, there exists $b = b(a) \in B$ such that $p(a,b) \leq H_p(A,B) + \varepsilon$.*

Using the partial Hausdorff distance H_p, Aydi *et al.* [9] proved the following fixed point theorem for multivalued mappings.

Theorem 1.10. *Let (X,p) be a complete partial metric space. If $T : X \to CB_p(X)$ is a mapping such that*

$$H_p(Tx,Ty) \leq kp(x,y)$$

for all $x,y \in X$, where $k \in (0,1)$. Then T has a fixed point.

The following theorem is a generalized version of Theorem 1.10, which is given by Altun and Minak in [6].

Theorem 1.11. *Let (X,p) be a complete partial metric space and let $T : X \to CB_p(X)$ be a multivalued map. Assume*

$$H_p(Tx,Ty) \leq \alpha(p(x,y))p(x,y)$$

for all $x,y \in X$, where α is an $\mathcal{MT}-$function (that is, it satisfies $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0,\infty)$). Then T has a fixed point.

Recently, Minak and Altun [21] generalized the above theorems as follows:

Theorem 1.12. *Let (X,p) be a complete partial metric space and $T : X \to CB_p(X)$ be a multivalued map such that*

$$H_p(Tx,Ty) \leq kp(x,y) + LP^w(y,Tx)$$

for all $x,y \in X$, where $k \in (0,1), L \geq 0$ and $P^w(y,Tx) = \inf\{p^w(y,z) : z \in Tx\}$. Then T has a fixed point.

Theorem 1.13. *Let (X,p) be a complete partial metric space and let $T : X \to CB_p(X)$ be a multivalued map such that there exist an $\mathcal{MT}-$function α and a constant $L \geq 0$ satisfying*

$$H_p(Tx,Ty) \leq \alpha(p(x,y))p(x,y) + LP^w(y,Tx)$$

for all $x,y \in X$. Then T has a fixed point.

2. Main results

We begin this section with the notion of a hybrid generalized multivalued contraction mapping on partial metric spaces.

Definition 2.1. Let (X,p) be a partial metric space, $f : X \to X$ and $T : X \to CB_p(X)$ be a multivalued operator. T is said to be multivalued f weakly Picard operator if and only if for each $x \in X$ and $fy \in Tx(y \in X)$, there exists a sequence $\{x_n\}$ in X such that

(1) $x_0 = x, x_1 = y$;
(2) $fx_{n+1} \in Tx_n$ for all $n = 0,1,2,\cdots$;
(3) the sequence $\{fx_n\}$ converges to fu, where u is the coincidence point of f and T.

Definition 2.2. Let $\{x_n\}$ be a sequence in X satisfying condition (1) and (2) in Definition 2.1, then the sequence $O_f(x_0) = \{fx_n : n = 1,2,\cdots\}$ is said to be an f-orbit of T at x_0.

Definition 2.3. Let (X,p) be a partial metric space, $f : X \to X$ and $T : X \to CB_p(X)$ be a multivalued operator. T is said to be a multivalued f weakly contraction or a multivalued $(f,\theta,L)-$weak contraction if and only if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H_p(Tx,Ty) \leq \theta p(fx,fy) + LP^w(fy,Tx) \tag{2.1}$$

for all $x,y \in X$, where $P^w(fy,Tx) = \inf\{p^w(fy,z) : z \in Tx\}$ and p^w as in (1.2).

Remark 2.4. Due to the symmetry of p and H_p, in order to check that T is a multivalued $(f,\theta,L)-$weak contraction on (X,p), we have also check to the dual of (2.1), that is to check that T verifies

$$H_p(Tx,Ty) \leq \theta p(fx,fy) + LP^w(fx,Ty). \tag{2.2}$$

Theorem 2.5. *Let (X,p) be a partial metric space, $f : X \to X$ and $T : X \to CB(X)$ be a multivalued (f,θ,L)-weak contraction such that $TX \subset fX$. Suppose fX is complete. Then*

(1) the set of coincidence points of f and $T, C(f,T)$, is nonempty.

(2) for any $x_0 \in X$, there exists an $f-$orbit $O_f(x_0) = \{fx_n : n = 1,2,\cdots\}$ of T at x_0 such that $fx_n \to fu$, where u is coincidence point of f and T. Further, if $ffu = fu$ then f and T have a common fixed point.

