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On strong and $\triangle$-convergence of modified S-iteration for uniformly continuous total asymptotically nonexpansive mappings in CAT(κ) spaces

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Abstract

In this paper, we obtain strong and Δ-convergence theorems of modified S-iteration for total asymptotically nonexpansive mappings in CAT(κ) spaces with $\kappa > 0$. Our results extend and improve the corresponding recent results announced by Panyanak [B. Panyanak, J. Inequal. Appl., 2014 (2014), 13 pages] and many authors.

Keywords: fixed point, total asymptotically nonexpansive mapping, $\triangle$ - convergence, CAT(κ) space, S-iteration.
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1. Introduction

The initials of the term CAT are in honor of E. Cartan, A. D. Alexanderov and V. A. Toponogov, who have made important contributions to the understanding of curvature via inequalities for the distance function. A CAT(κ) space is a geodesic metric space which no geodesic triangle is fatter than the corresponding

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comparison triangle in a model space with constant curvature κ, for $\kappa \in \mathbb{R}$. It is a generalization of a simply-connected Riemannian manifold with sectional curvature $\leq \kappa$.

Kirk ([18, 19]) first studied the theory of fixed point in CAT(κ) spaces. Later on, many authors generalized the notion of CAT(κ) given in [18, 19], mainly focusing on CAT(0) spaces (see e.g., [1, 9, 10, 12, 16, 20, 22, 29, 26, 30]). The results of a CAT(0) space can be applied to any CAT(κ) space with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for every $\kappa' \geq \kappa$ (see in [5]). Although, CAT(κ) spaces for $\kappa > 0$, were studied by some authors (see e.g., [13, 15, 25]).

Alber et al. [3] first introduced the total asymptotically nonexpansive mappings in Banach spaces. He generalizes the concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [14] as well as the concept of nearly asymptotically nonexpansive mappings was introduced by Sahu [27]. Recently, Panyanak [25] studied the existence theorems, the demiclosed principle, Δ-convergence and strongly convergence theorems for uniformly continuous total asymptotically nonexpansive mappings in CAT(κ) spaces. Moreover, there were many authors who have studied about this mappings, (see e.g., [4, 7, 8, 17, 25, 31, 33, 34, 35, 36, 37]).

The S-iteration process was introduced by Agarwal, O'Regan and Sahu [2] in a Banach space. They showed that their process was independent of those of Mann and Ishikawa and converges faster than both of theses (see in [2]).

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T(y_n), \\ y_n = (1-\beta_n)x_n + \beta_n T(x_n), n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0, 1).

In 1991, Schu [28] considered the following modified Mann iteration process which is a generalization of the Mann iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n(x_n), n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in (0, 1).

In 1994, Tan and Xu [32] studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n(y_n), \\ y_n = (1-\beta_n)x_n + \beta_n T^n(x_n), n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. This iteration process reduces to the modified Mann iteration process when $\beta_n = 0$ for all $n \in \mathbb{N}$.

In 2007, Agarwal, O'Regan and Sahu [2] introduced the following modified S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n(y_n), \\ y_n = (1-\beta_n)x_n + \beta_n T^n(x_n), n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. Note that (1.4) is independent of (1.3) (and hence of (1.2)). Also, (1.4) reduces to (1.1) when $n = 1$.

Recently, Kumam, Saluja and Nashine [21] studied modified S-iteration process and investigated the existence and convergence theorems in the setting of CAT(0) spaces for a class of mappings which is wider than that of asymptotically nonexpansive mappings as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n S^n(y_n), \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n(x_n), n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0,1]$, for all $n \geq 1$.

Motivated and inspired by (1.4) and (1.5) we proposed the algorithm as follows.

Let K be a nonempty closed convex subset of a complete CAT(κ) space X and $T : K \to K$ be uniformly continuous total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n T^n(y_n), \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n(x_n), n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$, for all $n \geq 1$.

The purpose of this paper was to prove strong and Δ-convergence of the modified S-iteration process for uniformly continuous total asymptotically nonexpansive mappings in CAT(κ) spaces. Our results extend and improve the corresponding recent results announced by [25]. This paper was organized as follows. In section 2 and 3, we present preliminaries and results of strong and Δ-convergence, respectively.

