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Coincidence points of generalized multivalued $(f, L)-$almost $F-$contraction with applications

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Abstract

Recently Abbas [M. Abbas, Coincidence points of multivalued $f-$almost nonexpansive mappings, Fixed Point Theory, 13 (1) (2012), 3–10] introduced the concept of $f-$almost contraction which generalizes the class of multivalued almost contraction mapping and obtained coincidence point results for this new class of mappings. We extend this notion to multivalued $f-$almost $F-$contraction mappings and prove the existence of coincidence points for such mappings. As a consequence, coincidence point results are obtained for generalized multivalued $f-$almost $F-$nonexpansive mappings which assume closed values only. Related common fixed point theorems are also proved. In the last section, applications of our results in dynamic programming and integral equations to show the existence and uniqueness of solutions are obtained. We present some remarks to show that our results provide extension as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

Keywords: Coincidence point, multivalued $f-$almost weak contraction, star shaped sets, integral equations, dynamic programming.
2010 MSC: 47H04, 47H10, 54H25, 60H25.

1. Introduction and Preliminaries

Let (X, d) be a metric space. Let $CB(X)$ $(CL(X))$ be the family of all nonempty closed and bounded (nonempty closed) subsets of X. For $A, B \in CL(X)$, define a set

$$E_{A,B} = \{\varepsilon > 0 : A \subseteq N_\varepsilon(B), B \subseteq N_\varepsilon(A)\}.$$

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The Hausdorff metric H on $CL(X)$ induced by metric d is given as:

$$H(A,B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \emptyset, \\ \infty & \text{if } E_{A,B} = \emptyset. \end{cases}$$

Let $f : X \to X$ and $T : X \to CL(X)$. A point x in X is called a fixed point of T if $x \in Tx$. The set of all fixed points of T is denoted by $F(T)$. Furthermore, a point x in X is called a coincidence point of f and T if $fx \in Tx$. The set of all such points is denoted by $C(f,T)$. If for some point x in X, we have $x = fx \in Tx$, then a point x is called a common fixed point of f and T. We denote set of all common fixed points of f and T by $F(f,T)$. A mapping $T : X \to CL(X)$ is said to be continuous at $p \in X$ if for any sequence $\{x_n\}$ in X with $\lim\limits_{n\to\infty} d(x_n, p) = 0$, we have $\lim\limits_{n\to\infty} H(Tx_n, Tp) = 0$.

Berinde [12] introduced the following concept of a weak contraction mapping.

Definition 1.1 ([12])**.** Let (X,d) be a metric space. A self mapping f on X is called weak contraction if there exist constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$d(fx, fy) \leq \theta d(x,y) + Ld(y, fx)$$

for every x, y in X.

For more discussion on weak contraction mappings, we refer to [15, 17] and references therein.

Berinde and Berinde [13] extended the notion of weak contraction mappings as follows:

Definition 1.2 ([13, 14])**.** A mapping $T : X \to CL(X)$ is called a multivalued weak contraction if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x,y) + Ld(y, Tx) \tag{1.1}$$

for every x, y in X.

Following definition of a generalized multivalued $(\theta, L)-$strict almost contraction mapping is due to Berinde and Păcurar [16].

Definition 1.3 ([16])**.** A mapping $T : X \to CL(X)$ is called generalized multivalued $(\theta, L)-$strict almost contraction mapping if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x,y) + L \min\{d(y,Tx), d(x,Ty), d(x,Tx), d(y,Ty)\} \tag{1.2}$$

for every x, y in X.

We have following fixed point theorem given in [16].

Theorem 1.4. *Let (X,d) be a complete metric space and $T : X \to CL(X)$ a generalized multivalued $(\theta, L)-$strict almost contraction mapping. Then $F(T) \neq \emptyset$, moreover for any $p \in F(T)$, T is continuous at p.*

Kamran [22] extended the notion of a multivalued weak contraction mapping to hybrid pair $\{f, T\}$ of single valued mapping f and multivalued mapping T.

Definition 1.5. Let (X,d) be a metric space and f a self map on X. A multivalued mapping $T : X \to CL(X)$ is called generalized multivalued $(f, \theta, L)-$weak contraction mapping if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(fx, fy) + Ld(fy, Tx) \tag{1.3}$$

for every x, y in X.

Abbas [1] extended the above definition as follows.

Definition 1.6 ([1]). Let (X, d) be a metric space and f a self map on X. A multivalued mapping $T : X \to CL(X)$ is called generalized multivalued $(f, \theta, L)-$almost contraction mapping if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta M(x, y) + LN(x, y) \tag{1.4}$$

for every x, y in X, where

$$\begin{aligned} M(x, y) &= \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\}, \\ N(x, y) &= \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}. \end{aligned}$$

Let $\mathcal{F}$ be the collection of all mappings $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy the following conditions:

C1 F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;

C2 For every sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

C3 There exist $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Recently Wardowski [31] introduced the following concept of $F-$contraction mappings.

Definition 1.7 ([31]). Let (X, d) be a metric space. A self map f on X is said to be an $F-$contraction on X if there exists $\tau > 0$ such that

$$d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(d(x, y)) \tag{1.5}$$

for all $x, y \in X$, where $F \in \mathcal{F}$.

Remark 1.8 ([31]). Every $F-$contraction mapping is continuous.

Abbas et al.([3]) extended the concept of $F-$ contraction mapping and obtained common fixed point results. Further in this direction, Abbas et al.([2]) introduced a notion of generalized $F-$contraction and employed their results to obtain a fixed point of a generalized nonexpansive mappings on star shaped subsets of normed linear spaces. Recently, Minak [25] proved some fixed point results for Ciric type generalized $F-$contractions on complete metric spaces.

Sgroi and Vetro [30] proved the following result to obtain fixed point of multivalued mappings as a generalization of Nadler's Theorem [24].

Theorem 1.9 ([30]). *Let (X, d) be a complete metric space and $T : X \to CL(X)$ a multivalued mapping. Assume that there exists an $F \in \mathcal{F}$ and $\tau \in \mathbb{R}_+$ such that*

$$2\tau + F(H(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx))$$

for all $x, y \in X$, with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Then T has a fixed point.

Acar et al. [4] proved the following result.

