Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5497–5509
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On the well-posedness of the generalized split quasi-inverse variational inequalities

Liang Cao[a], Hua Kong[b], Sheng-Da Zeng[b,c,*]

[a] *Guangxi University of Finance and Economics, Nanning, Guangxi 530003, P. R. China.*

[b] *Data Recovery Key Laboratory of Sichuan Province, College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641100, P. R. China.*

[c] *Institute of Computer Science, Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Lojasiewicza 6, 30-348 Krakow, Poland.*

Communicated by C. Park

Abstract

In this paper, a generalized split quasi-inverse variational inequality ((GSQIVI), for short) is considered and investigated in Hilbert spaces. Since the well-posedness results, not only show us the qualitative properties of problem (GSQIVI), but also it gives us an outlook to the convergence analysis of the solutions for (GSQIVI). Therefore, we first introduce the concepts concerning with the approximating sequences, well-posedness and well-posedness in the generalized sense of (GSQIVI). Then, under those definitions, we establish several metric characterizations and equivalent conditions of well-posedness for the (GSQIVI) by using the measure of noncompactness theory and the generalized Cantor theorem.

Keywords: Generalized split quasi-inverse variational inequality, measure of noncompactness, well-posedness, Painlevé-Kuratowski limits.
2010 MSC: 49J53, 49K40, 90C33, 90C46.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. The norm and the scalar product of H_1 (or H_2) are denoted by $\|\cdot\|_{H_1}$ (or $\|\cdot\|_{H_2}$) and $\langle\cdot,\cdot\rangle_{H_1}$ (or $\langle\cdot,\cdot\rangle_{H_2}$), respectively. In the sequel, the norm convergence is denoted by "$\to$" and the weak convergence by "$\rightharpoonup$". Let $f : H_1 \to H_1$, $g : H_2 \to H_2$ and $h : H_1 \times H_2 \to \mathbb{R}$ be

*Corresponding author
Email addresses: bdhzxcaoliang@163.com (Liang Cao), konghua2008@126.com (Hua Kong), shdzeng@hotmail.com; zengshengda@163.com (Sheng-Da Zeng)

Received 2016-07-19

given functions, let $F : H_1 \rightrightarrows H_1$ and $G : H_2 \rightrightarrows H_2$ be two set-valued mappings with nonempty, closed and convex values, and let $\phi : H_1 \to \mathbb{R}$ and $\psi : H_2 \to \mathbb{R}$ be two weakly continuous functionals.

With these data, in this work, we study the following generalized split quasi-inverse variational inequality (GSQIVI):

Find $x^* \in H_1$ and $y^* \in H_2$ such that,

$$\begin{cases} f(x^*) \in F(x^*), \\ g(y^*) \in G(y^*), \\ h(x^*, y^*) = 0, \\ \langle f' - f(x^*), x^*\rangle_{H_1} + \phi(f') - \phi(f(x^*)) \geq 0, \ \forall f' \in F(x^*), \\ \langle g' - g(y^*), y^*\rangle_{H_2} + \psi(g') - \psi(g(y^*)) \geq 0, \ \forall g' \in G(y^*). \end{cases} \tag{1.1}$$

Particularly, if $H_1 = \mathbb{R}^m, H_2 = \mathbb{R}^n$, $F(x) \equiv C$ and $G(y) \equiv Q$ for all $(x, y) \in H_1 \times H_2$, $\phi(f') = \psi(g') \equiv 0$ for all $(f', g') \in H_1 \times H_2$ and $h(x, y) = \|y - Ax\|$, where $A : \mathbb{R}^m \to \mathbb{R}^n$ is a bounded linear function, and C, Q are closed and convex subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Then (GSQIVI) becomes to the following split inverse variational inequality (SIVI) problem:

Find $x^* \in \mathbb{R}^m$ and $y^* \in \mathbb{R}^n$ such that,

$$\begin{cases} f(x^*) \in C, \\ g(y^*) \in Q, \\ y^* = Ax^*, \\ \langle f' - f(x^*), x^*\rangle \geq 0, \ \forall f' \in C, \\ \langle g' - g(y^*), y^*\rangle \geq 0, \ \forall g' \in Q, \end{cases} \tag{1.2}$$

which has been studied by Hu and Fang in [12].

Inverse variational inequalities in finite dimensional Euclidean spaces were first introduced and investigated by He et al. [8, 9]. In they works [8, 9], they have pointed out that there are many control problems appearing in economics, transportation, and management science and energy networks can be modeled as the inverse variational inequalities, but they are difficult to be formulated as the classical variational inequalities. Since then, many results concerning with the inverse variational inequalities were obtained, for example, a proximal point based algorithm for solving the inverse variational inequality was established by He et al. in [7]. Scrimali [23] studied the time-dependent spatial price equilibrium control problem and modeled it as an evolutionary inverse variational inequality. Recently, Li, Wang and Huang proved some existence theorems of Carathéodory weak solutions for the differential inverse variational inequality in Euclidean spaces, and an application to the time-dependent spatial price equilibrium control problem was demonstrated. For more related works, we also can refer to [13, 18, 25, 26] and the references therein.

On the other hand, the issue of well-posedness is one of the most important and interesting subjects in the study of various problems. The classical concept of well-posedness for the minimization problem, which has been known as the Tykhonov well-posedness, is due to Tykhonov [24], which requires the existence and uniqueness of solution to the global minimization problems and the convergence of every minimizing sequence toward the unique solution. However, in many practical situations, the solution may not be unique for an optimization problem. Thus, the concept of well-posedness in the generalized sense was introduced, which means the existence of solutions and the convergence of some subsequence of every minimizing sequence toward a solution. The concept of well-posedness is motivated by the numerical methods producing approximating sequences. So, many authors were devoted to generalizing the concept of well-posedness of optimization problems (see [1, 3]), variational inequalities (see [4–6, 19]), fixed point problems (see [17]), equilibrium problems (see [14, 15, 20–22]) and inclusion problems (see [2]) etc.

