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Generalized Lefschetz fixed point theorems in extension type spaces

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Abstract

Several Lefschetz fixed point theorems for compact type self maps in new classes of spaces are presented in this paper.

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1. Introduction

In this paper we present many generalizations of the Lefschetz fixed point theorem in a variety of extension type spaces. These spaces are generalization of spaces considered in [5, 6, 8, 10, 11, 12].

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Suppose X and Y are topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \ Fix\,F \neq \emptyset \ \text{ for all } \ F \in \mathcal{X}(Z,Z)\}$$

where $Fix\,F$ denotes the set of fixed points of F.

The class $\mathcal{A}$ of maps is defined by the following properties:

(i) $\mathcal{A}$ contains the class $\mathcal{C}$ of single valued continuous functions;

(ii) each $F \in \mathcal{A}_c$ is upper semicontinuous and closed valued; and

(iii) $B^n \in \mathcal{F}(\mathcal{A}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n : \ \|x\| \leq 1\}$.

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Remark 1.1. The class $\mathcal{A}$ is essentially due to Ben-El-Mechaiekh and Deguire [3]. $\mathcal{A}$ includes the class of maps $\mathcal{U}$ of Park ($\mathcal{U}$ is the class of maps defined by (i), (iii) and (iv). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued). Thus if each $F \in \mathcal{A}_c$ is compact valued the class $\mathcal{A}$ and $\mathcal{U}$ coincide.

We next consider the class $\mathcal{U}_c^\kappa(X,Y)$ (respectively $\mathcal{A}_c^\kappa(X,Y)$) of maps $F : X \to 2^Y$ such that for each F and each nonempty compact subset K of X there exists a map $G \in \mathcal{U}_c(K,Y)$ (respectively $G \in \mathcal{A}_c(K,Y)$) such that $G(x) \subseteq F(x)$ for all $x \in K$.

Recall $\mathcal{U}_c^\kappa$ is closed under compositions. The class $\mathcal{U}_c^\kappa$ include (the Kakutani maps, the acyclic maps, the O'Neill maps, the approximable maps and the maps admissible with respect to Gorniewicz.

For a subset K of a topological space X, we denote by $Cov_X(K)$ the set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$. Given two maps single valued $f, g : X \to Y$ and $\alpha \in Cov(Y)$, f and g are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We say f and g are α-homotopic if there is a homotopy $h_h : X \to Y$ $(0 \le t \le 1)$ joining f and g such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0,1]$.

The following results can be found in [1, Lemma 1.2 and 4.7].

Theorem 1.2. *Let X be a regular topological space and $F : X \to 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subseteq Cov_X(\overline{F(X)})$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 1.3. labelRemark 1.2. From Theorem 1.2 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [2, pp. 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [9, pp. 199] so such refinements form a cofinal family of open covers). Note also uniform spaces are regular (in fact completely regular) [4, pp. 431] (see also [4, pp. 434]). Note in Theorem 1.2 if F is compact valued then the assumption that X is regular can be removed. For convenience in this paper we will apply Theorem 1.2 only when the space is uniform.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;

(ii) p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\,p^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\, \tilde{E} < \infty$;

for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if

(i) all f_q are admissible and

(ii) almost all $\tilde{E}_q$ are trivial.

For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q \, Tr\,(f_q).$$

A linear map $f : E \to E$ of a vector space E into itself is called weakly nilpotent provided for every $x \in E$ there exists n_x such that $f^{n_x}(x) = 0$. Assume that $E = \{E_q\}$ is a graded vector space and $f = \{f_q\} : E \to E$ is an endomorphism. We say that f is weakly nilpotent iff f_q is weakly nilpotent for every q. It is well known [6, pp 53] that any weakly nilpotent endomorphism $f : E \to E$ is a Leray endomorphism and $\Lambda(f) = 0$.

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$.

With Čech homology functor extended to a category of morphisms (see [7, pp. 364]) we have the following well known result (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

we define the induced map

$$H\,(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$).

Recall the following result [5, 6, pp. 227].

Theorem 1.4. *If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then*

$$(\psi \,\circ\, \phi)_\star = \psi_\star \,\circ\, \phi_\star.$$

Two morphisms $\phi, \psi \in M(X, Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0, 1], Y)$ such that $\chi(x, 0) = \phi(x)$, $\chi(x, 1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0, i_1 : X \to X \times [0, 1]$ are defined by $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$). Recall the following result [6, pp. 231]: If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map

(ii) $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Definition 1.5. A upper semicontinuous map $\phi : X \to Y$ is said to be strongly admissible [6, 7] (and we write $\phi \in Ads(X, Y)$) provided there exists a selected pair (p, q) of ϕ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$.

Definition 1.6. A map $\phi \in Ads(X, X)$ is said to be a Lefschetz map if for each selected pair $(p, q) \subset \phi$ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

When we talk about $\phi \in Ads$ it is assumed that we are also considering a specified selected pair (p, q) of ϕ with $\phi(x) = q\,(p^{-1}(x))$.