Proof. Suppose $q > 1$ with $q\theta < 1$. Let $x_0 \in X$ and $y_0 = f(x_0)$. Since $Tx_0 \subset fX$, there exists a point $x_1 \in X$ such that $y_1 = f(x_1) \in Tx_0$. If $H_p(Tx_0,Tx_1) = 0$, then $f(x_1) \in Tx_0 = Tx_1$, i.e., $x_1 \in C(f,T)$. Let $H_p(Tx_0,Tx_1) \neq 0$, then Lemma 1.8 guarantees a point $y_2 \in Tx_1$ such that $p(y_1,y_2) \leq qH_p(Tx_0,Tx_1)$. Since $Tx_1 \subset fX$, there exists a point $x_2 \in X$ such that $y_2 = f(x_2) \in Tx_1$, i.e.,

$$p(fx_1,fx_2) \leq qH_p(Tx_0,Tx_1).$$

Using (2.1), we get

$$\begin{aligned} p(fx_1,fx_2) &\leq qH_p(Tx_0,Tx_1) \\ &\leq q[\theta p(fx_0,fx_1) + LP^w(fx_1,Tx_0)] \\ &= q\theta p(fx_0,fx_1) \end{aligned} \tag{2.3}$$

since $P^w(fx_1,Tx_0) = \inf\{p^w(fx_1,z) : z \in Tx_0\} = 0$. We take $h = q\theta$ thus

$$p(fx_1,fx_2) \leq hp(fx_0,fx_1).$$

If $H_p(Tx_1,Tx_2) = 0$, then $f(x_2) \in Tx_1 = Tx_2$, i.e., $x_2 \in C(f,T)$. Let $H_p(Tx_1,Tx_2) \neq 0$, then Lemma 1.8 guarantees a point $y_3 \in Tx_2$ such that

$$p(fx_2,fx_3) \leq hp(fx_1,fx_2).$$

Continuing in this manner, we obtain a sequence $\{x_n\}$ in X such that

$$p(fx_n,fx_{n+1}) \leq hp(fx_{n-1},fx_n), n = 1,2,\cdots.$$

So, we inductively obtain

$$p(fx_n,fx_{n+1}) \leq h^n p(fx_0,fx_1).$$

Using the modified triangular inequality for the partial metric, for any $m,n \in N$ with $m > n$ we obtain

$$\begin{aligned} (fx_m,fx_n) &\leq p(fx_n,fx_{n+1}) + p(fx_{n+1},fx_{n+2}) + \cdots + p(fx_{m-1},fx_m) \\ &\leq h^n p(fx_0,fx_1) + h^{n+1}p(fx_0,fx_1) + \cdots + h^{m-1}p(fx_0,fx_1) \\ &\leq h^n \frac{h^n}{1-h} p(fx_0,fx_1). \end{aligned} \tag{2.4}$$

Letting $n \to \infty$ in (2.4), we get $p(fx_m, fx_n) \to 0$, since $0 < h < 1$. By the definition of p^s, we get

$$p^s(fx_m, fx_n) \leq 2p(fx_m, fx_n).$$

So it is obvious that $p^s(fx_m, fx_n) \to 0$ as $n, m \to \infty$, since $p(fx_m, fx_n) \to 0$. This shows that $\{fx_n\}$ is a Cauchy sequence in (fX, p^s). Since (fX, p) is complete, (fX, p^s) is also complete by Lemma 1.3(2). Therefore, there exists a point $u \in X$ such that $fx_n \to fu$ with respect to the metric p^s, that is $\lim_{n\to\infty} p^s(fx_n, fu) = 0$.

By (1.1), we have

$$p(fu, fu) = \lim_{n\to\infty} p(fx_n, fu) = \lim_{n,m\to\infty} p(fx_m, fx_n) = 0. \tag{2.5}$$

Now,

$$\begin{aligned} P(fu, Tu) &\leq p(fu, fx_{n+1}) + P(fx_{n+1}, Tu) \\ &\leq p(fu, fx_{n+1}) + H_p(Tx_n, Tu) \\ &\leq p(fu, fx_{n+1}) + \theta p(fx_n, fu) + LP^w(fu, Tx_n) \\ &\leq p(fu, fx_{n+1}) + \theta p(fx_n, fu) + Lp^w(fu, fx_{n+1}). \end{aligned}$$

Letting $n \to \infty$ in the above inequality we get (note that p^s and p^w are equivalent metrics) $P(fu, Tu) = 0$. Therefore, from (2.5), we obtain $P(fu, Tu) = p(fu, fu)$. Thus, from Lemma 1.4, we have $fu \in Tu$, since Tu is closed.