Indeed, there are very few researchers extending the well-posedness to inverse variational inequalities. In 2008, Hu and Fang [10] firstly introduced the well-posedness for inverse variational inequalities and presented some basic results concerning with the well-posed inverse variational inequality. Then, the Levitin-Polyak well-posedness of inverse variational inequality also was studied by Hu and Fang in [11]. Recently, Hu and

Fang in [12] investigated the well-posedness of split inverse variational inequality in Euclidean spaces. By following this line, in this article, we propose the well-posedness and the well-posedness in the generalized sense of (GSQIVI) (1.1). By using these concepts, we show the relation between metric characterizations and well-posedness of (GSQIVI). In addition, we prove the solution set of (GSQIVI) is compact, if the problem is well-posed in the generalized sense. However, it needs to be pointed out that we extend the recent results in [12].

This paper is organized as follows. In Section 2, we will recall some basic preliminaries needed in the sequel. In Section 3 and Section 4, the equivalence results between the well-posedness, well-posedness in the generalized sense and some corresponding metric characterizations will be obtained.

2. Preliminaries

In this section, we collect a few notions and results to be used later in the paper.

Definition 2.1. Let A be a nonempty subset of a Banach space E. The measure, say μ, of noncompactness for the set A is defined by

$$\mu(A) := \inf\{\epsilon > 0 : A = \bigcup_{i=1}^{n} A_i, \operatorname{diam}(A_i) < \epsilon, i = 1, 2, ..., n\},$$

where $\operatorname{diam}(A_i)$ means the diameter of the set A_i.

Definition 2.2. Let A, B be nonempty subsets of E. The Hausdorff metric $\mathcal{H}(\cdot,\cdot)$ between A and B is defined by

$$\mathcal{H}(A, B) := \max\{e(A, B), e(B, A)\},$$

where $e(A, B) := \sup_{a\in A} d(a, B)$ with $d(a, B) := \inf_{b\in B} \|a - b\|_E$.

Remark 2.3. Particularly, when $\{A_n\}$ is a sequence of nonempty subsets of E, we say that A_n converges to $A \subset E$ in the sense of Hausdorff metric, if and only if, $\mathcal{H}(A_n, A) \to 0$.

In addition, we recall the concepts of Painlevé-Kuratowski limits.

Definition 2.4. Let E be a Banach space. The Painlevé-Kuratowski strong limit inferior and weak limit superior of a sequence $\{A_n\} \subseteq E$ are defined as follows:

$$\begin{aligned}
&\mathrm{s} - \liminf A_n := \{x \in E : \exists x_n \in A_n, n \in \mathbb{N}, \text{ with } x_n \to x \text{ in } E\},\\
&\mathrm{w} - \limsup A_n := \{x \in E : \exists n_k \uparrow +\infty, n_k \in \mathbb{N}, \exists x_{n_k} \in A_{n_k}, k \in \mathbb{N}, \text{ with } x_{n_k} \rightharpoonup x \text{ in } E\}.
\end{aligned}$$

Remark 2.5. Indeed, if there is a subset A of E, such that

$$\mathrm{w} - \limsup A_n = A = \mathrm{s} - \liminf A_n,$$

then we call $\{A_n\}$ Mosco convergence to the set A. However, when sequence $\{A_n\}$ satisfies

$$A = \mathrm{s} - \liminf A_n,$$

then, we call $\{A_n\}$ Lower Semi-Mosco convergence to the set A.

Furthermore, we introduce some quantitative properties of set-valued mappings.

Definition 2.6. A set-valued mapping $A : E \rightrightarrows Y$ from a Banach space E to a Banach space Y is called

(i) (s, w)-closed, if for any $x_n \to x$ in E, $y_n \rightharpoonup y$ in Y with $y_n \in A(x_n)$, one has $y \in A(x)$, that is

$$w - \limsup A(x_n) \subseteq A(x), \text{ as } x_n \to x \text{ in } E.$$

(ii) (s,s)-lower semicontinuous, if for any $x_n \to x$ in E, and for any $y \in A(x)$, there exists a sequence $\{y_n\}$ with $y_n \in A(x_n)$ such that $y_n \to y$ in Y, that is,

$$A(x) \subseteq s - \liminf A(x_n), \text{ as } x_n \to x \text{ in } E.$$

(iii) (s,w)-subcontinuous, if for every sequence $\{x_n\}$ strong converging in E, every sequence $\{y_n\} \subseteq Y$ with $y_n \in S(x_n)$ has a weak convergent subsequence in Y.

For the convenience of readers, we now denote a special case of (GSQIVI) (1.1), which will be considered in Section 3 and Section 4.

Let C and Q be two closed and convex subsets of H_1 and H_2, respectively, where H_1 and H_2 are defined in Section 1. However, if $F(x) \equiv C$ and $G(y) \equiv Q$ for each $(x,y) \in H_1 \times H_2$, then (GSQIVI) in (1.1) reduces to the generalized split inverse variational inequality problem ((GSIVI), for short) as follows:

Find $x^* \in C$ and $y^* \in Q$ such that,

$$\begin{cases} f(x^*) \in C, \\ g(y^*) \in Q, \\ h(x^*, y^*) = 0, \\ \langle f' - f(x^*), x^* \rangle_{H_1} + \phi(f') - \phi(f(x^*)) \geq 0, \ \forall f' \in C, \\ \langle g' - g(y^*), y^* \rangle_{H_2} + \psi(f') - \psi(g(y^*)) \geq 0, \ \forall g' \in Q. \end{cases} \tag{2.1}$$

3. The characterizations of well-posedness for (GSQIVI)

In this section, we establish the metric characterizations and equivalent conditions of well-posedness of (GSQIVI). Firstly, we introduce the concept of approximating sequence of (GSQIVI) as follows.