Remark 1.7. In fact since we specify the pair (p, q) of ϕ it is enough to say ϕ is a Lefschetz map if $\phi_\star = q_\star\, p_\star^{-1} : H(X) \to H(X)$ is a Leray endomorphism. However for the examples of ϕ, X known in the literature [6] the more restrictive condition in Definition 1.6 works. We note [6, pp 227] that $\phi_\star$ does not depend on the choice of diagram from $[(p, q)]$, so in fact we could specify the morphism.

If $\phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [6, 7]) $\mathbf{\Lambda}\,(\phi)$ (or $\mathbf{\Lambda}_X\,(\phi)$) by

$$\mathbf{\Lambda}\,(\phi) = \Lambda(q_\star\, p_\star^{-1}).$$

If we do not wish to specify the selected pair (p, q) of ϕ then we would consider the Lefschetz set $\mathbf{\Lambda}\,(\phi) = \{\Lambda(q_\star\, p_\star^{-1}) : \phi = q\,(p^{-1})\}$.

Definition 1.8. A Hausdorff topological space X is said to be a Lefschetz space (for the class Ads) provided every compact $\phi \in Ads(X, X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq 0$ implies ϕ has a fixed point.

Definition 1.9. A upper semicontinuous map $\phi : X \to Y$ with closed values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ.

Definition 1.10. A map $\phi \in Ad(X, X)$ is said to be a Lefschetz map if for each selected pair $(p, q) \subset \phi$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

If $\phi : X \to X$ is a Lefschetz map, we define the Lefschetz set $\mathbf{\Lambda}\,(\phi)$ (or $\mathbf{\Lambda}_X\,(\phi)$) by

$$\mathbf{\Lambda}\,(\phi) = \left\{\Lambda(q_\star\, p_\star^{-1}) : \ (p, q) \subset \phi\right\}.$$

Definition 1.11. A Hausdorff topological space X is said to be a Lefschetz space (for the class Ad) provided every compact $\phi \in Ad(X, X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies ϕ has a fixed point.

Remark 1.12. Many examples of Lefschetz spaces (for the class Ad or Ads) can be found in [5, 6, 7, 8, 12].

Definition 1.13. A multivalued map $F : X \to K(Y)$ ($K(Y)$ denotes the class of nonempty compact subsets of Y) is in the class $\mathcal{A}_m(X, Y)$ if (i). F is continuous, and (ii). for each $x \in X$ the set $F(x)$ consists of one or m acyclic components; here m is a positive integer. We say F is of class $\mathcal{A}_0(X, Y)$ if F is upper semicontinuous and for each $x \in X$ the set $F(x)$ is acyclic.

Definition 1.14. A decomposition $(F_1, ..., F_n)$ of a multivalued map $F : X \to 2^Y$ is a sequence of maps

$$X = X_0 \overset{F_1}{\to} X_1 \overset{F_2}{\to} X_2 \overset{F_3}{\to} \;\; ..... \;\; \overset{F_{n-1}}{\to} X_{n-1} \overset{F_n}{\to} X_n = Y,$$

where $F_i \in \mathcal{A}_{m_i}(X_{i-1}, X_i)$, $F = F_n \circ .... \circ F_1$. One can say that the map F is determined by the decomposition $(F_1, ..., F_n)$. The number n is said to be the length of the decomposition $(F_1, ..., F_n)$. We will denote the class of decompositions by $\mathcal{D}(X, Y)$.

Definition 1.15. An upper semicontinuous map $F : X \to K(Y)$ is permissible provided it admits a selector $G : X \to K(Y)$ which is determined by a decomposition $(G_1, ..., G_n) \in \mathcal{D}(X, Y)$. We denote the class of permissible maps from X into Y by $\mathcal{P}(X, Y)$.

Let X be a Hausdorff topological space and let a map Φ be determined by $(\Phi_1, ..., \Phi_k) \in \mathcal{D}(X, X)$. Then Φ is said to be a Lefschetz map if the induced homology homomorphism [6, pp 262, 263] $(\Phi_1, ..., \Phi_k)_\star : H(X) \to H(X)$ is a Leray endomorphism.

If $\Phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [6]) $\mathbf{\Lambda}(\Phi)$ (or $\mathbf{\Lambda}_X(\Phi)$) by

$$\mathbf{\Lambda}(\Phi) = \Lambda((\Phi_1, ..., \Phi_k)_\star).$$

A Hausdorff topological space X is said to be a Lefschetz space (for the class $\mathcal{D}$) provided every compact $\Phi : X \to K(X)$ determined by a decomposition $(\Phi_1, ..., \Phi_k) \in \mathcal{D}(X, X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq 0$ implies Φ has a fixed point.

A map $\Phi \in \mathcal{P}(X, X)$ is said to be a Lefschetz map provided every selector $G : X \to K(X)$ of Φ which is determined by $(G_1, ..., G_k) \in \mathcal{D}(X, X)$ is such that $(G_1, ..., G_k)_\star : H(X) \to H(X)$ is a Leray endomorphism.

If $\Phi \in \mathcal{P}(X, X)$ is a Lefschetz map as described above then we define the Lefschetz set $\mathbf{\Lambda}(\Phi)$ (or $\mathbf{\Lambda}_X(\Phi)$) by

$$\mathbf{\Lambda}(\Phi) = \{\Lambda((G_1, ..., G_k)_\star) : \ (G_1, ..., G_k) \in \mathcal{D}(X, X) \quad \text{and} \ (G_1, ..., G_k) \ \text{determines a selection of} \ \Phi\}.$$

A Hausdorff topological space X is said to be a Lefschetz space (for the class $\mathcal{P}$) provided every compact $\Phi \in \mathcal{P}(X, X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies Φ has a fixed point.