Theorem 1.10 ([4]). *Let (X, d) be a complete metric space and $T : X \to K(X)$ (Compact subsets of X). Assume that there exist an $F \in \mathcal{F}$ and $\tau \in \mathbb{R}_+$ such that*

$$x, y \in X, H(Tx, Ty) > 0 \Longrightarrow \tau + F(H(Tx, Ty)) \leq F(M(x, y))$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

Moreover if T or F is continuous, then T has a fixed point.

Recently, Altun et al. [5] proved the following result.

Theorem 1.11 ([5]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that there exist an $F \in \digamma$ and $\tau, \lambda \in \mathbb{R}_+$ such that*

$$x, y \in X, H(Tx, Ty) > 0 \text{ implies that } \tau + F(H(Tx, Ty)) \leq F(d(x, y) + \lambda d(y, Tx)).$$

Then the mapping T is multivalued weakly Picard operator.

For the definition of multivalued weakly Picard operator and the related results, we refer to [13].

Now, we give the following definition.

Definition 1.12. Let f be a self map on metric space X and $T : X \to CL(X)$ a multivalued mapping, then T is called generalized multivalued $(f, L)-$almost $F-$contraction mapping if there exist $F \in \digamma$ and $\tau \in \mathbb{R}_+$ and $L \geq 0$ such that

$$2\tau + F(H(Tx, Ty)) \leq F(M(x, y) + LN(x, y)) \tag{1.6}$$

for every x, y in X, with $Tx \neq Ty$ and

$$\begin{aligned} M(x, y) &= \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\}, \\ N(x, y) &= \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}). \end{aligned}$$

Remark 1.13. Take $F(x) = \ln x$ in the Definition 1.12. Then (1.6) becomes

$$2\tau + \ln(H(Tx, Ty)) \leq \ln(M(x, y) + LN(x, y),$$

that is

$$\begin{aligned} H(Tx, Ty)) &\leq e^{-2\tau} M(x, y) + e^{-2\tau} LN(x, y) \\ &= \theta_1 M(x, y) + L_1 N(x, y), \end{aligned}$$

where $\theta_1 = e^{-2\tau} \in (0, 1)$ and $L_1 = e^{-2\tau} L \geq 0$, so we get generalized multivalued $(f, \theta_1, L_1)-$almost contraction mapping introduced by Abbas [1].

Remark 1.14. Take $\alpha = \frac{1}{4}$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{4}$, $\delta = \frac{1}{8} = L$. Note that $\alpha + \beta + \gamma + 2\delta = 1$. Then a contraction condition in Theorem 1.9 becomes

$$\begin{aligned} 2\tau + F(H(Tx, Ty)) &\leq F\left(\frac{1}{4}\left(d(x, y) + (d(x, Tx) + d(y, Ty)) + \frac{d(x, Ty) + d(y, Tx)}{2}\right)\right) \\ &\leq F\left(\frac{1}{4}(4M(x, y))\right) = F((M(x, y) + 0N(x, y))) \end{aligned}$$

for all $x, y \in X$, with $Tx \neq Ty$. Thus, for $L = 0$ and $f = I$ (Identity map) in

$$\begin{aligned} M(x, y) &= \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\}, \\ N(x, y) &= \min\{d(fx, fy), d(fx, Tx), d(fy, Ty)\}, \end{aligned}$$

a contraction condition in Theorem 1.10 is an $(f, 0)-$almost $F-$contraction which is a special case of generalized multivalued $(f, L)-$almost $F-$contraction (for $L = 0$ and $\tau = 2\tau_1$).

Let $f : X \longrightarrow X$ and $T : X \longrightarrow CL(X)$ a multivalued mapping.

Definition 1.15. The pair (f, T) is called (e) *commuting* if $Tfx = fTx$ for all $x \in X$ (f) *weakly compatible* if they commute at their coincidence points, that is, $fTx = Tfx$ whenever $x \in C(f, T)$ ([21]).

The map f is called T- *weakly commuting* at $x \in X$ if $f^2x \in Tfx$. If hybrid pair (f, T) is weakly compatible at $x \in C(f, T)$, then f is $T-$weakly commuting at x and hence $f^n(x) \in C(f, T)$. However the converse is not true in general. For detailed discussion on above mentioned notions and their implications, we refer to [6], [18], [19, 20, 21], and references therein.

Definition 1.16. A mapping $T : X \to CL(X)$ is called a closed mapping if

$$G(T) = \{(x, y) : x \in X, y \in Tx\} \text{ (graph of } T)$$

is a closed subset of $X \times X$.

Note that, a mapping T is closed if and only if it satisfies the following condition:

For two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X with $y_n \in T(x_n)$ for each $n \in \mathbb{N}$, $x_n \to x$ and $y_n \to y$, we have $y \in T(x)$.

2. Coincidence and Common Fixed Point Theorems

Throughout this section, we assume that the mapping F is right continuous.

We start with the following.

Theorem 2.1. *Let X be a metric space, $f : X \to X$ and $T : X \to CL(X)$ be a generalized multivalued $(f, L)-$almost $F-$contraction with $\overline{T(X)} \subseteq f(X)$. If $\overline{T(X)}$ is complete, then $C(f, T) \neq \emptyset$ provided that either F is continuous or T is closed multivalued mapping. Moreover $F(f, T) \neq \emptyset$ if one of the following conditions holds:*

(a) *for some $x \in C(f, T)$, f is $T-$weakly commuting at x, $f^2x = fx$.*

(b) *f and T are weakly compatible on $C(f, T)$, f is continuous, and $\lim\limits_{n\to\infty} f^nx$ exists for some $x \in C(f, T)$ provided that F is continuous or T is closed multivalued mapping.*

(c) *for some $z \in C(f, T)$, f is continuous at z, and $\lim\limits_{n\to\infty} f^ny = z$ for some $y \in X$.*

(d) *$f(C(f, T))$ is a singleton subset of $C(f, T)$.*

Proof. We first note that, by Remark 1.13, $H(Tx, Ty) < \infty$ for all $x, y \in X$.