Let $z = fu \in Tu$; then $fz = ffu = fu = z$. Using the notion of multivalued $(f, \theta, L)-$weak contraction, we get

$$\begin{aligned} H_p(Tu, Tz) &\leq \theta p(fu, fz) + LP^w(fz, Tu) \\ &= \theta p(fu, fu) + LP^w(fu, Tu) = 0. \end{aligned}$$

It follows from $P(fz, Tz) = P(fu, Tz) \leq H_p(Tu, Tz)$, that $P(fz, Tz) = 0$. Therefore, from (2.5), we obtain $P(fz, Tz) = p(fu, fu) = p(fz, fz)$. Thus, from Lemma 1.4, we have $z = fz \in Tz$, since Tz is closed. Thus f and T have a common fixed point. This completes the proof. □

Remark 2.6. Substituting $f = I$, the identity map on X, we get at once Theorem 1.12.

Now, we give a more general result on a partial metric space. For this we need the following lemma.

Lemma 2.7 ([15]). *Let $\alpha : [0, \infty) \to [0, 1)$ be an $\mathcal{MT}-$function, then the function $\beta : [0, \infty) \to [0, 1)$ defined as $\beta(t) = \frac{1+\alpha(t)}{2}$ is also an $\mathcal{MT}-$function.*

Definition 2.8. Let (X, p) be a partial metric space, $f : X \to X$ and $T : X \to CB_p(X)$ be a multivalued operator. T is said to be a generalized multivalued f weakly contraction or a generalized multivalued $(f, \alpha, L)-$weak contraction if and only if there exist a constant $L \geq 0$ and an $\mathcal{MT}-$ function α such that

$$H_p(Tx, Ty) \leq \alpha(p(fx, fy))p(fx, fy) + LP^w(fy, Tx) \tag{2.6}$$

for all $x, y \in X$, where $P^w(fy, Tx) = \inf\{p^w(fy, z) : z \in Tx\}$ and p^w as in (1.2).

Theorem 2.9. *Let (X, p) be a partial metric space, $f : X \to X$ and $T : X \to CB(X)$ be a generalized multivalued (f, α, L)-weak contraction such that $TX \subset fX$. Suppose fX is a complete subspace of X. Then f and T have a coincidence point $u \in X$. Further, if $ffu = fu$ then f and T have a common fixed point.*

Proof. Define a function $\beta : [0, \infty) \to [0, 1)$ as $\beta(t) = \frac{1+\alpha(t)}{2}$, then from Lemma 2.7 $\beta(t)$ is also an $\mathcal{MT}-$function. Let $x, y \in X$ be two arbitrary points with $fx \neq fy$, $u \in Tx$ and $\varepsilon = \frac{1-\alpha(p(fx,fy))}{2} p(fx, fy) > 0$ (note that since $fx \neq fy$ then $p(fx, fy) > 0$), then from Lemma 1.9 we can find $v \in Ty$ such that $p(u, v) \leq H_p(Tx, Ty) + \varepsilon$. Therefore, from (2.6) we have

$$\begin{aligned} p(u,v) &\le H_p(Tx,Ty) + \frac{1-\alpha(p(fx,fy))}{2}p(fx,fy) \\ &\le \alpha(p(fx,fy))p(fx,fy) + LP^w(fy,Tx) + \frac{1-\alpha(p(fx,fy))}{2}p(fx,fy) \\ &= \frac{1+\alpha(p(fx,fy))}{2}p(fx,fy) + LP^w(fy,Tx) \\ &= \beta(p(fx,fy))p(fx,fy) + LP^w(fy,Tx). \end{aligned} \tag{2.7}$$