2. Fixed Point Theory

By a space we mean a Hausdorff topological space. Let X be a space and $F : X \to 2^X$.

Definition 2.1. We say $X \in locGNES$ (w.r.t. Ad and F) if there exists a Lefschetz space (for the class Ad) U, a set $V \subseteq X$ with $\overline{F(V)} \subseteq V$, a single valued continuous map $r : U \to W$ where $W = \overline{F(V)}$, and a compact valued map $\Phi \in Ad(W, U)$ with $r\,\Phi = id_W$.

Theorem 2.2. *Let $X \in locGNES$ (w.r.t. Ad and F) and let U, V, W, r and Φ be as described in Definition 2.1. Assume $F \in Ad(V, V)$ and $F|_W$ is a compact map. Then $\mathbf{\Lambda}(F|_W)$ is well defined. Also $\mathbf{\Lambda}(F|_W) \neq \{0\}$ guarantees that $F|_W$ has a fixed point (i.e. F has a fixed point in W).*

Proof. Let $G = \Phi\, F|_W\, r$. We first note that $F|_W \in Ad(W, W)$ since for any selected pair (p_0, q_0) of $F|_V$ then $(\overline{p_0}, \overline{q_0}) \subset F|_W$; here $\overline{p_0}$, $\overline{q_0} : p_0^{-1}(W) \to W$ are given by $\overline{p_0}(z) = p_0(z)$, $\overline{q_0}(z) = q_0(z)$ for $z \in p_0^{-1}(W)$. Next note that $G \in Ad(U, U)$ is a compact map.

Let (p, q) be a selected pair for $F|_W$ and (p_1, q_1) be a selected pair of Φ. Now since $F|_W\, r \in Ad(U, W)$ then [6, Section 40] guarantees that there exists a selected pair (p', q') of $F|_W\, r$ with

$$(q')_\star\, (p')_\star^{-1} = q_\star\, p_\star^{-1}\, r_\star. \tag{2.1}$$

Also there exists [6, Section 40] a selected pair $(\overline{p}, \overline{q})$ of G with

$$(\overline{q})_\star\, (\overline{p})_\star^{-1} = (q_1)_\star\, (p_1)_\star^{-1}\, (q')_\star\, (p')_\star^{-1}. \tag{2.2}$$

Now (2.1) and (2.2) imply

$$(\overline{q})_\star\, (\overline{p})_\star^{-1} = (q_1)_\star\, (p_1)_\star^{-1}\, q_\star\, p_\star^{-1}\, r_\star, \tag{2.3}$$

and notice as well since $r\,\Phi = id_W$ that

$$q_\star\, p_\star^{-1}\, r_\star\, (q_1)_\star\, (p_1)_\star^{-1} = q_\star\, p_\star^{-1}. \tag{2.4}$$

Now since U is a Lefschetz space (for the class Ad) then $(\overline{q})_\star\,(\overline{p})_\star^{-1}$ is a Leray endomorphism. Now [5, page 214, see (1.3) or see the diagram below] (here $E' = U' = H(U)$, $E'' = W' = H(W)$, $u = (q')_\star\,(p')_\star^{-1}$, $v = (q_1)_\star\,(p_1)_\star^{-1}$, $f' = (\overline{q})_\star\,(\overline{p})_\star^{-1}$ and $f'' = q_\star\,p_\star^{-1}$ and note (2.1), (2.3) and (2.4))

$$\begin{tikzcd}
E' \arrow[r, "u"] \arrow[d, "f'"'] & E'' \\
E' \arrow[r, "u"] & E'' \arrow[u, "f''"'] \arrow[ul, "v"]
\end{tikzcd}$$

guarantees that $q_\star\,p_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,((\overline{q})_\star\,(\overline{p})_\star^{-1})$. Thus $\mathbf{\Lambda}\,(F|_W)$ is well defined.

Next suppose $\mathbf{\Lambda}\,(F|_W) \neq \{0\}$. Then there exists a selected pair (p,q) as described above with $\Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Let $\overline{p}$ and $\overline{q}$ be as described above with $\Lambda\,((\overline{q})_\star\,(\overline{p})_\star^{-1}) = \Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Now since U is a Lefschetz space (for the class Ad) there exists $x \in U$ with $x \in \overline{q}\,(\overline{p})^{-1}(x)$ i.e. $x \in G(x)$. Let $y = r(x)$, so $y \in r\,\Phi\,F|_W\,(y)$ i.e. $y \in r\,\Phi\,(q)$ for some $q \in F|_W\,(y)$. Note $q \in W = \overline{F(V)}$. Now since $r\,\Phi = id_W$ we have $y \in F|_W\,(y)$. □

Remark 2.3. From the proof above we see that the assumption $F \in Ad(V,V)$ in the statement of Theorem 2.2 could be replaced by the assumption $F \in Ad(W,W)$. Note also if $F \in Ad(X,X)$ then automatically $F \in Ad(V,V)$ (and $F \in Ad(W,W)$).