Now we shall show that $C(f, T) \neq \emptyset$. Indeed, let x_0 be a given point in X. Since $Tx_0 \subseteq f(X)$, we can choose an element $x_1 \in X$ such that $fx_1 \in Tx_0$. If $H(Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$, so $x_1 \in C(f, T)$. Assume $H(Tx_0, Tx_1) > 0$. Since, by hypothesis, F is right continuous at $H(Tx_0, Tx_1)$, there exists $h > 1$ such that

$$F(hH(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau.$$

Since $fx_1 \in Tx_0$ we deduce that $d(fx_1, Tx_1) \leq H(Tx_0, Tx_1)$, and thus there exists $y_1 \in Tx_1$ such that

$$d(fx_1, y_1) < hH(Tx_0, Tx_1).$$

Pick an element x_2 in X such that $fx_2 = y_1$. Then, above inequality becomes

$$d(fx_1, fx_2) < hH(Tx_0, Tx_1).$$

If $fx_1 = fx_2$, then $fx_1 \in Tx_1$. In this case x_1 becomes a coincidence point of f and T and the proof is finished. Assume that $fx_1 \neq fx_2$, that is, $d(fx_1, fx_2) > 0$. Since F is strictly increasing we obtain

$$F(d(fx_1, fx_2)) < F(hH(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau.$$

As T is generalized multivalued (f, L)- almost F- contraction, it follows that

$$\begin{aligned}
F(d(fx_1, fx_2)) &< F(H(Tx_0, Tx_1)) + \tau \\
&\leq F(M(x_0, x_1) + LN(x_0, x_1)) - 2\tau + \tau \\
&= F(\max\{d(fx_0, fx_1), d(fx_0, Tx_0), d(fx_1, Tx_1), \frac{d(fx_0, Tx_1) + d(fx_1, Tx_0)}{2}\} \\
&\quad +L\min\{d(fx_0, Tx_0), d(fx_1, Tx_1), d(fx_0, Tx_1), d(fx_1, Tx_0)\}) - \tau \\
&\leq F(\max\{d(fx_0, fx_1), d(fx_0, fx_1), d(fx_1, fx_2), \frac{d(fx_0, fx_2) + d(fx_1, fx_1)}{2}\} \\
&\quad +L\min\{d(fx_0, fx_1), d(fx_1, fx_2), d(fx_0, fx_2), d(fx_1, fx_1)\}) - \tau \\
&\leq F(\max\{d(fx_0, fx_1), d(fx_1, fx_2), \frac{d(fx_0, fx_1) + d(fx_1, fx_2)}{2}\}) - \tau \\
&= F(\max\{d(fx_0, fx_1), d(fx_1, fx_2)\}) - \tau.
\end{aligned}$$

So we have

$$\tau + F(d(fx_2, fx_1)) < F(\max\{d(fx_0, fx_1), d(fx_1, fx_2)\}.$$

Continuing this way, we can obtain a sequence $\{x_n\}$ in X such that $fx_{n+1} \in Tx_n \subseteq T(X)$ and it satisfies

$$\tau + F(d(fx_n, fx_{n+1})) < F(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}) \tag{2.1}$$

for all $n \in \mathbb{N}$. Since F is strictly increasing, therefore

$$d(fx_n, fx_{n+1}) < \max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}.$$

If

$$\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}) = d(fx_n, fx_{n+1})$$

for some n, then,

$$d(fx_n, fx_{n+1}) < d(fx_n, fx_{n+1})$$

gives a contradiction. So we have

$$d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n). \tag{2.2}$$

Consequently

$$\tau + F(d(fx_n, fx_{n+1})) < F(d(fx_{n-1}, fx_n)), \tag{2.3}$$

for all $n \in \mathbb{N}$. If $\lambda_n = d(fx_n, fx_{n+1})$, then we obtain that

$$F(\lambda_n) < F(\lambda_{n-1}) - \tau < ... < F(\lambda_0) - n\tau$$

On taking limit as $n \to \infty$, we have $\lim\limits_{n\to\infty} F(\lambda_n) = -\infty$. By (C1), we get $\lim\limits_{n\to\infty} \lambda_n = 0$. By (C3) there exists an $r \in (0, 1)$ such that

$$\lim_{n\to\infty} \lambda_n^r F(\lambda_n) = -\infty.$$

Hence it follows that

$$\lambda_n^r F(\lambda_n) - \lambda_n^r F(\lambda_0) \leq \lambda_n^r F(\lambda_0) - n\lambda_n^r \tau - \lambda_n^r F(\lambda_0) = -n\lambda_n^r \tau.$$

On taking limit as n tends to ∞, we obtain $\lim\limits_{n\to\infty} n\lambda_n^r = 0$, that is, $\lim\limits_{n\to\infty} n^{1/r}\lambda_n = 0$. This implies that $\sum\limits_{n=1}^{\infty} \lambda_n$ is convergent and hence the sequence $\{fx_n\}_{n\geq 1}$ is a Cauchy sequence in $T(X) \subseteq \overline{T(X)}$. As $\overline{T(X)}$ is complete, so there is $p \in \overline{T(X)}$ such that $\lim\limits_{n\to\infty} fx_n = p$. Now $\overline{T(X)} \subseteq f(X)$ implies that there exists u^* in X such that $fu^* = p$.

Next we prove that $fu^* \in Tu^*$. Indeed, assume the contrary, then $d(fu^*, Tu^*) > 0$ because Tu^* is closed. Since F is strictly increasing, we deduce from Remark 1.13 that

$$H(Tx_n, Tu^*) < M(x_n, u^*) + LN(x_n, u^*)$$

for all $n \in \mathbb{N}$. Therefore

$$d(fx_{n+1}, Tu^*) \leq H(Tx_n, Tu^*) < M(x_n, u^*) + LN(x_n, u^*)$$

so, by Remark 1.13,

$$\begin{aligned} 2\tau + F(d(fx_{n+1}, Tu^*)) &\leq 2\tau + F(H(Tx_n, Tu^*)) \\ &\leq F(M(x_n, u^*) + LN(x_n, u^*)) \end{aligned}$$

for all $n \in \mathbb{N}$.

Next suppose that F is continuous. Since

$$\lim_{n\to\infty} d(fx_n, Tu^*) = d(fu^*, Tu^*)$$

we deduce that

$$\lim_{n\to\infty} M(x_n, u^*) = d(fu^*, Tu^*).$$

Moreover

$$\lim_{n\to\infty} N(x_n, u^*) = 0$$

so, by continuity of F,

$$2\tau + F(d(fu^*, Tu^*)) \leq F(d(fu^*, Tu^*)$$

which provides a contradiction. We conclude that $d(fu^*, Tu^*) = 0$, and thus $fu^* \in Tu^*$.