Now, let $x_0 \in X$ and $y_0 = fx_0$. Since $Tx_0 \subset fX$, there exists a point $x_1 \in X$ such that $y_1 = f(x_1) \in Tx_0$. If $y_0 = y_1$, i.e., $fx_0 = fx_1$, then $fx_0 \in Tx_0$, that is x_0 is a coincidence point of f and T and so the proof is complete. Let $fx_0 \neq fx_1$, then from (2.7) there exists $y_2 = f(x_2) \in Tx_1$ such that

$$\begin{aligned} p(y_1,y_2) = p(fx_1,fx_2) &\le \beta(p(fx_0,fx_1))p(fx_0,fx_1) + LP^w(fx_1,Tx_0) \\ &= \beta(p(fx_0,fx_1))p(fx_0,fx_1). \end{aligned}$$

If $y_1 = y_2$, i.e., $fx_1 = fx_2$, then $fx_1 \in Tx_1$, that is x_1 is a coincidence point of f and T and so the proof is complete. Let $fx_1 \neq fx_2$, then from (2.7) there exists $y_3 = f(x_3) \in Tx_2$ such that

$$\begin{aligned} p(y_2,y_3) = p(fx_2,fx_3) &\le \beta(p(fx_1,fx_2))p(fx_1,fx_2) + LP^w(fx_2,Tx_1) \\ &= \beta(p(fx_1,fx_2))p(fx_1,fx_2). \end{aligned}$$

By continuing this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = fx_n \in Tx_{n-1}$ and

$$p(y_n,y_{n+1}) = p(fx_n,fx_{n+1}) \le \beta(p(fx_{n-1},fx_n))p(fx_{n-1},fx_n)$$

for all $n \in N$. Since $\beta(t) < 1$ for all $t \in [0,\infty)$ then $p(y_n,y_{n+1})$ is a nonincreasing sequence of nonnegative real numbers. Hence $p(y_n,y_{n+1})$ converges to some $\lambda \ge 0$. Since $\beta(t)$ is an $\mathcal{MT}-$function, then $\limsup_{s\to t^+} \beta(s) < 1$ and $\beta(\lambda) < 1$. Therefore, there exists $r \in [0,1)$ and $\varepsilon > 0$ such that $\beta(s) \le r$ for all $s \in [\lambda, \lambda+\varepsilon)$. Since $p(y_n,y_{n+1}) \downarrow \lambda$, we can take $k_0 \in N$ such that $\lambda \le p(y_n,y_{n+1}) \le \lambda+\varepsilon$ for all $n \in N$ with $n \ge k_0$.

$$p(y_{n+1},y_{n+2}) = p(fx_{n+1},fx_{n+2}) \le \beta(p(fx_n,fx_{n+1}))p(fx_n,fx_{n+1}) \le rp(fx_n,fx_{n+1}) = rp(y_n,y_{n+1})$$

for all $n \in N$ with $n \ge k_0$, then we have

$$\begin{aligned} \sum_{n=1}^{\infty} p(y_n,y_{n+1}) &\le \sum_{n=1}^{k_0} p(y_n,y_{n+1}) + \sum_{n=k_0+1}^{\infty} p(y_n,y_{n+1}) \\ &= \sum_{n=1}^{k_0} p(y_n,y_{n+1}) + \sum_{n=k_0}^{\infty} p(y_{n+1},y_{n+2}) \\ &\le \sum_{n=1}^{k_0} p(y_n,y_{n+1}) + \sum_{n=k_0}^{\infty} rp(y_n,y_{n+1}) \\ &\le \sum_{n=1}^{k_0} p(y_n,y_{n+1}) + \sum_{n=1}^{\infty} r^n p(y_{k_0},y_{k_0+1}) < \infty. \end{aligned}$$

Then for $m,n \in N$ with $m > n$, by omitting the negative term in the modified triangular inequality we obtain

$$\begin{aligned} p(y_n,y_m) &\le p(y_n,y_{n+1}) + p(y_{n+1},y_{n+2}) + \cdots + p(y_{m-1},y_m) \\ &= \sum_{i=n}^{m-1} p(y_i,y_{i+1}) \end{aligned}$$

$$\leq \sum_{i=n}^{\infty} p(y_i, y_{i+1}) \to 0 \text{ as } n \to \infty.$$

Therefore, we have $\lim_{n\to\infty} p(y_n, y_m) \to 0$, that is $\{y_n = fx_n\}$ is a Cauchy sequence in (fX, p). Since (fX, p) is complete, (fX, p^s) is also complete by Lemma 1.3 (2). So, there exists a point $u \in X$ such that $fx_n \to fu$ with respect to the metric p^s, that is $\lim_{n\to\infty} p^s(fx_n, fu) = 0$.