Remark 2.4. From the proof of Theorem 2.2 we see that we can replace the condition that U is a Lefschetz space with the assumption that the compact map $\Phi\,F|_W\,r \in Ad(U,U)$ is a Lefschetz map and $\mathbf{\Lambda}(\Phi\,F|_W\,r) \neq \{0\}$ implies $\Phi\,F|_W\,r$ has a fixed point.

Remark 2.5. One could also replace Ad maps with Ads maps in the above presentation.

Remark 2.6. One could also obtain a result for the class $\mathcal{P}$ (or the class $\mathcal{D}$) if some extra technical assumptions are assumed. We leave the details to the reader (for ideas here we refer the reader to [10]).

Remark 2.7. One could also obtain a fixed point result (with no reference to Lefschetz maps or sets) for the class $\mathcal{U}_c^\kappa$ if we assume $G \in \mathcal{U}_c^\kappa(U,U)$ (as described in Theorem 2.2) has a fixed point.

Definition 2.8. We say $X \in locGANES$ (w.r.t. Ad and F) if there exists a set $V \subseteq X$ with $\overline{F(V)} \subseteq V$ and for each $\alpha \in Cov_W\,(\overline{F(W)})$, here $W = \overline{F(V)}$, there exists a Lefschetz space (for the class Ad) U_α, a single valued continuous map $r_\alpha : U_\alpha \to W$ and a compact valued map $\Phi_\alpha \in Ad(W, U_\alpha)$ such that $r_\alpha\,\Phi_\alpha : W \to 2^W$ and $i : W \to W$ are strongly α-close (by this we mean for each $x \in K$ there exists $V_x \in \alpha$ with $r_\alpha\,\Phi_\alpha(x) \subseteq V_x$ and $x = i(x) \in V_x$) and $(r_\alpha)_\star\,(q_\alpha^0)_\star\,(p_\alpha^0)_\star^{-1} = i_\star$ for any selected pair (p_α^0, q_α^0) of Φ_α.

Theorem 2.9. *Let $X \in locGANES$ (w.r.t. Ad and F) be a uniform space and let V, W, α, U_α, r_α and Φ_α be as described in Definition 2.8. Assume $F \in Ad(W,W)$ and $F|_W$ is a compact map. Then $\mathbf{\Lambda}\,(F|_W)$ is well defined. Also $\mathbf{\Lambda}\,(F|_W) \neq \{0\}$ guarantees that $F|_W$ has a fixed point (i.e. F has a fixed point in W).*

Proof. Let $G_\alpha = \Phi_\alpha\,F|_W\,r_\alpha$. Note $G_\alpha \in Ad(U_\alpha, U_\alpha)$ is a compact map. Let (p,q) be a selected pair for $F|_W$ and (p_α^0, q_α^0) be a selected pair of Φ_α. Now since $F|_W\,r_\alpha \in Ad(U_\alpha, W)$ then [6, Section 40] guarantees that there exists a selected pair (p'_α, q'_α) of $F|_W\,r_\alpha$ with

$$(q'_\alpha)_\star\,(p'_\alpha)_\star^{-1} = q_\star\,p_\star^{-1}\,(r_\alpha)_\star. \tag{2.5}$$

Also there exists [6, Section 40] a selected pair $(\overline{p}_\alpha, \overline{q}_\alpha)$ of G_α with

$$(\overline{q}_\alpha)_\star\,(\overline{p}_\alpha)_\star^{-1} = (q_\alpha^0)_\star\,(p_\alpha^0)_\star^{-1}\,(q'_\alpha)_\star\,(p'_\alpha)_\star^{-1} \tag{2.6}$$

so (2.5) and (2.6) imply

$$(\overline{q}_\alpha)_\star\,(\overline{p}_\alpha)_\star^{-1} = (q_\alpha^0)_\star\,(p_\alpha^0)_\star^{-1}\,q_\star\,p_\star^{-1}\,(r_\alpha)_\star. \tag{2.7}$$

Notice as well by assumption (see Definition 2.8) that

$$q_\star\,p_\star^{-1}\,(r_\alpha)_\star\,(q_\alpha^0)_\star\,(p_\alpha^0)_\star^{-1} = q_\star\,p_\star^{-1}. \tag{2.8}$$

Now since U_α is a Lefschetz space (for the class Ad) then $(\overline{q}_\alpha)_\star\,(\overline{p}_\alpha)_\star^{-1}$ is a Leray endomorphism. Now [5, page 214, see (1.3)] (here $E' = U'_\alpha$, $E'' = W'$, $u = (q'_\alpha)_\star\,(p'_\alpha)_\star^{-1}$, $v = (q_\alpha^0)_\star\,(p_\alpha^0)_\star^{-1}$, $f' = (\overline{q}_\alpha)_\star\,(\overline{p}_\alpha)_\star^{-1}$ and $f'' = q_\star\,p_\star^{-1}$ and note (2.5), (2.7) and (2.8)) guarantees that $q_\star\,p_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,((\overline{q}_\alpha)_\star\,(\overline{p}_\alpha)_\star^{-1})$. Thus $\mathbf{\Lambda}\,(F|_W)$ is well defined.