Now suppose that T is closed multivalued mapping. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fx_{n+1} = fu^*$ and $fx_{n+1} \in Tfx_n$, we have $fu^* \in Tfu^*$, that is, $fu^* \in C(f,T)$ and hence $C(f,T) \neq \emptyset$.

Now let (a) holds, that is for $x \in C(f,T)$, f is $T-$weakly commuting at x. So we get $f^2x \in Tfx$. By the given hypothesis $fx = f^2x$ and hence $fx = f^2x \in Tfx$. Consequently $fx \in F(f,T)$.

Suppose (b) holds when F is continuous: let $y = \lim_{n\to\infty} f^n x$ for some x in $C(f,T)$. Since f is continuous, this implies that y is a fixed point of f. That is $y = fy$. Furthermore $f^{n+1}x \in C(f,T)$ for all $n \geq 1$ and hence $f^{n+1}x \in Tf^n x$. Exactly as in the first part of the proof (taking $p = u^* = y$) we deduce that $d(y, Ty) = 0$. Hence $y = fy \in Ty$ and $F(f,T) \neq \emptyset$.

Suppose (b) holds when T is closed: Since $\lim_{n\to\infty} f^n x = \lim_{n\to\infty} f^{n+1}x = y$ and $f^{n+1}x \in Tf^n x$ then $y \in Ty$. Consequently $y = fy \in Ty$.

(c) Suppose for some $z \in C(f,T)$, f is continuous at z and $\lim_{n\to\infty} f^n x = z$ for some $x \in X$. Then $z = fz \in Tz$, and $F(f,T) \neq \emptyset$.

(d) Since $f(C(f,T)) = \{x\}$ (say) and $x \in C(f,T)$, this implies that $x = fx \in Tx$. Thus $F(f,T) \neq \emptyset$. □

In Theorem 1.9 underlying space is a complete metric space but in above theorem we do not assume the completeness of underlying space, instead we take the completeness of $\overline{T(X)}$. In Theorem 1.10 authors assume that a mapping T is compact valued but we prove the result when T is closed valued mapping.

Theorem 2.1 generalizes Theorem 3.4 of [30] and Theorem 2.2 of [4].

Corollary 2.2. *Let X be a metric space, $T : X \longrightarrow CL(X)$ generalized multivalued $(f, \theta_1, L_1)-$almost contraction with $\overline{T(X)} \subseteq X$ for $\theta_1 = e^{-2\tau} \in (0,1)$ and $L_1 = Le^{-2\tau}$, where $\tau > 0$. Suppose that $\overline{T(X)}$ is complete. Then $C(f,T) \neq \emptyset$. Moreover, $F(f,T) \neq \emptyset$ if one of the conditions from (a)-(d) in Theorem 2.1 holds.*

Proof. The result follows if we take $F(x) = \ln x$ in Theorem 2.1. $\square$

Corollary 2.3. *Let X be a metric space, $T : X \longrightarrow CL(X)$ generalized multivalued $(\theta_1, L_1)-$almost contraction with $\overline{T(X)} \subseteq X$ for $\theta_1 = e^{-2\tau} \in (0,1)$ and $L_1 = Le^{-2\tau}$, where $\tau > 0$. Suppose that $\overline{T(X)}$ is complete. Then T has a fixed point.*

Proof. Take $f = I$ (identity map on X) in Corollary 2.2. $\square$

Remark 2.4. Theorem 2.1 generalizes the results proved in [1, 13, 16, 22, 24].

If we take T as single valued self map in Theorem 2.1, then we get the following corollary. We will apply this corollary to show the existence and uniqueness of common and bounded solution of functional equations arising in dynamic programming. We shall also give an application of this corollary in finding the solution of volterra type system of integral equations.

Corollary 2.5. *Let X be a metric space, $f, T : X \to X$ two mappings with $\overline{T(X)} \subseteq f(X)$. Assume that there exist $\tau > 0$ and $L \geq 0$ such that*

$$2\tau + F(d(Tx, Ty)) \leq F(M(x, y) + LN(x, y))$$

for every x, y in X, with $Tx \neq Ty$ and

$$\begin{aligned} M(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}, \\ N(x,y) &= \min\{d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}). \end{aligned}$$

If $\overline{T(X)}$ is complete, then $C(f,T) \neq \emptyset$ provided that either F is continuous or T is continuous. Moreover if for some $x \in C(f,T)$, f and T are commuting at x, then $f^2x = fx$, $F(f,T)$ is nonempty and singleton.

Proof. From Theorem 2.1 it follows that $C(f,T) \neq \emptyset$. Let $x \in C(f,T)$, that is $fx = Tx$. Since f and T are commuting therefore we obtain that $f^2x = fTx = Tfx$. Now we claim that $fx = f^2x$. If $fx \neq f^2x$. Then we have

$$\begin{aligned} 2\tau + F(d(fx, f^2x)) &= 2\tau + F(d(Tx, Tfx)) \\ &\leq F(M(x, fx) + LN(x, fx)) \\ &= F(\max\{d(fx, ffx), d(fx, Tx), d(ffx, Tfx), \frac{d(fx, Tfx) + d(ffx, Tx)}{2}\} \\ &\quad +L\min\{d(fx, Tx), d(ffx, Tfx), d(fx, Tfx), d(ffx, Tx)\}). \\ &\leq F(\max\{d(fx, ffx), d(fx, fx), d(ffx, ffx), \frac{d(fx, ffx) + d(ffx, fx)}{2}\} \\ &\quad +L\min\{d(fx, fx), d(ffx, ffx), d(fx, ffx), d(ffx, fx)\}). \\ &\leq F(d(fx, ffx)). \end{aligned}$$

This implies that $\tau \leq 0$, a contradiction. So $fx = f^2x$. Consequently $fx = f^2x = Tfx$. Now we prove the uniqueness of common fixed point of f and T. Suppose that there exist u and w in $F(f,T)$ such that $u \neq w$. Then by given assumption, we have

$$\begin{aligned} 2\tau + F(d(fu, fw)) &= 2\tau + F(d(Tu, Tw)) \\ &\leq F(M(u, w) + LN(u, w)) \\ &= F(\max\{d(fu, fw), d(fu, Tu), d(fw, Tw), \frac{d(fu, Tw) + d(fw, Tu)}{2}\} \\ &\quad +L\min\{d(fu, Tu), d(fw, Tw), d(fu, Tw), d(fw, Tu)\}). \\ &\leq F(\max\{d(fu, fw), d(fu, fu), d(fw, fw), \frac{d(fu, fw) + d(fw, fu)}{2}\} \\ &\quad +L\min\{d(fu, fu), d(fw, fw), d(fu, fw), d(fw, fu)\}) \\ &\leq F(d(fu, fw)). \end{aligned}$$

This implies that $\tau \leq 0$, a contradiction. So $u = w$. □

Note that Corollary 2.5 generalizes Theorem 2.4 in [32]. Now we present an example to validate Theorem 2.1.