2. Preliminaries

In this section , we divide the content of preliminaries into three parts as follows.

2.1. CAT(κ) spaces and property

Let (X,ρ) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map γ from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $\gamma(0) = x, \gamma(l) = y$, and $\rho(\gamma(t),\gamma(t')) = |t-t'|$ for all $t,t' \in [0,l]$. In particular, γ is an isometry and $\rho(x,y) = l$. The image $\gamma([0,l])$ of γ is called a *geodesic segment* joining x and y. When it is unique this geodesic segment is denoted by $[x,y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that

$$\rho(x,z) = (1-\alpha)\rho(x,y) \text{ and } \rho(y,z) = \alpha\rho(x,y).$$

In this case, we write $z = \alpha x \oplus (1-\alpha)y$. The space (X,ρ) is said to be a geodesic space ($D-$ *geodesic space*) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be uniquely geodesic ($D-$ *uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x,y \in X$ (for $x,y \in X$ with $\rho(x,y) < D$). A subset K of X is said to be convex if K includes every geodesic segment joining any two of its points. The set K is said to be *bounded* if

$$\operatorname{diam}(K) := \sup\{\rho(x,y) : x,y \in K\} < \infty.$$

Now we introduce the model spaces M^n_κ, for more details on these spaces the reader is referred to [5]. Let $n \in N$. We denote by $\mathbb{E}^n$ the metric space $\mathbb{R}^n$ endowed with the usual Euclidean distance. We denote by $(\cdot|\cdot)$ the Euclidean scalar product in $\mathbb{R}^n$, that is,

$$(x|y) = x_1y_1 + ... + x_ny_n \text{ where } x = (x_1, ..., x_n), y = (y_1, ..., y_n).$$

Let $\mathbb{S}_n$ denote the $n-$*dimensional sphere* defined by

$$\mathbb{S}^n = \{x = x_1, ..., x_{n+1} \in \mathbb{R}^{n+1} : (\cdot|\cdot) = 1\},$$

with metric $d_{\mathbb{S}^n} = arccos(x|y), x, y \in \mathbb{S}^n$.

Let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1, ..., u_{n+1})$ and $v = (v_1, ..., v_{n+1})$ the real number $\langle u|v\rangle$ defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_iv_i.$$

Let $\mathbb{H}^n$ denote the *hyperbolic* $n-$*space* defined by

$$\mathbb{H}^n = \{u = (u_1, u_2, ..., u_{n+1}) \in \mathbb{E}^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 1\}$$

with metric $d_{\mathbb{H}^n}$ such that

$$\cosh d_{\mathbb{H}^n}(x, y) = -\langle x|y\rangle, x, y \in \mathbb{H}^n.$$

Definition 2.1. Given $\kappa \in \mathbb{R}$, we denote by M^n_κ the following metric spaces:

(1) if $\kappa = 0$ then M^n_0 is the Euclidean space $\mathbb{E}^n$;

(2) if $\kappa > 0$ then M^n_κ is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{\kappa}$;

(3) if $\kappa < 0$ then M^n_κ is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $1/\sqrt{-\kappa}$.

A *geodesic triangle* $\Delta(x, y, z)$ in a geodesic space (X, ρ) consists of three points x, y, z in X (the *vertices* of Δ) and three geodesic segments between each pair of vertices (the *edges* of Δ). A comparison triangle for a geodesic triangle $\Delta(x, y, z)$ in (X, ρ) is a triangle $\Delta(\overline{x}, \overline{y}, \overline{z})$ in M^2_κ such that

$$\rho(x, y) = d_{M^2_\kappa}(\overline{x}, \overline{y}), \rho(x, z) = d_{M^2_\kappa}(\overline{x}, \overline{z}) \text{ and } \rho(z, x) = d_{M^2_\kappa}(\overline{z}, \overline{x}).$$

If $\kappa \leq 0$ then such a comparison triangle always exists in M^2_κ. If $\kappa > 0$ then such a triangle exists whenever $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a comparison point for $p \in [x, y]$ if $\rho(x, p) = d_{M^2_\kappa}(\overline{x}, \overline{p})$.