Next suppose $\mathbf{\Lambda}\,(F|_W) \neq \{0\}$. Then there exists a selected pair (p, q) as described above with $\Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Let $\overline{p}_\alpha$ and $\overline{q}_\alpha$ be as described above with $\Lambda\,((\overline{q}_\alpha)_\star\,(\overline{p}_\alpha)_\star^{-1}) = \Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Now since U_α is a Lefschetz space (for the class Ad) there exists $x \in U_\alpha$ with $x \in \overline{q}_\alpha\,(\overline{p}_\alpha)^{-1}(x)$ i.e. $x \in G_\alpha(x)$. Let $y = r_\alpha(x)$, so $y \in r_\alpha\,\Phi_\alpha\,F|_W\,(y)$ i.e. $y \in r_\alpha\,\Phi_\alpha\,(q)$ for some $q \in F|_W\,(y)$. Note $q \in W$. Now since $r_\alpha\,\Phi_\alpha : W \to 2^W$ and $i : W \to W$ are strongly α-close there exists $V_0 \in \alpha$ with

$$r_\alpha\,\Phi_\alpha(q) \subseteq V_0 \quad \text{and} \quad q \in V_0.$$

Thus $y \in V_0$ since $y \in r_\alpha\,\Phi_\alpha(q)$ and also note $q \in F|_W\,(y)$ and $q \in V_0$. Thus

$$y \in V_0 \quad \text{and} \quad F|_W\,(y) \cap V_0 \neq \emptyset.$$

As a result $F|_W$ has an α-fixed point (for $\alpha \in Cov_W\,(\overline{F(W)})$) so Theorem 1.2 guarantees that $F|_W$ has a fixed point. □

Remark 2.10. In the proof of Theorem 2.9 the condition $(r_\alpha)_\star\,(q_\alpha^0)_\star\,(p_\alpha^0)_\star^{-1} = i_\star$ for any selected pair (p_α^0, q_α^0) of Φ_α in Definition 2.8 was only used to establish (2.8). Suppose for example $r_\alpha\,\Phi_\alpha : W \to W$ (is single valued) and $i : W \to W$ are α-homotopic. Then [6, pp. 202] guarantees that $(r_\alpha\,\Phi_\alpha)_\star = i_\star$ and so for any selected pair (p_α^0, q_α^0) of Φ_α there exists a selected pair (p_α^2, q_α^2) of $r_\alpha\,\Phi_\alpha$ with $i_\star = (r_\alpha\,\Phi_\alpha)_\star = (q_\alpha^2)_\star\,(p_\alpha^2)_\star^{-1} = (r_\alpha)_\star\,(q_\alpha^0)_\star\,(p_\alpha^0)_\star^{-1})$. Another example follows from [6, pp. 202] if $r_\alpha\,\Phi_\alpha : W \to 2^W$ (is acyclic) and $i : W \to W$ are α-homotopic (homotopic in the sense of [6, pp. 202]).

Remark 2.11. From the proof above we see that the assumption $F \in Ad(W, W)$ in the statement of Theorem 2.9 could be replaced by the assumption $F \in Ad(V, V)$. Note also if $F \in Ad(X, X)$ then automatically $F \in Ad(V, V)$ (and $F \in Ad(W, W)$). Of course X being a uniform space could be replaced by W being a uniform space in the statement of Theorem 2.9.

Remark 2.12. From the proof in Theorem 2.9 we see that we can replace the condition that U_α is a Lefschetz space for each $\alpha \in Cov_W\,(\overline{F(W)})$ with the assumption that for each $\alpha \in Cov_W\,(\overline{F(W)})$ the compact map $\Phi_\alpha\,F|_W\,r_\alpha \in Ad(U_\alpha, U_\alpha)$ is a Lefschetz map and $\mathbf{\Lambda}(\Phi_\alpha\,F|_W\,r_\alpha) \neq \{0\}$ implies $\Phi_\alpha\,F|_W\,r_\alpha$ has a fixed point.

Remark 2.13. One could also replace Ad maps with Ads maps in the above presentation.

Remark 2.14. One could also obtain a result for the class $\mathcal{P}$ (or the class $\mathcal{D}$) if some extra technical assumptions are assumed. We leave the details to the reader (for ideas here we refer the reader to [10]).

Remark 2.15. One could also obtain a fixed point result (with no reference to Lefschetz maps or sets) for the class $\mathcal{U}_c^\kappa$ (assuming as well that $F|_W$ is upper semicontinuous with closed values) if we assume $G_\alpha \in \mathcal{U}_c^\kappa(U_\alpha, U_\alpha)$ (as described in Theorem 2.9) has a fixed point.

Now we discuss a more general situation motivated in part by [11]. Again X is a space and $F : X \to 2^X$.

Definition 2.16. We say $X \in locGMNES$ (w.r.t. Ad and F) if there exists a Lefschetz space (for the class Ad) U, a set $V \subseteq X$ with $\overline{F(V)} \subseteq V$ and $F|_W \in Ad(W,W)$ (here $W = \overline{F(V)}$), a compact map $\Phi \in Ad(U,W)$, a compact valued map $\Psi \in Ad(W,U)$ with $\Phi\,\Psi\,(x) \subseteq F|_W\,(x)$ for $x \in W$, and such that if (p,q) is a selected pair of $F|_W$ then there exists a selected pair (p_1,q_1) of Φ and a selected pair (p',q') of Ψ with $(q_1)_\star\,(p_1)_\star^{-1}\,(q')_\star\,(p')_\star^{-1} = q_\star\,p_\star^{-1}$.