Example 2.6. Let $X = [1, \infty)$ be the usual metric space. Define $f : X \to X$, and $T : X \to CL(X)$ by $fx = x^2$ and $Tx = [x + 2, \infty)$ for all $x \in X$. Note that $\overline{T(X)} = T(X) = [3, \infty)$, so $\overline{T(X)}$ is complete. It is easy to check that for all $x, y \in X$ with $Tx \neq Ty$ (equivalently with $x \neq y$), one has

$$2\tau + F(H(Tx, Ty)) \leq F(M(x, y))$$

where $\tau = \ln \sqrt{2}$, and $F(\alpha) = \ln \alpha$. So we can apply Theorem 2.1. In fact $C(f, T) = [2, \infty)$. Observe also that $F(f, T) = \emptyset$.

Definition 2.7. Let $f : Y \longrightarrow Y$ be a self map, and $T : Y \longrightarrow CL(Y)$ a multivalued mapping, where Y is a subset of a normed linear space X. Then the mapping $f - T$ is called *demiclosed at* 0 if whenever a sequence $\{x_n\}$ in Y converges weakly to x_0 in Y and $\{y_n\} \subseteq (f - T)x_n$ converges to 0 strongly, then $0 \in (f - T)x_0$.

Definition 2.8. Let Y be a subset of a normed space X and $q \in Y$. It is said to be

(a) *$q-$starshaped or starshaped with respect to q* if $\lambda x + (1 - \lambda)q \in Y$ for all $x \in Y$ and $\lambda \in [0, 1]$;

(b) *convex* if $\lambda x + (1 - \lambda)y \in Y$ for all $x, y \in Y$ and $\lambda \in [0, 1]$.

Definition 2.9. Let f be a self map on a normed space X and $Y \subseteq X$, then f is called

(c) *affine on Y* if Y is convex and $f(\lambda x + (1 - \lambda)y) = \lambda fx + (1 - \lambda)fy$ for all $x, y \in Y$ and $\lambda \in [0, 1]$;

(d) *$q-$affine on Y* if Y is $q-$starshaped and $f(\lambda x + (1 - \lambda)q) = \lambda fx + (1 - \lambda)q$ for all $x \in Y$ and $\lambda \in [0, 1]$.

Definition 2.10. Let Y be a $q-$starshaped subset of a normed space X, $f : Y \longrightarrow Y$ and $T : Y \longrightarrow CL(Y)$. A pair (f, T) satisfies the coincidence point condition on a closed subset A of Y if, whenever $\{x_n\}$ is a sequence in A such that $\lim_{n\to\infty} d(fx_n, Tx_n) = 0$ then $fu \in Tu$ for some $u \in A$. A map T satisfies the fixed point condition on $A \in CL(Y)$ if, whenever $\{x_n\}$ is a sequence in A such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $u \in Tu$ for some $u \in A$.

We also define

$$\delta(fy, Tx) = \inf\{d(fy, T_\lambda x) : 0 \leq \lambda \leq 1\},$$

where $T_\lambda x = \lambda Tx + (1 - \lambda)q$.

Definition 2.11. Let $(X, \|\cdot\|)$ be normed space, $f : X \longrightarrow X$ and $T : X \longrightarrow CL(X)$. If there exist an $F \in \digamma$ and $L \geq 0$ such that the pair (f, T) satisfies

$$\begin{aligned} F(H(Tx, Ty)) \leq\ & F(\max\{\|fx - fy\|, \delta(fx, Tx), \delta(fy, Ty), \frac{\delta(fx, Ty) + \delta(fy, Tx)}{2}\} \\ & + L \min\{\delta(fx, Tx), \delta(fy, Ty), \delta(fx, Ty), \delta(fy, Tx)\}) \end{aligned} \tag{2.4}$$

for all $x, y \in Y$ with $Tx \neq Ty$. Then T is called an F_f- nonexpansive.

Theorem 2.12. *Let Y be a subset of a normed space X, $f : Y \longrightarrow Y$ and $T : Y \longrightarrow CL(Y)$ be an F_f-nonexpansive. Suppose that Y is $q-$starshaped, $f(Y) = Y$ [resp. f is $q-$affine on Y], $T(Y)$ is bounded, $\overline{T(Y)}$ is complete, $\overline{T(Y)} \subseteq f(Y)$. Then $C(f, T) \neq \emptyset$ provided that either F is continuous or T is closed multivalued mapping. Moreover $F(f, T) \neq \emptyset$ if one of the conditions (a)-(d) of Theorem 2.1 holds.*

Proof. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$. For $n \geq 1$, let

$$T_n(x) = T_{\lambda_n}(x) = \lambda_n Tx + (1-\lambda_n)q$$

for all x in Y. As Y is $q-$starshaped, $\overline{T(Y)}$ is complete, $\overline{T(Y)} \subseteq f(Y)$, and $f(Y) = Y$ [resp. f is $q-$affine on Y], we have $\overline{T_n(Y)} \subseteq f(Y)$ and $\overline{T_n(Y)}$ is complete for each $n \geq 1$. Now consider,

$$\begin{aligned} F(H(T_n x, T_n y)) &= F(H(T_{\lambda_n}(x), T_{\lambda_n}(y)) \\ &= F(H(\lambda_n Tx + (1-\lambda_n)q, \lambda_n Ty + (1-\lambda_n)q)) \\ &= F(\lambda_n H(Tx, Ty)). \end{aligned}$$