Definition 3.1. A sequence $\{(x_n, u_n)\}$ in $H_1 \times H_2$ is called an approximating sequence of (GSQIVI), if and only if there exists a positive sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\begin{cases} d_{H_1}(f(x_n), F(x_n)) \leq \epsilon_n, \\ d_{H_2}(g(y_n), G(y_n)) \leq \epsilon_n, \\ |h(x_n, y_n)| \leq \epsilon_n, \\ \langle f' - f(x_n), x_n \rangle_{H_1} + \phi(f') - \phi(f(x_n)) \geq -\epsilon_n, \ \forall f' \in F(x_n), \\ \langle g' - g(y_n), y_n \rangle_{H_2} + \psi(g') - \psi(g(y_n)) \geq -\epsilon_n, \ \forall g' \in G(y_n). \end{cases}$$

Definition 3.2. Let S be the solution set of (GSQIVI). We say that (GSQIVI) is strongly (respectively, weakly) well-posed, if and only if $S = \{(x^*, y^*)\}$ is a singleton and every approximating sequence $\{(x_n, y_n)\}$ of (GSQIVI) strongly (respectively, weakly) converges to (x^*, y^*).

Remark 3.3. However, if $F(x) \equiv C$ and $G(y) \equiv Q$ for each $x \in H_1$ and $y \in H_2$, where C and Q are both closed convex subsets of H_1 and H_2, respectively, then (GSQIVI) in (1.1) turns into (GSIVI) (2.1). Then, the approximating sequence

$$\begin{cases} f(x_n) \in C, \\ g(y_n) \in Q, \\ |h(x_n, y_n)| \leq \epsilon_n, \\ \langle f' - f(x_n), x_n \rangle_{H_1} + \phi(f') - \phi(f(x_n)) \geq -\epsilon_n, \ \forall f' \in C, \\ \langle g' - g(y_n), y_n \rangle_{H_2} + \psi(g') - \psi(g(y_n)) \geq -\epsilon_n, \ \forall g' \in Q, \end{cases}$$

is called to be an approximating sequence (GSIVI). Moreover, when $H_1 = \mathbb{R}^m$, $H_2 = \mathbb{R}^n$, $\phi(f') \equiv 0$, $\psi(g') \equiv 0$ for all $f' \in H_1$, $g' \in H_2$, and $h(x,y) = \|y - Ax\|$, where $A : \mathbb{R}^m \to \mathbb{R}^n$ is a bounded linear function, then we also have approximating sequences of (SIVI), which have been introduced by Hu and Fang in [12].

Besides, for (SIVI), since H_1 and H_2 are two finite spaces, then strongly well-posed and weakly well-posed are consistent (they are both called well-posed). For more details, one can see [12, Definitions 3.1 and 3.2].

For any $\epsilon > 0$, we introduce the following approximating solution set of (GSQIVI)

$$\Omega(\epsilon) = \Big\{ (x,y) \in H_1 \times H_2 : \ d_{H_1}(f(x), F(x)) \leq \epsilon, \ d_{H_2}(g(y), G(y)) \leq \epsilon, |h(x,y)| \leq \epsilon$$
$$\langle f' - f(x), x\rangle_{H_1} + \phi(f') - \phi(f(x)) \geq -\epsilon, \ \forall f' \in F(x),$$
$$\text{and } \langle g' - g(x), y\rangle_{H_2} + \psi(g') - \psi(g(x)) \geq -\epsilon, \ \forall g' \in G(y)\Big\}.$$

In the sequel, we assume that:

(A_1) $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are both two demicontinuous functions, and $h : H_1 \times H_2 \to \mathbb{R}$ is continuous;

(A_2) $F : H_1 \rightrightarrows H_1$ is (s,w)-closed, (s,s)-lower semicontinuous and (s,w)-subcontinuous with closed, convex values;

(A_3) $G : H_2 \rightrightarrows H_2$ is (s,w)-closed, (s,s)-lower semicontinuous and (s,w)-subcontinuous with closed, convex values;

(A_4) $\phi : H_1 \to \mathbb{R}$ and $\psi : H_2 \to \mathbb{R}$ are two weak continuous functions.

Theorem 3.4. *Let $F : H_1 \rightrightarrows H_1$ and $G : H_2 \rightrightarrows H_2$ be two set-valued mappings. Then (GSQIVI) is strongly well-posed, if and only if the solution set S of (GSQIVI) is nonempty and*

$$\lim_{\epsilon \to 0} \operatorname{diam}\Big(\Omega(\epsilon)\Big) = 0. \tag{3.1}$$

Proof. ($\Rightarrow$) : Suppose that (GSQIVI) is strongly well-posed. Then, by the definition of well-posedness of (GSQIVI), we know that (GSQIVI) has a unique solution $(x^*, y^*) \in H_1 \times H_2$, thus $S := \{(x^*, y^*)\} \neq \emptyset$. We now demonstrate that (3.1) holds.

Suppose to the contrary that $\operatorname{diam}(\Omega(\epsilon))$ does not tend to 0 as $\epsilon \to 0$. Hence, there are a constant $\beta > 0$, a positive sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ and $(x_n^{(1)}, y_n^{(1)}), (x_n^{(2)}, y_n^{(2)}) \in \Omega(\epsilon_n)$ such that

$$\|(x_n^{(1)}, y_n^{(1)}) - (x_n^{(2)}, y_n^{(2)})\|_{H_1 \times H_2} > \beta > 0, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

This means that $\{(x_n^{(1)}, y_n^{(1)})\}$ and $\{(x_n^{(2)}, y_n^{(2)})\}$ are both approximating sequences of the problem (GSQIVI). Therefore, according to the strong well-posedness of (GSQIVI), we get

$$\lim_{n\to\infty} (x_n^{(1)}, y_n^{(1)}) = \lim_{n\to\infty} (x_n^{(2)}, y_n^{(2)}) = (x^*, y^*), \text{ in } H_1 \times H_2. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} 0 < \beta &< \|(x_n^{(1)}, y_n^1) - (x_n^{(2)}, y_n^{(2)})\|_{H_1 \times H_2} \\ &\leq \|(x_n^{(1)}, y_n^1) - (x^*, y^*)\|_{H_1 \times H_2} + \|(x_n^{(2)}, y_n^{(2)}) - (x^*, y^*)\|_{H_1 \times H_2} \\ &\to 0, \end{aligned}$$