Theorem 2.17. *Let $X \in locGMNES$ (w.r.t. Ad and F) and let U, V, W, Φ and Ψ be as described in Definition 2.16. Then $\mathbf{\Lambda}\,(F|_W)$ is well defined. Also $\mathbf{\Lambda}\,(F|_W) \neq \{0\}$ guarantees that $F|_W$ has a fixed point (i.e. F has a fixed point in W).*

Proof. Let $G = \Psi\,\Phi$. Note $G \in Ad(U,U)$ is a compact map (note the image of a compact set under Ψ is compact). Let (p,q) be a selected pair of $F|_W$. Then from Definition 2.16 there exists a selected pair (p_1,q_1) of Φ and a selected pair (p',q') of Ψ with

$$(q_1)_\star\,(p_1)_\star^{-1}\,(q')_\star\,(p')_\star^{-1} = q_\star\,p_\star^{-1}. \tag{2.9}$$

There exists [6, Section 40] a selected pair $(\overline{p},\overline{q})$ of G with

$$(\overline{q})_\star\,(\overline{p})_\star^{-1} = (q')_\star\,(p')_\star^{-1}\,(q_1)_\star\,(p_1)_\star^{-1} \tag{2.10}$$

Now U is a Lefschetz space (for the class Ad) so $(\overline{q})_\star\,(\overline{p})_\star^{-1}$ is a Leray endomorphism. Now [5, page 214, see (1.3)] (here $E' = U'$, $E'' = W'$, $v = (q')_\star\,(p')_\star^{-1}$, $u = (q_1)_\star\,(p_1)_\star^{-1}$, $f' = (\overline{q})_\star\,(\overline{p})_\star^{-1}$ and $f'' = q_\star\,p_\star^{-1}$ and note (2.9) and (2.10)) guarantees that $q_\star\,p_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,((\overline{q})_\star\,(\overline{p})_\star^{-1})$. Thus $\mathbf{\Lambda}\,(F|_W)$ is well defined.

Next suppose $\mathbf{\Lambda}\,(F|_W) \neq \{0\}$. Then there exists a selected pair (p,q) as described above with $\Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Let $\overline{p}$ and $\overline{q}$ be as described above with $\Lambda\,((\overline{q})_\star\,(\overline{p})_\star^{-1}) = \Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Now since U is a Lefschetz space (for the class Ad) there exists $x \in U$ with $x \in \overline{q}\,(\overline{p})^{-1}(x)$ i.e. $x \in G(x) = \Psi\,\Phi(x)$. Then there exists a $y \in \Phi\,(x)$ such that $x \in \Psi\,(y)$. As a result $y \in \Phi\,(x) \in \Phi\,\Psi(y) \subseteq F|_W\,(y)$. □

Remark 2.18. From the proof above we see that the assumption $F \in Ad(W,W)$ in Definition 2.16 could be replaced by the assumption $F \in Ad(V,V)$. Note also if $F \in Ad(X,X)$ then automatically $F \in Ad(V,V)$ (and $F \in Ad(W,W)$).

Remark 2.19. One could also replace Ad maps with Ads maps in the above presentation. One could also obtain a result for the class $\mathcal{P}$ (or the class $\mathcal{D}$) if some extra technical assumptions are assumed. One could also obtain a fixed point result (with no reference to Lefschetz maps or sets) for the class $\mathcal{U}_c^\kappa$ (assuming as well that Ψ is upper semicontinuous with compact values) if we assume $G \in \mathcal{U}_c^\kappa(U,U)$ (as described in Theorem 2.17) has a fixed point.

Remark 2.20. From the proof above we see that we can replace the condition that U is a Lefschetz space with the assumption that the compact map $\Psi\,\Phi \in Ad(U,U)$ is a Lefschetz map and $\mathbf{\Lambda}(\Psi\,\Phi) \neq \{0\}$ implies $\Psi\,\Phi$ has a fixed point.

Definition 2.21. We say $X \in locGMANES$ (w.r.t. Ad and F) if there exists a set $V \subseteq X$ with $\overline{F(V)} \subseteq V$ and $F|_W \in Ad(W,W)$ (here $W = \overline{F(V)}$), and for each $\alpha \in Cov_W\,(\overline{F(W)})$ there exists a Lefschetz space (for the class Ad) U_α, a compact map $\Phi_\alpha \in Ad(U_\alpha,W)$, a compact valued map $\Psi_\alpha \in Ad(W,U_\alpha)$ such that for each $x \in U_\alpha$ and $y \in \Phi_\alpha\,(x)$ with $x \in \Psi_\alpha\,(y)$ there exists $U_{x,y} \in \alpha$ with $y \in U_{x,y}$ and $F|_W\,(y) \cap U_{x,y} \neq \emptyset$ and such that if (p,q) is a selected pair of $F|_W$ then there exists a selected pair $(p_{1,\alpha},q_{1,\alpha})$ of Φ_α and a selected pair (p'_α,q'_α) of Ψ_α with $(q_{1,\alpha})_\star\,(p_{1,\alpha})_\star^{-1}\,(q'_\alpha)_\star\,(p'_\alpha)_\star^{-1} = q_\star\,p_\star^{-1}$.