As F is strictly increasing and $\lambda_n < 1$ for each $n \geq 1$, so we obtain

$$F(H(T_n x, T_n y)) = F(\lambda_n H(Tx, Ty)) < F(H(Tx, Ty)).$$

This implies

$$\mu_n = F(H(Tx, Ty)) - F(H(T_n x, T_n y)) > 0.$$

Since $n \geq 1$ is fixed, therefore by Archimedean property there exists $n_{\mu_n} \in \mathbb{N}$ for each $n \geq 1$ such that

$$0 < \frac{1}{n_{\mu_n}} < \mu_n.$$

If $\tau = \dfrac{1}{2n_{\mu_n}}$, then $0 < 2\tau_n < \mu_n$ for each $n \geq 1$. So we have

$$0 < 2\tau_n < F(H(Tx, Ty)) - F(H(T_n x, T_n y)).$$

Thus

$$\begin{aligned} 2\tau_n + F(H(T_n x, T_n y)) &< F(H(Tx, Ty)) \\ &\leq F(\max\{\|fx - fy\|, \delta(fx, Tx), \delta(fy, Ty), \frac{\delta(fx, Ty) + \delta(fy, Tx)}{2}\} \\ &\quad + \min\{\delta(fx, Tx), \delta(fy, Ty), \delta(fx, Ty), \delta(fy, Tx)\}) \\ &\leq F(\max\{\|fx - fy\|, d(fx, T_n x), d(fy, T_n y), \frac{d(fx, T_n y) + d(fy, T_n x)}{2}\} \\ &\quad + L\min\{d(fx, T_n x), d(fy, T_n y), d(fx, T_n y), d(fy, T_n x)\}) \end{aligned}$$

holds for all $x, y \in Y$. Consequently, each T_n is a generalized multivalued $(f, L)-$almost $F-$contraction on Y. Hence, from Theorem 2.1 we conclude that

$$fx_n \in Tx_n = \lambda_n Tx_n + (1-\lambda_n)q$$

for some $x_n \in Y$. As

$$fx_n = \lambda_n y_n + (1-\lambda_n)q$$

for some $y_n \in Tx_n \subseteq T(Y)$. As $T(Y)$ is bounded, and

$$\|fx_n - y_n\| = (1-\lambda_n)\|q - y_n\| \leq (1-\lambda_n)(\|q\| + \|y_n\|),$$

so $\lim_{n\to\infty} \|fx_n - y_n\| = 0$ because $\lim_{n\to\infty} \lambda_n = 1$. Hence

$$\lim_{n\to\infty} d(fx_n, Tx_n) \leq \lim_{n\to\infty} \|fx_n - y_n\| = 0.$$

Since the pair (f, T) satisfies the coincidence point condition on Y, there exists a $u \in Y$ such that $fu \in Tu$. Thus $C(f, T) \neq \emptyset$. Using arguments similar to those given in the proof of Theorem 2.1, it can be shown that $F(f, T) \neq \emptyset$ if one of the conditions (a)-(d) of Theorem 2.1 holds. □

Remark 2.13. Clearly an F_f-nonexpansive multivalued map T is $f-$almost nonexpansive in [1]. Thus $F = \ln x$ in inequality (2.4) yields $f-$almost nonexpansive, so Theorem 2.12 improves and generalizes Theorem 2.3 in [1], Corollary 2.5 in [18], Corollaries 3.2, 3.4 in [20], Theorems 2.2-2.5 in [23], and Theorem 3 in [29].

Example 2.14. Let l_1 be the linear space of all summable sequences of real numbers. Then the pair $(l_1, \|\cdot\|_1)$ is a Banach space. For each $x = (x_1, x_2, x_3, ...), y = (y_1, y_2, y_3, ...) \in l_1$, define $\|x\|_1 = \sum\limits_{n=1}^{\infty} |x_n|$, and $\|x - y\|_1 = \sum\limits_{n=1}^{\infty} |x_n - y_n|$. Let $Y = \{x \in l_1 : \|x\|_1 \leq 1\} \subseteq l_1$. Define the mappings $f : Y \to Y$ and $T : Y \to CL(Y)$ by

$$\begin{aligned} f(x) &= (x_1, x_2, 0, 0, ...) \\ T(x) &= \{(0, 0, 0, ...), kf(x)\} \end{aligned}$$

for some $k \in (0, \frac{1}{2})$. Clearly Y is starshaped with star center $z = (0, 0, 0, ...)$, f is $z-$affine and $\overline{T(Y)}$ is bounded. It is easy to see that $\overline{T(Y)} \subseteq f(Y)$. Since $f(Y)$ is homeomorphic to the compact subset D of $\mathbb{R}^2$, where $D = \{(u, v) : |u| + |v| \leq 1\}$, therefore $f(Y)$ is compact. Since $(l_1, \|\cdot\|_1)$ is Hausdorff space, therefore $f(Y)$ is closed and so $f(Y)$ is complete. This further implies that $\overline{T(Y)}$ is complete. Now for $x, y \in Y$ with $Tx \neq Ty$, we obtain that

$$\begin{aligned} H(Tx, Ty) &= \|kf(x) - kf(y)\|_1 \\ &= k(|x_1 - y_1| + k|x_2 - y_2| \\ &\leq (|x_1 - y_1| + |x_2 - y_2|) \\ &= \|f(x) - f(y)\|_1 \\ &\leq (\max\{\|fx - fy\|_1, \delta(fx, Tx), \delta(fy, Ty), \frac{\delta(fx, Ty) + \delta(fy, Tx)}{2}\} \\ &\quad +L\min\{\delta(fx, Tx), \delta(fy, Ty), \delta(fx, Ty), \delta(fy, Tx)\}) \end{aligned}$$

for any $L \geq 0$. If we set $F(x) = \ln x$, then we have

$$\begin{aligned} F(H(Tx, Ty)) \leq\ & F(\max\{\|fx - fy\|, \delta(fx, Tx), \delta(fy, Ty), \frac{\delta(fx, Ty) + \delta(fy, Tx)}{2}\} \\ &+L\min\{\delta(fx, Tx), \delta(fy, Ty), \delta(fx, Ty), \delta(fy, Tx)\})). \end{aligned}$$

Thus all the conditions of Theorem 2.12 are satisfied. Hence $C(f, T) \neq \emptyset$. In particular $C(f, T) = F(f, T) = \{0\}$, where $0 = (0, 0, 0, ...) \in Y$.