that generates a contradiction. So, (3.1) holds.
($\Leftarrow$): Conversely, suppose that (3.1) and $S \neq \emptyset$ hold. For each $\epsilon > 0$, due to $S \subset \Omega(\epsilon)$, we directly imply that S is singleton point set by using (3.1). Therefore, we denote $S = \{(x^*, y^*)\}$. Let sequence $\{(x_n, u_n)\} \subseteq H_1 \times H_2$ be an approximating sequence of the problem (GSQIVI). Then, there exists a positive sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\begin{cases} d_{H_1}(f(x_n), F(x_n)) \leq \epsilon_n, \\ d_{H_2}(g(y_n), G(y_n)) \leq \epsilon_n, \\ |h(x_n, y_n)| \leq \epsilon_n, \\ \langle f' - f(x_n), x_n\rangle_{H_1} + \phi(f') - \phi(f(x_n)) \geq -\epsilon_n, \ \forall f' \in F(x_n), \\ \langle g' - g(y_n), y_n\rangle_{H_2} + \psi(g') - \psi(g(y_n)) \geq -\epsilon_n, \ \forall g' \in G(y_n). \end{cases}$$

Thereby, we get $(x_n, u_n) \in \Omega(\epsilon_n)$ for all $n \in \mathbb{N}$. On the other hand, we also have $(x^*, y^*) \in \Omega(\epsilon_n)$ for each $n \in \mathbb{N}$. Hence, we calculate

$$\lim_{n\to\infty} \|(x_n, y_n) - (x^*, y^*)\|_{H_1\times H_2} \leq \lim_{n\to\infty} \operatorname{diam}\Bigg(\Omega(\epsilon_n)\Bigg) = 0,$$

which implies that $\{(x_n, y_n)\}$ strongly converges to (x^*, y^*). Consequently, (GSQIVI) is strongly well-posed. □

By the proof of Theorem 3.4, we can see that $S \neq \emptyset$ plays a significant role. In fact, under some suitable conditions, it can be replaced by $\Omega(\epsilon)$.

Theorem 3.5. *Assume that* (A_1)*-*(A_4) *hold, then (GSQIVI) is strongly well-posed, if and only if*

$$\Omega(\epsilon) \neq \emptyset, \forall \epsilon > 0, \text{ and } \lim_{\epsilon\to 0} \operatorname{diam}\Bigg(\Omega(\epsilon)\Bigg) = 0. \tag{3.4}$$

Proof. The necessity of proof is obvious from Theorem 3.4. So, we only need to prove the sufficiency.

Assume that the condition (3.4) holds. Due to $S \subset \Omega(\epsilon)$ for each $\epsilon > 0$, then it is obvious that the problem (GSQIVI) at most one solution. Let $\{(x_n, y_n)\}$ be an approximating sequence of (GSQIVI). Then there exists a positive sequence $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ as $n \to \infty$ such that

$$\begin{cases} d_{H_1}(f(x_n), F(x_n)) \leq \epsilon_n, \\ d_{H_2}(g(y_n), G(y_n)) \leq \epsilon_n, \\ |h(x_n, y_n)| \leq \epsilon_n, \\ \langle f' - f(x_n), x_n\rangle_{H_1} + \phi(f') - \phi(f(x_n)) \geq -\epsilon_n, \forall f' \in F(x_n), \\ \langle g' - g(y_n), y_n\rangle_{H_2} + \psi(g') - \psi(g(y_n)) \geq -\epsilon_n, \forall g' \in G(y_n). \end{cases}$$

By the definition of Ω, we have $(x_n, y_n) \in \Omega(\epsilon_n)$ for each $n \in \mathbb{N}$. By virtue of (3.4), we know that $\{(x_n, y_n)\}$ is a Cauchy sequence. So, we assume $(x_n, y_n) \to (x^*, y^*)$ in $H_1 \times H_2$. We now illustrate that (x^*, y^*) is the unique solution of the problem (GSQIVI).

Step 1: $f(x^*) \in F(x^*)$, $g(y^*) \in G(y^*)$ and $h(x^*, y^*) = 0$.

For each $n \in \mathbb{N}$, due to

$$|h(x_n, y_n)| \leq \epsilon_n,$$

and $(x_n, y_n) \to (x^*, y^*)$ in $H_1 \times H_2$, $\epsilon_n \to 0$, then we have

$$|h(x^*, y^*)| = \lim_{n\to\infty} |h(x_n, y_n)| \leq \lim_{n\to\infty} \epsilon_n = 0,$$

by the continuity of h, thus $h(x^*, y^*) = 0$. Furthermore, we also get $f(x_n) \rightharpoonup f(x^*)$ and $g(x_n) \rightharpoonup g(x^*)$ as $n \to \infty$, thanks to (A_1). Next, we will show

$$d_{H_1}(f(x^*), F(x^*)) \leq \liminf_{n\to\infty} d_{H_1}(f(x_n), F(x_n)) \leq \lim_{n\to\infty} \epsilon_n = 0. \tag{3.5}$$

Assume by contradiction that the left inequality of (3.5) does not hold. Therefore, there is a constant $\gamma > 0$ such that

$$\liminf_{n\to\infty} d_{H_1}(f(x_n), F(x_n)) < \gamma < d_{H_1}(f(x^*), F(x^*)).$$

This implies that there exist a subsequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ and a sequence $\{w_{n_k}\}$ with $w_{n_k} \in F(x_{n_k})$, for all $n_k \in \mathbb{N}$, such that

$$\|f(x_{n_k}) - w_{n_k}\|_{H_1} < \gamma, \ \forall n_k \in \mathbb{N}.$$

Since F is (s, w)-subcontinuous, without loss of generality, assume that $w_{n_k} \rightharpoonup w^*$ in H_1. Besides, we obtain $w^* \in F(x^*)$ because F is (s, w)-closed. Above all, we get

$$\begin{aligned} \gamma < d_{H_1}(f(x^*), F(x^*)) &\leq \|f(x^*) - w^*\|_{H_1} \\ &\leq \liminf_{k\to\infty} \|f(x_{n_k}) - w_{n_k}\|_{H_1} \\ &\leq \gamma, \end{aligned}$$

which is a contradiction. Thus $f(x^*) \in F(x^*)$. By applying the same arguments, we also get $g(y^*) \in G(y^*)$.