Theorem 2.22. *Let $X \in locGMANES$ (w.r.t. Ad and F) be a uniform space and let V, W, α, U_α, Ψ_α and Φ_α be as described in Definition 2.4. Then $\mathbf{\Lambda}\,(F|_W)$ is well defined. Also $\mathbf{\Lambda}\,(F|_W) \neq \{0\}$ guarantees that $F|_W$ has a fixed point (i.e. F has a fixed point in W).*

Proof. Let $G_\alpha = \Psi_\alpha\, \Phi_\alpha$. Note $G_\alpha \in Ad(U_\alpha, U_\alpha)$ is a compact map. Let (p, q) be a selected pair of $F|_W$. Then from Definition 2.21 there exists a selected pair $(p_{1,\alpha}, q_{1,\alpha})$ of Φ_α and a selected pair (p'_α, q'_α) of Ψ_α with

$$(q_{1,\alpha})_\star\, (p_{1,\alpha})_\star^{-1}\, (q'_\alpha)_\star\, (p'_\alpha)_\star^{-1} = q_\star\, p_\star^{-1}. \tag{2.11}$$

There exists [6, Section 40] a selected pair $(\overline{p}_\alpha, \overline{q}_\alpha)$ of G_α with

$$(\overline{q}_\alpha)_\star\, (\overline{p}_\alpha)_\star^{-1} = (q'_\alpha)_\star\, (p'_\alpha)_\star^{-1}\, (q_{1,\alpha})_\star\, (p_{1,\alpha})_\star^{-1} \tag{2.12}$$

Now U_α is a Lefschetz space (for the class Ad) so $(\overline{q}_\alpha)_\star\, (\overline{p}_\alpha)_\star^{-1}$ is a Leray endomorphism. Now [5, page 214, see (1.3)] (here $E' = U'_\alpha$, $E'' = W'$, $v = (q'_\alpha)_\star\, (p'_\alpha)_\star^{-1}$, $u = (q_{1,\alpha})_\star\, (p_{1,\alpha})_\star^{-1}$, $f' = (\overline{q}_\alpha)_\star\, (\overline{p}_\alpha)_\star^{-1}$ and $f'' = q_\star\, p_\star^{-1}$ and note (2.11) and (2.12)) guarantees that $q_\star\, p_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q_\star\, p_\star^{-1}) = \Lambda\,((\overline{q}_\alpha)_\star\, (\overline{p}_\alpha)_\star^{-1})$. Thus $\mathbf{\Lambda}\,(F|_W)$ is well defined.

Next suppose $\mathbf{\Lambda}\,(F|_W) \neq \{0\}$. Then there exists a selected pair (p, q) as described above with $\Lambda\,(q_\star\, p_\star^{-1}) \neq 0$. Let $\overline{p}_\alpha$ and $\overline{q}_\alpha$ be as described above with $\Lambda\,((\overline{q}_\alpha)_\star\, (\overline{p}_\alpha)_\star^{-1}) = \Lambda\,(q_\star\, p_\star^{-1}) \neq 0$. Now since U_α is a Lefschetz space (for the class Ad) there exists $x \in U_\alpha$ with $x \in \overline{q}_\alpha\, (\overline{p}_\alpha)^{-1}(x)$ i.e. $x \in G_\alpha(x)$. As a result there exists a $y \in \Phi_\alpha(x)$ with $x \in \Psi_\alpha\,(y)$. Then (from Definition 2.21) there exists $V \in \alpha$ with

$$y \in V \quad \text{and} \quad F|_W\,(y) \cap V \neq \emptyset.$$

As a result $F|_W$ has an α-fixed point (for $\alpha \in Cov_W\,(\overline{F(W)})$) so Theorem 1.2 guarantees that $F|_W$ has a fixed point. □

Remark 2.23. From the proof above we see that the assumption $F \in Ad(W, W)$ in Definition 2.21 could be replaced by the assumption $F \in Ad(V, V)$. Note also if $F \in Ad(X, X)$ then automatically $F \in Ad(V, V)$ (and $F \in Ad(W, W)$). Of course X being a uniform space could be replaced by W being a uniform space in the statement of Theorem 2.22.

Remark 2.24. One could also replace Ad maps with Ads maps in the above presentation. One could also obtain a result for the class $\mathcal{P}$ (or the class $\mathcal{D}$) if some extra technical assumptions are assumed. One could also obtain a fixed point result (with no reference to Lefschetz maps or sets) for the class $\mathcal{U}_c^\kappa$ (assuming as well that Ψ_α is upper semicontinuous with compact values) if we assume $G_\alpha \in \mathcal{U}_c^\kappa(U_\alpha, U_\alpha)$ (as described in Theorem 2.22) has a fixed point.