Corollary 2.15. *Let Y be a subset of a normed space X, $f : Y \longrightarrow Y$ and $T : Y \longrightarrow CL(Y)$ be an F_f-nonexpansive. Suppose that Y is $q-$starshaped, $f(Y) = Y$ [resp. f is $q-$affine on Y], $\overline{T(Y)}$ is complete, $\overline{T(Y)} \subseteq f(Y)$. Assume that one of the following conditions holds:*

(e) $T(Y)$ is bounded and $(f - T)(Y)$ is closed.

(f) Y is weakly compact and $f-T$ is demiclosed at 0. Then $C(f, T) \neq \emptyset$ provided that either F is continuous or T is closed multivalued mapping.

Moreover $F(f, T) \neq \emptyset$ if one of the conditions (a)-(d) of Theorem 2.1 holds.

Proof. (e) As in the proof of Theorem 2.12, we obtain that $\lim\limits_{n\to\infty}(fx_n - y_n) = 0$, where $y_n \in Tx_n$. Since $(f-T)(Y)$ is closed, $0 \in (f-T)(Y)$. Hence the pair (f, T) satisfies the coincidence point condition on Y and result follows from Theorem 2.12. (f) As in the proof of Theorem 2.12 we obtain that $\lim\limits_{n\to\infty}(fx_n - y_n) = 0$, where $y_n \in Tx_n$. By the weak compactness of Y, there is a subsequence $\{x_m\}$ of the sequence $\{x_n\}$ such that $\{x_m\}$ converges weakly to $y \in Y$ as m tends to ∞. Since $f-T$ is demiclosed at 0, therefore $0 \in (f-T)y$. Hence the pair (f, T) satisfies coincidence point condition on Y and result follows from Corollary 2.3. □

Corollary 2.16. *Let Y be a subset of a normed space X and $T : Y \longrightarrow CL(Y)$ be an F_I-nonexpansive, where I is identity map. Suppose that Y is $q-$starshaped, $T(Y)$ is bounded, $\overline{T(Y)}$ is complete, $\overline{T(Y)} \subseteq Y$. Then T has a fixed point.*

3. Applications

(1) Existence and uniqueness of common solution of system of functional equations in dynamic programming:

Decision space and a state space are two basic components of dynamic programming problem. State space is a set of states including initial states, action states and transitional states. So a state space is set of parameters representing different states. A decision space is the set of possible actions that can be taken to solve the problem. These general settings allow us to formulate many problems in mathematical optimization and computer programming. In particular the problem of dynamic programming related to multistage process reduces to the problem of solving functional equations

$$p(x) = \sup_{y\in D}\{g(x,y) + G_1(x,y,p(\xi(x,y)))\}, \text{ for } x \in W \tag{3.1}$$

$$q(x) = \sup_{y\in D}\{g'(x,y) + G_2(x,y,q(\xi(x,y)))\}, \text{ for } x \in W, \tag{3.2}$$

where U and V are Banach spaces, $W \subseteq U$ and $D \subseteq V$ and

$$\begin{aligned} \xi &: W \times D \longrightarrow W \\ g, g' &: W \times D \longrightarrow \mathbb{R} \\ G_1, G_2 &: W \times D \times \mathbb{R} \longrightarrow \mathbb{R}, \end{aligned}$$

for more details on dynamic programming we refer to [8, 9, 10, 11, 28]. Suppose that W and D are the state and decision spaces respectively. We aim to give the existence and uniqueness of common and bounded solution of functional equations given in (3.1) and (3.2). Let $B(W)$ denotes the set of all bounded real valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x\in W} |h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space endowed with the metric d defined as

$$d(h,k) = \sup_{x\in W} |hx - kx| . \tag{3.3}$$

Suppose that the following conditions hold:

$(C1)$: $G_1, G_2, g,$ and g' are bounded.
$(C2)$: For $x \in W$, $h \in B(W)$ and $b > 0$, define

$$Kh(x) = \sup\nolimits_{y\in D}\{g(x,y) + G_1(x,y,h(\xi(x,y)))\}, \tag{3.4}$$

$$Jh(x) = \sup\nolimits_{y\in D}\{g'(x,y) + G_2(x,y,h(\xi(x,y)))\}. \tag{3.5}$$

Moreover assume that there exist $\tau > 0$ and $L \geq 0$ such that for every $(x,y) \in W \times D$, $h,k \in B(W)$ and $t \in W$ implies

$$|G_1(x,y,h(t)) - G_1(x,y,k(t))| \leq e^{-2\tau}[M(h(t),k(t)) + LN(h(t),k(t))] \tag{3.6}$$

where

$$\begin{aligned} M((h(t),k(t)) &= \max\{d(Jh(t),Jk(t)), d(Jk(t),Kk(t)), d(Jh(t),Kh(t)), \\ &\quad \frac{d(Jh(t),Kk(t)) + d(Jk(t),Kh(t))}{2}\}, \\ N((h(t),k(t)) &= \min\{d(h(t),Kh(t)), d(k(t),Kk(t)), d(h(t),Kk(t)), d(k(t),Kh(t))\}. \end{aligned}$$

$(C3)$: For any $h \in B(W)$, there exists $k \in B(W)$ such that for $x \in W$

$$Kh(x) = Jk(x).$$

$(C4)$: There exists $h \in B(W)$ such that

$$Kh(x) = Jh(x) \text{ implies that } JKh(x) = KJh(x).$$

Theorem 3.1. *Assume that the conditions $(C1)-(C4)$ are satisfied. If $J(B(W))$ is a closed convex subspace of $B(W)$, then the functional equations (3.1) and (3.2) have a unique, common and bounded solution.*

Proof. Note that $(B(W), d)$ is a complete metric space. By $(C1)$, J, K are self-maps of $B(W)$. The condition $(C3)$ implies that $K(B(W)) \subseteq J(B(W))$. It follows from $(C4)$ that J and K commute at their coincidence points. Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Choose $x \in W$ and $y_1, y_2 \in D$ such that

$$Kh_j < g(x, y_j) + G_1(x, y_j, h_j(x_j)) + \lambda, \tag{3.7}$$

where $x_j = \xi(x, y_j)$, $j = 1, 2$. Further from (3.4) and (3.5), we have

$$\begin{aligned} Kh_1 &\geq g(x, y_2) + G_1(x, y_2, h_1(x_2)) && (3.8)\\ Kh_2 &\geq g(x, y_1) + G_1(x, y_1, h_2(x_1)). && (3.9)\end{aligned}$$