Step 2. $(x^*, y^*, f(x^*), g(y^*))$ satisfies the following results

$$\langle f' - f(x^*), x^* \rangle_{H_1} + \phi(f) - \phi(f(x^*)) \geq 0,\ \forall f' \in F(x^*), \tag{3.6}$$

$$\langle g' - g(y^*), y^* \rangle_{H_2} + \psi(g) - \psi(g(y^*)) \geq 0,\ \forall g' \in G(y^*). \tag{3.7}$$

In fact, we just prove that $(x^*, f(x^*))$ satisfies (3.6), since the proof of (3.7) is similar to (3.6).

For any $f' \in F(x^*)$, due to (s, w)-lower semicontinuity of F, then there exists a sequence $\{f_n\}$ with $f_n \in F(x_n)$ such that $f_n \rightharpoonup f'$ in H_1. According to the continuity of $f(\cdot)$ and $\phi(\cdot)$, we have

$$\begin{aligned} 0 &= \limsup_{n\to\infty} -\epsilon_n \\ &\leq \limsup_{n\to\infty} \Big[\langle f_n - f(x_n), x_n \rangle_{H_1} + \phi(f_n) - \phi(f(x_n)) \Big] \\ &\leq \langle f' - f(x^*), x^* \rangle_{H_1} + \phi(f') - \phi(f(x^*)). \end{aligned}$$

Hence, we have

$$\langle f' - f(x^*), x^* \rangle_{H_1} + \phi(f') - \phi(f(x^*)),\ \forall f' \in F(x^*).$$

To conclude, we obtain $S = \{(x^*, y^*)\}$ and $(x_n, y_n) \to (x^*, y^*)$, thus (GSQIVI) is strongly well-posed. □

For (GSIVI), we also have:

Theorem 3.6. *Assume that the following conditions hold:*

(i) $f : H_1 \to H_1$ *and* $g : H_2 \to H_2$ *are demicontinuous and* $h : H_1 \times H_2 \to \mathbb{R}$ *is continuous;*

(ii) $\phi : H_1 \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ *and* $\psi : H_2 \to \mathbb{R} := \mathbb{R} \cup \{+\infty\}$ *are both two property convex and lower semicontinuous functions.*

Then (GSIVI) is strongly well-posed, if and only if

$$\Omega(\epsilon) \neq \emptyset, \forall \epsilon > 0, \text{ and } \lim_{\epsilon\to 0} \operatorname{diam}(\Omega(\epsilon)) = 0.$$

Proof. We need only to prove the sufficiency, because the proof of necessity is similar to the arguments in Theorem 3.4.

Let $\{(x_n, y_n)\}$ be an approximating sequence of (GSIVI). Then there is $\epsilon_n > 0$ with $\epsilon_n \to 0$ as $n \to \infty$ such that (see Remark 3.3)

$$\begin{cases} f(x_n) \in C, \\ g(y_n) \in Q, \\ |h(x_n, y_n)| \leq \epsilon_n, \\ \langle f' - f(x_n), x_n \rangle_{H_1} + \phi(f') - \phi(f(x_n)) \geq -\epsilon_n, \forall f' \in C, \\ \langle g' - g(y_n), y_n \rangle_{H_2} + \psi(g') - \psi(g(y_n)) \geq -\epsilon_n, \forall g' \in Q. \end{cases} \tag{3.8}$$

It is similar to the proof of Theorem 3.5 that $(x_n, y_n) \to (x^*, y^*)$, $f(x_n) \rightharpoonup f(x^*)$, $g(y_n) \rightharpoonup g(y^*)$ and $h(x_n, y_n) \to h(x^*, y^*) = 0$. Since C is closed and convex, then we have $f(x^*) \in C$. $g(y^*) \in Q$ is also obtained.

On the other hand, for each $f' \in C$ according to (3.8) and lower semicontinuity of ϕ we get

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} -\epsilon_n \\ &\leq \limsup_{n\to\infty} \langle f' - f(x_n), x_n \rangle_{H_1} + \phi(f') - \phi(f(x_n)) \\ &\leq \langle f' - f(x^*), x^* \rangle_{H_1} + \phi(f') - \phi(f(x^*)). \end{aligned}$$

This implies

$$\langle f' - f(x^*), x^* \rangle_{H_1} + \phi(f') - \phi(f(x^*)) \geq 0, \ \forall f' \in C.$$

Also, we apply the same process to obtain

$$\langle g' - g(y^*), y^* \rangle_{H_2} + \psi(f') - \psi(g(y^*)) \geq 0, \ \forall g' \in Q.$$

Above all, we conclude $(x_n, y_n) \to (x^*, y^*)$, where (x^*, y^*) is uniform solution of (GSIVI) (2.1). This finishes the proof of this theorem. □

Indeed, by applying Theorems 3.4 and 3.6, we can obtain the following corollaries, which have been proved by Hu and Fang in [12].

Corollary 3.7 ([12, Theorem 3.1]). *The problem (SIVI)* (1.2) *is well-posed, if and only if the solution set* S *of* (1.2) *is nonempty and* $\mathrm{diam}(T(\epsilon)) \to 0$*, as* $\epsilon \to 0$*, where for each* $\epsilon > 0$*,* $T(\epsilon)$ *is defined as follows:*

$$T(\epsilon) := \Big\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^n : \ f(x) \in C, \ g(y) \in Q, \ \|y - Ax\| \leq \epsilon, \\ \langle f(x) - f', x \rangle \leq \epsilon, \ \forall f' \in C, \ \textit{and} \ \langle g(y) - g', y \rangle \leq \epsilon, \ \forall g' \in Q \Big\}.$$

Corollary 3.8 ([12, Theorem 3.2]). *Let* $f : \mathbb{R}^m \to \mathbb{R}^m$ *and* $g : \mathbb{R}^n \to \mathbb{R}^n$ *be two continuous functions, and* A *be a bound linear operator. Then the problem (SIVI)* (1.2) *is well-posed, if and only if*

$$T(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0, \quad \textit{and} \quad \textit{diam}\Big(T(\epsilon)\Big) \to 0, \ \textit{as} \ \epsilon \to 0.$$

4. The characterizations of well-posedness in the generalized sense for (GSQIVI)

In this section, we establish the metric characterizations and equivalent conditions of well-posedness of (GSQIVI) in the generalized sense.