By (1.1), we have

$$p(fu, fu) = \lim_{n\to\infty} p(fx_n, fu) = \lim_{n,m\to\infty} p(fx_m, fx_n) = 0. \tag{2.8}$$

Now,

$$\begin{aligned} P(fu, Tu) &\leq p(fu, fx_{n+1}) + P(fx_{n+1}, Tu) \\ &\leq p(fu, fx_{n+1}) + H_p(Tx_n, Tu) \\ &\leq p(fu, fx_{n+1}) + \alpha(p(fx_n, fu))p(fx_n, fu) + LP^w(fu, Tx_n) \\ &\leq p(fu, fx_{n+1}) + \alpha(p(fx_n, fu))p(fx_n, fu) + Lp^w(fu, fx_{n+1}) \\ &\leq p(fu, fx_{n+1}) + p(fx_n, fu) + Lp^w(fu, fx_{n+1}). \end{aligned}$$

Letting $n \to \infty$ in the above inequality we get (note that p^s and p^w are equivalent metrics) $P(fu, Tu) = 0$. Therefore, from (2.8), we obtain $P(fu, Tu) = p(fu, fu)$. Thus, from Lemma 1.4, we have $fu \in Tu$, since Tu is closed.

Let $z = fu \in Tu$; then $fz = ffu = fu = z$. Using the notion of generalized multivalued $(f, \alpha, L)-$weak contraction, we get

$$\begin{aligned} H_p(Tu, Tz) &\leq \alpha(p(fu, fz))p(fu, fz) + LP^w(fz, Tu) \\ &= \alpha(p(fu, fu))p(fu, fu) + LP^w(fu, Tu) = 0. \end{aligned}$$

From $P(fz, Tz) = P(fu, Tz) \leq H_p(Tu, Tz)$, then $P(fz, Tz) = 0$. Therefore, from (2.8), we obtain $P(fz, Tz) = p(fu, fu) = p(fz, fz)$. Thus, from Lemma 1.4, we have $z = fz \in Tz$, since Tz is closed. Thus f and T have a common fixed point. This completes the proof. □

Remark 2.10. Substituting $f = I$, the identity map on X, we get at once Theorem 1.13.

Finally, we introduce an example satisfying the hypotheses of Theorem 2.9 to support the usability of our results. In doing so, we are essentially inspired by Aydi, Abbas and Vetro [10].

Example 2.11. Let $X = \{0, 1, 2, 3\}$, be endowed with the partial metric $p : X \times X \to R^+$ defined by

$$p(0,0) = p(1,1) = p(2,2) = 0,\ p(3,3) = \frac{1}{5},\ p(0,1) = p(1,0) = \frac{2}{5},\ p(0,2) = p(2,0) = \frac{1}{3},$$

$$p(1,2) = p(2,1) = \frac{2}{3},\ p(0,3) = p(3,0) = \frac{1}{2},\ p(1,3) = p(3,1) = \frac{3}{5},\ p(2,3) = p(3,2) = \frac{7}{10}.$$

Also define the mappings $f : X \to X$ and $T : X \to CB_p(X)$ by

$$fx = \begin{cases} 0 & \text{if } x \in \{0,1\} \\ 1 & \text{if } x = 2 \\ 2 & \text{if } x = 3 \end{cases}, \quad Tx = \begin{cases} \{0\} & \text{if } x \in \{0,1,2\} \\ \{1,2\} & \text{if } x = 3 \end{cases}$$

and the $\mathcal{MT}-$function $\alpha : [0, \infty) \to [0, 1)$ by $\alpha(t) = \frac{6t}{5+2t^2}$ for any $t \geq 0$ and $L = 1$. Note that Tx is closed and bounded for all $x \in X$ under the given partial metric p. We shall show that (2.6) holds for all $x, y \in X$. We distinguish the following cases:

(1) If $x, y \in \{0, 1, 2\}$, then $H_p(Tx, Ty) = H_p(\{0\}, \{0\}) = 0$ and (2.6) is obviously satisfied.