A geodesic triangle $\Delta(x, y, z)$ in X is said to satisfy the CAT(κ) inequality if for any $p, q \in \Delta(x, y, z)$ and for their comparison points $\overline{p}, \overline{q} \in \Delta(\overline{x}, \overline{y}, \overline{z})$, one has

$$\rho(p, q) \leq d_{M^2_\kappa}(\overline{p}, \overline{q}).$$

Definition 2.2. If $\kappa \leq 0$, then X is called a CAT(κ) *space* if and only if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality. If $\kappa > 0$, then X is called a CAT(κ) *space* if and only if X is D_κ -geodesic and any geodesic triangle $\Delta(x, y, z)$ in X with $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Notice that in a CAT(0) space (X, ρ), if $x, y, z \in X$ then the CAT(0) inequality implies

$$\rho^2(x, \frac{1}{2}y \oplus \frac{1}{2}z) \leq \frac{1}{2}\rho^2(x, y) + \frac{1}{2}\rho^2(x, z) - \frac{1}{4}\rho^2(y, z). \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [6]. This inequality is extended by Dhompongsa and Panyanak [11] as

$$\rho^2(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)\rho^2(x, y) + \alpha\rho^2(x, z) - (1-\alpha)\alpha\rho^2(y, z). \tag{CN*}$$

for all $\alpha \in [0, 1]$ and $x, y, z \in X$. In fact, if X is a geodesic space then the following statements are equivalent:

(1) X is a CAT(0) space;

(2) X satisfies (CN);

(3) X satisfies (CN*).

Let $R \in (0, 2]$. Recall that a geodesic space (X, ρ) is said to be $R-convex$ for R (see [24]) if for any three points $x, y, z \in X$, we have

$$\rho^2(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)\rho^2(x, y) + \alpha\rho^2(x, z) - \frac{R}{2}(1-\alpha)\alpha\rho^2(y, z). \tag{2.1}$$

It follows from (CN*) that a geodesic space (X, ρ) is a CAT(0) space if and only if (X, ρ) is $R-convex$ for $R = 2$. The following lemma is a consequence of Proposition 3.1 in [24].

Lemma 2.3. *Let $\kappa > 0$ and (X, ρ) be a CAT(κ) space with $diam(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Then (X, ρ) is $R-convex$ for $R = (\pi - 2\varepsilon)tan(\varepsilon)$.*

The following lemma is also needed.

Lemma 2.4 ([5])**.** *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with $diam(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Then*

$$\rho((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)\rho(x, z) + \alpha\rho(y, z),$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$.

2.2. Δ-convergence for total asymptotically nonexpansive mappings in CAT(κ) spaces

We now collect some elementary facts about CAT(κ) spaces. Most of them are proved in the setting of CAT(1) spaces. For completeness, we state the results in CAT(κ) with $\kappa > 0$.

Let $\{x_n\}$ be a bounded sequence in a CAT(κ) space (X, ρ). For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \rho(x, \{x_n\}).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from [13] that in a CAT(κ) space X with $diam(X) < \frac{2\pi}{2\sqrt{\kappa}}$, $A(\{x_n\})$ consists of exactly one point. We now give the concept of Δ-convergence and collect some of its basic properties.

Definition 2.5 ([20, 23])**.** A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.6 ([26])**.** *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with $diam(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Then the following statements hold:*

(i) every sequence in X has a Δ-convergence subsequence;

(ii) if $\{x_n\} \subseteq X$ and $\Delta - lim_n x_n = x$, then $x \in \bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, ...\}$, where $\overline{conv(A)} = \bigcap\{B : B \supseteq A$ and B is closed and convex$\}$.

By the uniqueness of asymptotic centers, we can obtain the following lemma (see [11]).