Definition 4.1. The problem (GSQIVI) is said to be strongly (respectively, weakly) well-posed in the generalized sense, if and only if the solution set S of (GSQIVI) is nonempty and for every approximating sequence $\{(x_n, u_n)\}$, has a subsequence which strongly (respectively, weakly) converges to some point of S.

Firstly, we have the following lemma.

Lemma 4.2. *If* (A_1)*-*(A_4) *are satisfied, then the approximating solution set of (GSQIVI), that is* $\Omega(\epsilon)$*, is closed for each* $\epsilon > 0$*.*

Proof. For any $\epsilon > 0$ fixed, let $\{(x_n, y_n)\} \subset \Omega(\epsilon)$ be such that $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$, i.e.,

$$\begin{cases} d_{H_1}(f(x_n), F(x_n)) \leq \epsilon, \\ d_{H_2}(g(y_n), G(y_n)) \leq \epsilon, \\ |h(x_n, y_n)| \leq \epsilon, \\ \langle f' - f(x_n), x_n \rangle_{H_1} + \phi(f') - \phi(f(x_n)) \geq -\epsilon, \forall f' \in F(x_n), \\ \langle g' - g(y_n), y_n \rangle_{H_2} + \psi(g') - \psi(g(y_n)) \geq -\epsilon, \forall g' \in G(y_n). \end{cases}$$

We now show $(x^*, y^*) \in \Omega(\epsilon)$.

Since for each (x, y), $F(x)$ and $G(y)$ are two closed convex subsets of H_1 and H_2, respectively, therefore, we can obtain that there exists $f_n \in F(x_n)$ such that $\|f(x_n) - f_n\|_{H_1} \leq \epsilon$ by $d_{H_1}(f(x_n), F(x_n)) \leq \epsilon$. According to the (s, w)-subcontinuity and (s, w)-closedness of F, we conclude that there exists a subsequence

$\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \rightharpoonup f^* \in F(x^*)$. Furthermore, $f(x_n) \rightharpoonup f(x^*)$ thanks to demicontinuity of f. Thus we get

$$\|f(x_{n_k}) - f_{n_k}\| \le \epsilon.$$

By taking liminf into above inequality, we get

$$\|f(x^*) - f^*\| \le \liminf_{n\to\infty} \|f(x_{n_k}) - f_{n_k}\| \le \epsilon.$$

This implies $d_{H_1}(f(x^*), F(x^*)) \le \epsilon$. In the same way, we also have $d_{H_2}(g(y^*), G(y^*)) \le \epsilon$. Also, by the continuity of h, we have

$$\begin{aligned}|h(x^*, y^*)| &\le |h(x^n, y^n)| + |h(x^n, y^n) - h(x^*, y^*)| \\ &\le \epsilon + |h(x^n, y^n) - h(x^*, y^*)| \to \epsilon, \text{ as } n \to \infty,\end{aligned}$$

thus,

$$|h(x^*, y^*)| \le \epsilon.$$

On the other hand, for each $n \in \mathbb{N}$, we have

$$\langle f' - f(x_n), x_n\rangle_{H_1} + \phi(f') - \phi(f(x_n)) \ge -\epsilon, \ \forall f' \in F(x_n).$$

Since F is (s, w)-lower semicontinuous, then for each $f^* \in F(x^*)$ there exists a sequence $\{f_n\}$ with $f_n \in F(x_n)$ such that $f_n \rightharpoonup f^*$. Thereby, we calculate

$$\begin{aligned}\langle f^* - f(x^*), x^*\rangle_{H_1} + \phi(f^*) - \phi(f(x^*)) &= \lim_{n\to\infty} \Big[\langle f' - f(x_n), x_n\rangle_{H_1} + \phi(f') - \phi(f(x_n))\Big] \\ &\ge -\epsilon,\end{aligned}$$

therefore,

$$\langle f^* - f(x^*), x^*\rangle_{H_1} + \phi(f^*) - \phi(f(x^*)) \ge -\epsilon, \quad \forall f^* \in F(x^*).$$

By the similar method, we also obtain

$$\langle g^* - g(y^*), y^*\rangle_{H_1} + \psi(g^*) - \psi(g(y^*)) \ge -\epsilon, \ \forall g^* \in G(y^*).$$

Consequently, we conclude $(x^*, y^*) \in \Omega(\epsilon)$, i.e., $\Omega(\epsilon)$ is closed for each $\epsilon > 0$. □

For (GSIVI) (2.1), we have the following lemma.

Lemma 4.3. *Assume that the following conditions hold:*

(i) $f : H_1 \to H_1$ *and* $g : H_2 \to H_2$ *are demicontinuous and* $h : H_1 \times H_2 \to \mathbb{R}$ *is continuous;*

(ii) $\phi : H_1 \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ *and* $\psi : H_2 \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ *are both two property convex and lower semicontinuous functions.*

Then the approximating solution set of (SQIVI), $\Omega'(\epsilon)$ *is closed for each* $\epsilon > 0$*, where* $\Omega'(\epsilon)$ *is defined as follows:*

$$\begin{aligned}\Omega'(\epsilon) := \Big\{&(x, y) \in H_1 \times H_2 : \ f(x) \in C, \ g(y) \in Q, \ |h(x, y)| \le \epsilon, \\ &\langle f(x) - f', x\rangle \le \epsilon, \ \forall f' \in C, \ \text{and } \langle g(y) - g', y\rangle \le \epsilon, \ \forall g' \in Q\Big\}.\end{aligned}$$

Theorem 4.4. *The problem (GSQIVI) is strongly well-posed in the generalized sense, if and only if the solution set* S *of (GSQIVI) is nonempty compact and*

$$\lim_{\epsilon\to 0^+} e(\Omega(\epsilon), S) = 0. \tag{4.1}$$

Proof. (⇒): Suppose that (GSQIVI) is strongly well-posed in the generalized sense. According to the definition of strong well-posedness in the generalized sense for (GSQIVI), we know that $S \neq \emptyset$ and $S \subseteq \Omega(\epsilon) \neq \emptyset$, for all $\epsilon > 0$.