Remark 2.25. From the proof above we see that we can replace the condition that U_α is a Lefschetz space for each $\alpha \in Cov_W\,(\overline{F(W)})$ with the assumption that for each $\alpha \in Cov_W\,(\overline{F(W)})$ the compact map $\Psi_\alpha\, \Phi_\alpha \in Ad(U_\alpha, U_\alpha)$ is a Lefschetz map and $\mathbf{\Lambda}(\Psi_\alpha\, \Phi_\alpha) \neq \{0\}$ implies $\Psi_\alpha\, \Phi_\alpha$ has a fixed point.

Definition 2.26. Let X be a space. A map $F \in Ad(X, X)$ is said to be a locally general compact absorbing contraction (written $F \in locGCAC(X, X)$ or $F \in locGCAC(X)$) if

(i) $X \in locGNES$ (w.r.t. Ad and F), and let U, V, W, r and Φ be as described in Definition 2.1, and $F|_W$ is a compact map;

(ii) for any selected pair (p, q) of F, $q''_\star\, (p'')_\star^{-1} : H(X, W) \to H(X, W)$ is a weakly nilpotent endomorphism (here p'', $q'' : (\Gamma, p^{-1}(W)) \to (X, W)$ are given by $p''(u) = p(u)$ and $q''(u) = q(u)$).

Remark 2.27. For a discussion on compact absorbing contractions see [10] and the books [6, Section 42] and [8, Section 15.5].

Our next result guarantees that $\mathbf{\Lambda}\,(F)$ is well defined.

Theorem 2.28. *Let X be a space and $F \in locGCAC(X, X)$ (and let U, V, W, r and Φ be as described in Definition 2.1). Then $\mathbf{\Lambda}\,(F)$ is well defined and if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.*

Proof. Let (p, q) be a selected pair for F so in particular $q\,p^{-1}(W) \subseteq F(W)$. Consider $F|_W$ and let $q',\, p' : p^{-1}(W) \to W$ be given by $p'(u) = p(u)$ and $q'(u) = q(u)$ (and note (p', q') is a selected pair for $F|_W$). Now since $X \in locGNES$ (w.r.t. Ad and F) then as in Theorem 2.2, $q'_\star\,(p')_\star^{-1}$ is a Leray endomorphism. Now (ii) and [6, Property 11.8, pp 53] guarantees that $q''_\star\,(p'')_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q''_\star\,(p'')_\star^{-1}) = 0$. Also [6, Property 11.5, pp 52] guarantees that $q_\star\,p_\star^{-1}$ is a Leray endomorphism (with $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,(q'_\star\,(p')_\star^{-1})$) so $\mathbf{\Lambda}\,(F)$ is well defined.

Next suppose $\mathbf{\Lambda}\,(F) \neq \{0\}$. Then there exists a selected pair (p, q) of F with $\Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Let (p', q') be as described above with $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,(q'_\star\,(p')_\star^{-1})$. Then $\Lambda\,(q'_\star\,(p')_\star^{-1}) \neq 0$ so since $X \in locGNES$ (w.r.t. Ad and F) there exists $x \in W$ with $x \in F|_W\,(x)$ i.e. $x \in F\,x$. $\square$

Definition 2.29. Let X be a space. A map $F \in Ad(X, X)$ is said to be a locally general approximative compact absorbing contraction (written $F \in locGACAC(X, X)$ or $F \in locGACAC(X)$) if $X \in locGANES$ (w.r.t. Ad and F), and let V, W, α, U_α, r_α and Φ_α be as described in Definition 2.8, and $F|_W$ is a compact map and (ii) in Definition 2.26 holds.

The same reasoning as in Theorem 2.28 (except Theorem 2.9 replaces Theorem 2.2) establishes the next result.

Theorem 2.30. *Let X be a uniform space and $F \in locGACAC(X, X)$ (and let V, W, α, U_α, r_α and Φ_α be as described in Definition 2.8). Then $\mathbf{\Lambda}\,(F)$ is well defined and if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.*

Definition 2.31. Let X be a space. A map $F \in Ad(X, X)$ is said to be a locally general absorbing contraction (written $F \in locGAC(X, X)$ or $F \in locGAC(X)$) if $X \in locGMNES$ (w.r.t. Ad and F), and let U, V, W, Φ and Ψ be as described in Definition 2.16, and (ii) in Definition 2.26 holds.

The same reasoning as in Theorem 2.28 (except Theorem2.17 replaces Theorem2.2) establishes the next result.

Theorem 2.32. *Let X be a space and $F \in locGAC(X, X)$ (and let U, V, W, Φ and Ψ be as described in Definition 2.16). Then $\mathbf{\Lambda}\,(F)$ is well defined and if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.*

Definition 2.33. Let X be a space. A map $F \in Ad(X, X)$ is said to be a locally general approximative absorbing contraction (written $F \in locGAAC(X, X)$ or $F \in locGAAC(X)$) if $X \in locGMANES$ (w.r.t. Ad and F), and let V, W, α, U_α, Φ_α and Ψ_α be as described in Definition 2.21, and (ii) in Definition 2.26 holds.

The same reasoning as in Theorem 2.28 (except Theorem 2.22 replaces Theorem 2.2) establishes the next result.

Theorem 2.34. *Let X be a uniform space and $F \in locGAAC(X, X)$ (and let V, W, α, U_α, Φ_α and Ψ_α be as described in Definition 2.21). Then $\mathbf{\Lambda}\,(F)$ is well defined and if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.*

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