Lemma 2.7. *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. If $\{x_n\}$ is a sequence in X with $A(\{x_n\}) = \{x\}$ and let $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{\rho(x_n, u)\}$ converges, then $x = u$.*

Definition 2.8. Let K be a nonempty subset of a CAT(κ) space (X, ρ). A mapping $T : K \to K$ is called total asymptotically nonexpansive if there exist nonnegative real sequences $\{\nu_n\}, \{\mu_n\}$ with $\nu_n \to 0, \mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : [0, 1) \to [0, 1)$ with $\psi(0) = 0$ such that

$$\rho(T^n(x), T^n(y)) \leq \rho(x, y) + \nu_n \psi(\rho(x, y)) + \mu_n \text{ for all } n \in N, x, y \in K.$$

A point $x \in K$ is called a *fixed point* of T if $x = T(x)$. We denote with $F(T)$ the set of fixed points of T. A sequence $\{x_n\}$ in K is called approximate fixed point sequence for T (AFPS in short) if

$$\lim_{n\to\infty} \rho(x_n, T(x_n)) = 0.$$

Lemma 2.9 ([32])**.** *Let $\{s_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying*

$$s_{n+1} \leq s_n + t_n \text{ for all } n \in N.$$

If $\sum_{n=1}^{\infty} t_n < \infty$ then $\lim_{n\to\infty} s_n$ exists.

2.3. Existence theorems, Demiclosed principle and Semi-compact

Theorem 2.10. *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ be a continuous total asymptotically nonexpansive mapping. Then T has a fixed point in K.*

Proof. See in [25] □

Theorem 2.11. *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for T such that $\Delta \to \lim_{n\to\infty} x_n = \omega$, then $\omega \in K$ and $\omega = T(\omega)$.*

Proof. See in [25] □

Definition 2.12. Let (X, ρ) be a metric space and K be its nonempty subset. Then $T : K \to K$ is said to be semi-compact if for a sequence x_n in K with $\lim_{n\to\infty} \rho(x_n, Tx_n) = 0$, there exists a subsequence x_{n_k} of x_n such that $x_{n_k} \to p \in K$.

3. Main results

In this section, we prove strong and Δ-convergence of the modified S-iteration process for total asymptotically nonexpansive mappings in a CAT(κ) spaces as follows.

Lemma 3.1. *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with diam$(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (1.6) where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ such that $\liminf_n \alpha_n\beta_n(1-\beta_n) > 0$. Then $\{x_n\}$ is an AFPS for T and $\lim_n \rho(x_n, p)$ exists for all $p \in F(T)$.*

Proof. We divide the proof of this lemma into two steps.

Step 1 : We will prove that $\lim_n \rho(x_n, p)$ exists.

It follows from Theorem 2.10 that $F(T) \neq \emptyset$. Let $p \in F(T)$ and $M = diam(K)$. Since T is total asymptotically nonexpansive, by Lemma 2.4 we have

$$\begin{aligned}
\rho(y_n, p) &= \rho((1-\beta_n)x_n \oplus \beta_n T^n(x_n), p) \\
&\leq (1-\beta_n)\rho(x_n, p) + \beta_n \rho(T^n(x_n), p) \\
&= (1-\beta_n)\rho(x_n, p) + \beta_n \rho(T^n(x_n), T^n(p)) \\
&\leq (1-\beta_n)\rho(x_n, p) + \beta_n\{\rho(x_n, p) + \nu_n\psi(M) + \mu_n\} \\
&\leq \rho(x_n, p) + \beta_n\nu_n\psi(M) + \beta_n\mu_n.
\end{aligned}$$

This implies that

$$\begin{aligned}
\rho(x_{n+1}, p) &= \rho((1-\alpha_n)T^n(x_n) \oplus \alpha_n T^n(y_n), p) \\
&\leq (1-\alpha_n)\rho(T^n(x_n), p) + \alpha_n\rho(T^n(y_n), T^n(p)) \\
&\leq (1-\alpha_n)\rho(T^n(x_n), T^n(p)) + \alpha_n\rho(T^n(y_n), T^n(p)) \\
&\leq (1-\alpha_n)\{\rho(x_n, p) + \nu_n\psi(\rho(x_n, p)) + \mu_n\} \\
&\quad + \alpha_n\{\rho(y_n, p) + \nu_n\psi(\rho(y_n, p)) + \mu_n\} \\
&\leq (1-\alpha_n)\{\rho(x_n, p) + \nu_n\psi(M) + \mu_n\} \\
&\quad + \alpha_n\{\rho(x_n, p) + \beta_n\nu_n\psi(M) + \beta_n\mu_n \\
&\quad + \nu_n\psi(M) + \mu_n\} \\
&\leq \rho(x_n, p) + \nu_n\psi(M) + (1 + \alpha_n\beta_n)\mu_n).
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \nu_n < 1$ and $\sum_{n=1}^{\infty} \mu_n < 1$, by Lemma 2.9 $\lim_{n\to\infty} \rho(x_n, p)$ exists.