We now show that the solution set S of (GSQIVI) is compact. To do so, we just prove that for each sequence of S, which has convergent subsequence such that this subsequence converges to S. Let sequence $\{(x_n, y_n)\}$ be arbitrary sequence of S. Then, we readily get $\{(x_n, y_n)\} \subset \Omega(\epsilon)$ for each $n \in \mathbb{N}$ and for all $\epsilon > 0$. It is clear that $\{(x_n, y_n)\}$ is an approximating sequence for (GSQIVI). Because (GSQIVI) is strongly well-posed in the generalized sense, then there exists a subsequence of $\{(x_n, y_n)\}$, which strongly converges to some point of S. This means that S is compact.

To finish the sufficiency, next, we demonstrate that (4.1) is satisfied. Suppose to the contrary that $e(\Omega(\epsilon), S)$ does not tend to 0 as $\epsilon \to 0$. So, for any positive sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$, there exist $\beta > 0$ and $(x'_n, y'_n) \in \Omega(\epsilon_n)$ such that

$$d_{H_1\times H_2}\Bigg((x'_n, y'_n), S\Bigg) > \beta,\ \forall n \in \mathbb{N}. \tag{4.2}$$

Obviously, $\{(x'_n, y'_n)\}$ is an approximating sequence of (GSQIVI). Then, by applying the well-posedness of (GSQIVI) in the generalized sense, there is a subsequence $\{(x'_{n_k}, y'_{n_k})\}$ of $\{(x'_n, y'_n)\}$ such that $\{(x'_{n_k}, y'_{n_k})\}$ converges strongly to some point of S. It yields

$$d_{H_1\times H_2}\Bigg((x'_{n_k}, y'_{n_k}), S\Bigg) \to 0 \text{ as } n_k \to \infty.$$

This contradicts (4.2), so, (4.1) holds.

(⇐): For the converse, suppose that S is nonempty compact and condition (4.1) holds. Let sequence $\{(x_n, y_n)\}$ be an approximating sequence of (GSQIVI). Therefore, there exists sequence $\{\epsilon_n\}$ with $0 < \epsilon_n \to 0$ as $n \to \infty$, such that $(x_n, y_n) \in \Omega(\epsilon_n)$ for all $n \in \mathbb{N}$. By (4.1), there is a sequence $\{(x^*_n, y^*_n)\}$ in S, such that

$$\|(x_n, y_n) - (x^*_n, y^*_n)\|_{H_1\times H_2} \to 0 \text{ as } n \to \infty.$$

Due to the compactness of S, there is a subsequence $\{(x^*_{n_k}, y^*_{n_k})\}$ of $\{(x^*_n, y^*_n)\}$ such that it converges strongly to some point $(x^*, y^*) \in S$. Hence, we have

$$\begin{aligned}&\|(x_{n_k}, y_{n_k}) - (x^*, y^*)\|_{H_1\times H_2}\\ &\qquad \le \|(x_{n_k}, y_{n_k}) - (x^*_{n_k}, y^*_{n_k})\|_{H_1\times H_2} + \|(x^*_{n_k}, y^*_{n_k}) - (x^*, y^*)\|_{H_1\times H_2} \to 0 \quad \text{as} \quad n_k \to \infty.\end{aligned}$$

Therefore, (GSQIVI) is strongly well-posed in the generalized sense. □

According to the proof of Theorem 4.4, we can see that the compactness of S plays a key role. In fact, under the suitable conditions, we can establish a metric characterization of strongly well-posed in the generalized sense, by using the measurable of non-compactness of approximating solution sets.

Theorem 4.5. *Assume that* (A_1)-(A_4) *hold. Then (GSQIVI) is strongly well-posed in the generalized sense, if and only if*

$$\Omega(\epsilon) \neq \emptyset,\ \forall \epsilon > 0 \ \textit{and} \quad \lim_{\epsilon\to 0} \mu(\Omega(\epsilon)) = 0. \tag{4.3}$$

Proof. (⇒): If (GSQIVI) is strongly well-posed in the generalized sense, then the solution set S of (GSQIVI) is nonempty. It follows from Theorem 4.4 that S is compact, hence

$$\mu(S) = 0.$$

Also, it is directly obtained by $S \subseteq \Omega(\epsilon) \neq \emptyset$ for any $\epsilon > 0$ that

$$\mathcal{H}(\Omega(\epsilon), S) = \max\{e(\Omega(\epsilon), S), e(S, \Omega(\epsilon))\} = e(\Omega(\epsilon), S), \qquad \forall \epsilon > 0.$$

Therefore, we have

$$\mu(\Omega(\epsilon)) \leq 2H(\Omega(\epsilon), S) + \mu(S) = 2\mathcal{H}(\Omega(\epsilon), S) = 2e(\Omega(\epsilon), S).$$

We apply the condition (4.1) in Theorem 4.4 that $\lim_{\epsilon\to 0} \mu(\Omega(\epsilon)) = 0$. This implies that (4.3) holds.
($\Leftarrow$): For the converse, assume that the condition (4.3) is satisfied. Then, it follows from Lemma 4.2 that $\Omega(\epsilon)$ is nonempty closed for any $\epsilon > 0$, and

$$\lim_{\epsilon\to 0} \mu(\Omega(\epsilon)) = 0.$$

Set $\Omega = \bigcap_{\epsilon>0} \Omega(\epsilon)$. By applying the generalized Cantor theorem in [16, p. 412], we have

$$\lim_{\epsilon\to 0} \mathcal{H}(\Omega(\epsilon), \Omega) = 0,$$

and Ω is nonempty compact. Indeed, we claim that

$$\Omega = S.$$

It is obvious that $S \subseteq \Omega$. Therefore, we only need to illustrate that $\Omega \subseteq S$.