Then (3.7) and (3.9) together with (3.6) imply

$$\begin{aligned} Kh_1(x) - Kh_2(x) &< G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2)) + \lambda \\ &\leq |G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2))| + \lambda \\ &\leq e^{-2\tau}(M((h(t), k(t)) + LN(h(t), k(t))) + \lambda. \end{aligned} \tag{3.10}$$

Then (3.7) and (3.8) together with (3.6) imply

$$\begin{aligned} Kh_2(x) - Kh_1(x) &\leq G_1(x, y_1, h_2(x_2)) - G_1(x, y_1, h_1(x_1)) \\ &\leq |G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2))| \\ &\leq e^{-2\tau}(M((h(t), k(t)) + LN(h(t), k(t))). \end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we have

$$|Kh_1(x) - Kh_2(x)| \leq e^{-2\tau}(M((h(t), k(t)) + LN(h(t), k(t))). \tag{3.12}$$

The inequality (3.12) implies

$$d(Kh_1(x) - Kh_2(x)) \leq e^{-2\tau}[(M((h(t), k(t)) + LN(h(t), k(t)))]. \tag{3.13}$$

$$2\tau + \ln[d(Kh_1(x) - Kh_2(x))] \leq \ln[(M((h(t), k(t)) + LN(h(t), k(t)))]. \tag{3.14}$$

Therefore by Corollary (2.5), the pair (K, J) has a common fixed point h^*, that is, $h^*(x)$ is unique, bounded and common solution of (3.1) and (3.2). □

(1) Existence and uniqueness of common solution of system of integral equations:

Now we discuss an application of fixed point theorem we proved in the previous section in solving the system of Volterra type integral equations. Such system is given by the following equations:

$$u(t) = \int_0^t K_1(t, s, u(s))ds + g(t), \tag{3.15}$$

$$w(t) = \int_0^t K_2(t, s, w(s))ds + f(t). \tag{3.16}$$

for $t \in [0, a]$, where $a > 0$. We find the solution of the system (3.15) and (3.16). Let $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on $[0, a]$. For $u \in C([0, a], \mathbb{R})$, define supremum norm as: $\|u\|_\tau = \sup_{t\in[0,a]} \{u(t)e^{-\tau t}\}$, where $\tau > 0$ is taken arbitrary. Let $C([0, a], \mathbb{R})$ be endowed with the metric

$$d_\tau(u, v) = \sup_{t\in[0,a]} \| \left|u(t) - v(t)\right| e^{-\tau t}\|_\tau \tag{3.17}$$

for all $u, v \in C([0, a], \mathbb{R})$. With these setting $C([0, a], \mathbb{R}, \|\cdot\|_\tau)$ becomes Banach space.

Now we prove the following theorem to ensure the existence of solution of system of integral equations. For more details on such applications we refer the reader to [7, 26].

Theorem 3.2. *Assume the following conditions are satisfied:*

(i) $K_1, K_2 : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ *and* $f, g : [0, a] \to \mathbb{R}$ *are continuous;*

(ii) Define

$$\begin{aligned} Tu(t) &= \int_0^t K_1(t, s, u(s))ds + g(t), \\ Su(t) &= \int_0^t K_2(t, s, u(s))ds + f(t). \end{aligned}$$

Suppose there exist $\tau \geq 1$ *and* $L \geq 0$*such that*

$$|K_1(t, s, u) - K_1(t, s, v)| \leq \tau e^{-2\tau}[M(u, v) + LN(u, v)]$$

for all $t, s \in [0, a]$ *and* $u, v \in C([0, a], \mathbb{R})$, *where*

$$\begin{aligned} M(u, v) &= \max\{|Su(t) - Sv(t)|, |Sv(t) - Tv(t)|, |Su(t) - Tu(t)|, \frac{|Su(t) - Tv(t)| + |Sv(t) - Tu(t)|}{2}\}, \\ N(u, v) &= \min\{|u(t) - Tu(t)|, |v(t) - Tv(t)|\ |u(t) - Tv(t)|, |v(t) - Tu(t)|\}; \end{aligned}$$

(iii) there exists $u \in C([0, a], \mathbb{R})$ *such that* $Tu(t) = Su(t)$ *implies* $TSu(t) = STu(t)$. *Then the system of integral equations given in (3.15) and (3.16) has a solution.*

Proof. By assumption (iii)

$$\begin{aligned} |Tu(t) - Tv(t)| &= \int_0^t |K_1(t, s, u(s) - K_1(t, s, v(s)))| \, ds \\ &\leq \int_0^t \tau e^{-2\tau}([M(u, v) + LN(u, v)]e^{-\tau s})e^{\tau s} ds \\ &\leq \int_0^t \tau e^{-2\tau}\|M(u, v) + LN(u, v)\|_\tau e^{\tau s} ds \\ &\leq \tau e^{-2\tau}\|M(u, v) + LN(u, v)\|_\tau \int_0^t e^{\tau s} ds \\ &\leq \tau e^{-2\tau}\|M(u, v) + LN(u, v)\|_\tau \frac{1}{\tau} e^{\tau t} \\ &\leq e^{-2\tau}\|M(u, v) + LN(u, v)\|_\tau e^{\tau t}. \end{aligned}$$

This implies

$$|Tu(t) - Tv(t)|\, e^{-\tau t} \leq e^{-2\tau} \|M(u, v) + LN(u, v)\|_\tau,$$

That is

$$\|Tu(t) - Tv(t)\|_\tau \leq e^{-2\tau} \|M(u, v) + LN(u, v)\|_\tau$$

which further implies

$$2\tau + \ln \|Tu(t) - Tv(t)\|_\tau \leq \ln \|M(u, v) + LN(u, v)\|_\tau.$$

So all the conditions of Corollary 2.5 are satisfied. Hence the system of integral equations given in (3.15) and (3.16) has a unique common solution. □

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