Step 2 : We will prove that $\lim_{n\to\infty} \rho(x_n, T(x_n)) = 0$.

Next,we show that $\{x_n\}$ is an AFPS for T. In view of (2.1), we have

$$\begin{aligned}
\rho^2(x_{n+1}, p) &= \rho^2((1-\alpha_n)T^n(x_n) \oplus \alpha_n T^n(y_n), p) \\
&\leq (1-\alpha_n)\rho^2(T^n(x_n), p) + \alpha_n\rho^2(T^n(y_n), p) - \frac{R}{2}\alpha_n(1-\alpha_n)\rho^2(T^n(x_n), T^n(y_n)) \\
&\leq (1-\alpha_n)\rho^2(T^n(x_n), T^n(p)) + \alpha_n\rho^2(T^n(y_n), T^n(p)) \\
&\leq (1-\alpha_n)[\rho(x_n, p) + (\nu_n\psi(\rho(x_n, p)) + \mu_n)]^2 + \alpha_n[\rho(y_n, p) + (\nu_n\psi(\rho(y_n, p)) + \mu_n)]^2 \\
&\leq (1-\alpha_n)[\rho^2(x_n, p) + 2\rho(x_n, p)(\nu_n\psi(\rho(x_n, p)) + \mu_n) + (\nu_n\psi(\rho(x_n, p)) + \mu_n)^2] \\
&\quad + \alpha_n[\rho^2(y_n, p) + 2\rho(y_n, p)(\nu_n\psi(\rho(y_n, p)) + \mu_n) + (\nu_n\psi(\rho(y_n, p)) + \mu_n)^2] \\
&\leq (1-\alpha_n)\rho^2(x_n, p) + (1-\alpha_n)[2\rho(x_n, p)(\nu_n\psi(\rho(x_n, p)) + \mu_n) + (\nu_n\psi(\rho(x_n, p)) + \mu_n)^2] \\
&\quad + \alpha_n\rho^2(y_n, p) + \alpha_n[2\rho(y_n, p)(\nu_n\psi(\rho(y_n, p)) + \mu_n) + (\nu_n\psi(\rho(y_n, p)) + \mu_n)^2].
\end{aligned}$$

This implies that

$$\rho^2(x_{n+1}) \leq (1-\alpha_n)\rho^2(x_n, p) + \alpha_n\rho^2(y_n, p) + A\nu_n + B\mu_n \quad \exists A, B \geq 0. \tag{3.1}$$

Again by (2.1) , we have

$$
\begin{aligned}
\rho^2(y_n,p) &= \rho^2((1-\beta_n)x_n \oplus \beta_n T^n((x_n),p)) \\
&\leq (1-\beta_n)\rho^2(x_n,p) + \beta_n\rho^2(T^n(x_n),T^n(p)) - \frac{R}{2}\beta_n(1-\beta_n)\rho^2(x_n,T^n(x_n)) \\
&\leq (1-\beta_n)\rho^2(x_n,p) + \beta_n[\rho(x_n,p)+\nu_n\psi(M)+\mu_n]^2 \\
&\quad -\frac{R}{2}\beta_n(1-\beta_n)\rho^2(x_n,T^n(x_n)) \\
&\leq \rho^2(x_n,p) + \beta_n[2\rho(x_n,p)(\nu_n\psi(M)+\mu_n)+(\nu_n\psi(M)+\mu_n)^2] \\
&\quad -\frac{R}{2}\beta_n(1-\beta_n)\rho^2(x_n,T^n(x_n)).
\end{aligned}
$$