(2) If $x = 0, y = 3$, then

$$\begin{aligned}\alpha(p(fx, fy))p(fx, fy) + LP^w(fy, Tx) &= \alpha(p(f0, f3))p(f0, f3) + P^w(f3, T0)\\ &= \alpha(p(0, 2))p(0, 2) + P^w(2, \{0\}) = \alpha(\frac{1}{3})\frac{1}{3} + \frac{1}{3}\\ &= \frac{18}{47} + \frac{1}{3} = \frac{101}{141} \geq \frac{2}{5} = H_p(\{0\}, \{1, 2\}) = H_p(T0, T3).\end{aligned}$$

(3) If $x = 1, y = 3$, then

$$\begin{aligned}\alpha(p(fx, fy))p(fx, fy) + LP^w(fy, Tx) &= \alpha(p(f1, f3))p(f1, f3) + P^w(f3, T1)\\ &= \alpha(p(0, 2))p(0, 2) + P^w(2, \{0\}) = \alpha(\frac{1}{3})\frac{1}{3} + \frac{1}{3}\\ &= \frac{18}{47} + \frac{1}{3} = \frac{101}{141} \geq \frac{2}{5} = H_p(\{0\}, \{1, 2\}) = H_p(T1, T3).\end{aligned}$$

(4) If $x = 2, y = 3$, then

$$\begin{aligned}\alpha(p(fx, fy))p(fx, fy) + LP^w(fy, Tx) &= \alpha(p(f2, f3))p(f2, f3) + P^w(f3, T2)\\ &= \alpha(p(1, 2))p(1, 2) + P^w(2, \{0\}) = \alpha(\frac{2}{3})\frac{2}{3} + \frac{1}{3}\\ &= \frac{24}{53} + \frac{1}{3} = \frac{125}{159} \geq \frac{2}{5} = H_p(\{0\}, \{1, 2\}) = H_p(T2, T3).\end{aligned}$$

(5) If $x = 3, y = 0$, then

$$\begin{aligned}\alpha(p(fx, fy))p(fx, fy) + LP^w(fy, Tx) &= \alpha(p(f3, f0))p(f3, f0) + P^w(f0, T3)\\ &= \alpha(p(2, 0))p(2, 0) + P^w(0, \{1, 2\}) = \alpha(\frac{1}{3})\frac{1}{3} + \frac{1}{3}\\ &= \frac{18}{47} + \frac{1}{3} = \frac{101}{141} \geq \frac{2}{5} = H_p(\{1, 2\}, \{0\}) = H_p(T3, T0).\end{aligned}$$

(6) If $x = 3, y = 1$, then

$$\begin{aligned}\alpha(p(fx, fy))p(fx, fy) + LP^w(fy, Tx) &= \alpha(p(f3, f1))p(f3, f1) + P^w(f1, T3)\\ &= \alpha(p(2, 0))p(2, 0) + P^w(0, \{1, 2\}) = \alpha(\frac{1}{3})\frac{1}{3} + \frac{1}{3}\\ &= \frac{18}{47} + \frac{1}{3} = \frac{101}{141} \geq \frac{2}{5} = H_p(\{1, 2\}, \{0\}) = H_p(T3, T1).\end{aligned}$$

(7) If $x = 3, y = 2$, then

$$\begin{aligned}\alpha(p(fx, fy))p(fx, fy) + LP^w(fy, Tx) &= \alpha(p(f3, f2))p(f3, f2) + P^w(f2, T3)\\ &= \alpha(p(2, 1))p(2, 1) + P^w(1, \{1, 2\}) = \alpha(\frac{2}{3})\frac{2}{3} + 0\\ &= \frac{24}{53} \geq \frac{2}{5} = H_p(\{1, 2\}, \{0\}) = H_p(T3, T2).\end{aligned}$$

(8) If $x = y = 3$, then then $H_p(Tx, Ty) = H_p(\{1, 2\}, \{1, 2\}) = 0$ and (2.6) is obviously satisfied. Thus, all the conditions of Theorem 2.9 are satisfied and $x = 0$ is a common fixed point of f and T in X.

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