Substituting this into (3.1), we get that

$$
\begin{aligned}
\rho^2(x_{n+1},p) &\leq (1-\alpha_n)\rho^2(x_n,p) + \alpha_n[\rho^2(x_n,p) + \beta_n[2\rho(x_n,p)(\nu_n\psi(M)+\mu_n) \\
&\quad +(\nu_n\psi(M)+\mu_n)^2] - \frac{R}{2}\beta_n(1-\beta_n)\rho^2(x_n,T^n(x_n))] + A\nu_n + B\mu_n, \\
&\leq \rho^2(x_n,p) + \alpha_n[\beta_n[2\rho(x_n,p)(\nu_n\psi(M)+\mu_n)+(\nu_n\psi(M)+\mu_n)^2] \\
&\quad -\frac{R}{2}\beta_n(1-\beta_n)\rho^2(x_n,T^n(x_n))] + A\nu_n + B\mu_n,
\end{aligned}
$$

yielding

$$\frac{R}{2}\alpha_n\beta_n(1-\beta_n)\rho^2(x_n,T^n(x_n)) \leq \rho^2(x_n,p) - \rho^2(x_{n+1},p) + C\nu_n + D\mu_n \ \ \exists C,D \geq 0.$$

Since $\sum_{n=1}^{\infty}\nu_n < \infty$ and $\sum_{n=1}^{\infty}\mu_n < \infty$, we have

$$\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)\rho^2(x_n,T^n(x_n)) < \infty.$$

This implies by $\liminf_{n\to\infty}\alpha_n\beta_n(1-\beta_n) > 0$ that

$$\lim_{n\to\infty}\rho(x_n,T^n(x_n)) = 0. \tag{3.2}$$

By the uniform continuity of T, we have

$$\lim_{n\to\infty}\rho(T(x_n),T^{n+1}(x_n)) = 0. \tag{3.3}$$

It follows from (3.2) and the definitions of x_{n+1} and y_n that

$$
\begin{aligned}
\rho(x_n,x_{n+1}) &= \rho(x_n,(1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n) \\
&\leq (1-\alpha_n)\rho(x_n,T^n(x_n)) + \alpha_n\rho(x_n,T^n(y_n)) \\
&\leq \rho(x_n,T^n(x_n)) + \rho(x_n,T^n(y_n)) \\
&\leq \rho(x_n,T^n(x_n)) + \rho(x_n,T^n(x_n)) + \rho(T^n(x_n),T^n(y_n)) \\
&\leq 2\rho(x_n,T^n(x_n)) + \rho(T^n(x_n),T^n(y_n)) \\
&= 2\rho(x_n,T^n(x_n)) + [\rho(x_n,y_n)+\nu_n\psi(M)+\mu_n] \\
&\leq 2\rho(x_n,T^n(x_n)) + \rho(x_n,(1-\beta_n)x_n \oplus \beta_n T^n(x_n)) + \nu_n\psi(M)+\mu_n \\
&\leq 2\rho(x_n,T^n(x_n)) + (1-\beta_n)\rho(x_n,x_n) + \beta_n\rho(x_n,T^n(x_n)) + \nu_n\psi(M)+\mu_n \\
&= 2\rho(x_n,T^n(x_n)) + \beta_n\rho(x_n,T^n(x_n)) + \nu_n\psi(M)+\mu_n \\
&= (2+\beta_n)\rho(x_n,T^n(x_n)) + \nu_n\psi(M)+\mu_n \to 0 \qquad as \quad n\to\infty.
\end{aligned} \tag{3.4}
$$

By (3.2),(3.3), and (3.4), we have

$$\begin{aligned}\rho(x_n,T(x_n)) &\le \rho(x_n,x_{n+1})+\rho(x_{n+1},T^n(x_{n+1}))\\ &\quad+\rho(T^{n+1}(x_{n+1}),T^{n+1}(x_n))+\rho(T^{n+1}(x_n),T(x_n))\\ &\le \rho(x_n,x_{n+1})+\rho(x_{n+1},T^n(x_{n+1}))+\rho(x_{n+1},x_n)\\ &\quad+\nu_{n+1}\psi(M)+\mu_{n+1}+\rho(T^{n+1}(x_n),T(x_n))\to 0 \qquad as \quad n\to\infty.\end{aligned}$$

□

Now, we are ready to prove our Δ - convergence theorem.

Theorem 3.2. *Let $\kappa > 0$ and (X,ρ) be a complete $CAT(\kappa)$ space with $diam(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty}\nu_n < \infty$ and $\sum_{n=1}^{\infty}\mu_n < \infty$ .Let $\{x_n\}$ be a sequence in K defined by* (1.6) *where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in* $(0,1)$ *such that* $\liminf_n \alpha_n\beta_n(1-\beta_n) > 0$. *Then $\{x_n\}$ Δ - converges to a fixed point of T.*

Proof. Let $\omega_\omega(\{x_n\}) := \bigcup A(\{u_n\})$ where the union is taken for all subsequences $\{u_n\}$ of $\{x_n\}$. We first show that $\omega_\omega(\{x_n\}) \subseteq F(T)$. Let $u \in \omega_\omega(\{x_n\})$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.6, there exists a subsequence $\{\upsilon_n\}$ of $\{u_n\}$ such that $\Delta-\lim_n \upsilon_n = \upsilon \in K$. By Lemma 3.1 and Theorem 2.11, we have $\upsilon \in F(T)$. Since $\lim_n \rho(x_n,\upsilon)$ exists, so $u = \upsilon$ by Lemma 2.7. This shows that $\omega_\omega(x_n) \subseteq F(T)$.

Next, we show that Δ-converges to a point in $F(T)$, it is sufficient to show that $\omega_\omega(\{x_n\})$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{xn\}) = \{x\}$. Since $u \in \omega_\omega(x_n) \subseteq F(T)$, by Lemma 3.1 $\lim_n \rho(x_n,u)$ exists. And by Lemma 2.7, we have $x = u$. This completes the proof.

□

As a consequence of Theorem 3.2, we obtain

Corollary 3.3 ([17])**.** *Let (X,ρ) be a complete $CAT(0)$ space, K be a nonempty bounded closed convex subset of X, and $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty}\nu_n < \infty$ and $\sum_{n=1}^{\infty}\mu_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by* (1.6) *where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in* $(0,1)$ *such that* $\liminf_n \alpha_n\beta_n(1-\beta_n) > 0$. *Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Now, we prove a strong convergence theorem for uniformly continuous total asymptotically nonexpansive semi-compact mappings.

Theorem 3.4. *Let $\kappa > 0$ and (X,ρ) be a complete $CAT(\kappa)$ space with $diam(X) \le \frac{\pi-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty}\nu_n < \infty$ and $\sum_{n=1}^{\infty}\mu_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by* (1.6) *where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in* $(0,1)$ *such that* $\liminf_n \alpha_n\beta_n(1-\beta_n) > 0$.*Suppose that T^m is semi-compact for some $m \in N$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, $\lim_n \rho(x_n,T(x_n)) = 0$. Since T is uniformly continuous, we have

$$\rho(x_n,T^m(x_n)) \le \rho(x_n,T(x_n))+\rho(T(x_n),T^2(x_n))+...+\rho(T^{m-1}(x_n),T^m(x_n))\to 0$$

as $n\to\infty$. That is, $\{x_n\}$ is an AFPS for T^m. By definition 2.12, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in K$ such that $\lim_{j\to\infty} x_{n_j} = p$. Again, by the uniform continuity of T, we have

$$\rho(T(p),p) \le \rho(T(p),T(x_{n_j}))+\rho(T(x_{n_j}),x_{n_j})+\rho(x_{n_j},p)\to 0 \quad as \quad j\to\infty.$$

That is, $p \in F(T)$. By Lemma 3.1, $\lim_n \rho(x_n, p)$ exists, thus p is the strong limit of the sequence $\{x_n\}$ itself.

□

Corollary 3.5 ([17]). *Let (X, ρ) be a complete $CAT(0)$ space, K be a nonempty bounded closed convex subset of X, and $T : K \to K$ be a uniformly continuous total asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (1.6) where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $\liminf_n \alpha_n \beta_n (1 - \beta_n) > 0$. Suppose that T^m is semi-compact for some $m \in N$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Remark 3.6. The results in this paper also hold for the class of weakly total asymptotically nonexpansive mappings in the following sense. A mapping $T : K \to K$ is called weakly total asymptotically nonexpansive if there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a nondecreasing function $\psi : [0, 1) \to [0, 1)$ such that

$$\rho(T^n(x), T^n(y)) = \rho(x, y) + \nu_n \psi(\rho(x, y)) + \mu_n \quad for \quad all \quad n \in \mathbb{N}, \quad x, y \in K.$$

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