For any $(x^*, y^*) \in \Omega$ and $\epsilon > 0$ we have $d_{H_1\times H_2}((x^*, y^*), \Omega(\epsilon)) = 0$. Then, for each $\epsilon_n \to 0$ as $n \to \infty$, there exists $(x_n, y_n) \in \Omega(\epsilon_n)$ such that

$$\|(x^*, y^*) - (x_n, y_n)\|_{H_1\times H_2} \leq \epsilon_n.$$

That implies $x_n \to x^*$ in H_1 and $y_n \to y^*$ in H_2. By the same arguments as before, we easily get $f(x_n) \rightharpoonup f(x^*)$, $g(y_n) \rightharpoonup g(y^*)$ and $h(x_n, y_n) \to h(x^*, y^*) = 0$.

Since F and G are (s, w)-closed and (s, w)-subcontinuous, we also obtain by the same arguments in Theorem 3.5 that

$$d_{H_1}(f(x^*), F(x^*)) \leq \liminf_{n\to\infty} d_{H_1}(f(x_n), F(x_n)) \leq \lim_{n\to\infty} \epsilon_n = 0,$$

and

$$d_{H_2}(g(y^*), G(y^*)) \leq \liminf_{n\to\infty} d_{H_2}(g(y_n), G(y_n)) \leq \lim_{n\to\infty} \epsilon_n = 0,$$

thus $f(x^*) \in F(x^*)$ and $g(y^*) \in G(y^*)$.

On the other hand, according to (s, w)-lower semicontinuity of F, for each $f^* \in F(x^*)$ there exists $f_n \in F(x_n)$ such that $f_n \rightharpoonup f^*$. Hence, we have

$$\begin{aligned} 0 &= \limsup_{n\to\infty} -\epsilon_n \\ &\leq \limsup_{n\to\infty} \Big[\langle f_n - f(x_n), x_n\rangle_{H_1} + \phi(f_n) - \phi(f(x_n))\Big] \\ &\leq \langle f^* - f(x^*), x^*\rangle_{H_1} + \phi(f^*) - \phi(f(x^*)). \end{aligned}$$

Thus

$$\langle f^* - f(x^*), x^*\rangle_{H_1} + \phi(f^*) - \phi(f(x^*)) \geq 0, \ \forall f^* \in F(x^*).$$

Also, we can obtain

$$\langle g^* - g(y^*), y^*\rangle_{H_2} + \psi(g^*) - \psi(g(y^*)) \geq 0, \ \forall g^* \in G(y^*).$$

Above all, we deduce $(x^*, y^*) \in S$.

Thus, $S = \Omega$. Then, $\lim_{\epsilon\to 0} \mathcal{H}(\Omega(\epsilon), S) = 0$ and $\lim_{\epsilon\to 0} e(\Omega(\epsilon), S) = 0$. It follows from the compactness of S and Theorem 4.4 that (GSQIVI) is strongly well-posed in the generalized sense. □

In fact, we also have

Theorem 4.6. *Assume that the following conditions hold:*

(i) $f : H_1 \to H_1$ *and* $g : H_2 \to H_2$ *are demicontinuous and* $h : H_1 \times H_2 \to \mathbb{R}$ *is continuous;*

(ii) $\phi : H_1 \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ *and* $\psi : H_2 \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ *are two property convex and lower semicontinuous functions.*

Then (GSIVI) (2.1) *is strongly well-posed in the generalized sense, if and only if*

$$\Omega(\epsilon) \neq \emptyset, \ \forall \epsilon > 0 \quad and \quad \lim_{\epsilon \to 0} \mu(\Omega(\epsilon)) = 0.$$

By the same method, we conclude the following results.

Corollary 4.7 ([12, Theorem 3.3])**.** *(SIVI) is generalized well-posed, if and only if the solution set S of (SIVI) is nonempty compact and*

$$\mathcal{H}\Big(T(\epsilon), S\Big) \to 0, \ \ as \ \epsilon \to 0.$$

Corollary 4.8 ([12, Theorem 3.4])**.** *Let* $f : \mathbb{R}^m \to \mathbb{R}^m$ *and* $g : \mathbb{R}^n \to \mathbb{R}^n$ *be two continuous functions, and A be a bound linear operator. Then (SIVI) is generalized well-posed, if and only if*

$$T(\epsilon) \neq \emptyset, \ \forall \epsilon > 0, \ \ and \ \mu\Big(T(\epsilon), S\Big) \to 0, \ \ as \ \epsilon \to 0.$$

Acknowledgment

This work was financially supported by the National Natural Science Foundation of China (Grant No. 11301257) and the Innovative Team Program of the Neijiang Normal University (Grant No. 13TD02).

References

[1] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994), 123–145 . 1

[2] L. C. Ceng, J. C. Yao, *Well-posedness of generalized mixed variational inequalities, inclusion problems and fixed-point problems*, Nonlinear Anal., **69** (2008), 4585–4603. 1

[3] K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann., **266** (1984), 519–537. 1

[4] Y.-P. Fang, N.-J. Huang, *Variational-like inequalities with generalized monotone mappings in Banach spaces*, J. Optim. Theory Appl., **118** (2003), 327–338. 1

[5] R. Glowinski, J. L. Lions, R. Trémoliéres, *Numerical analysis of variational inequalities*, Translated from the French, Studies in Mathematics and its Applications, North-Holland Publishing Co., Amsterdam-New York, (1981).

[6] S. M. Guu, J. Li, *Vector variational-like inequalities with generalized bifunctions defined on nonconvex sets*, Nonlinear Anal., **71** (2009), 2847–2855. 1

[7] B. S. He, X.-Z. He, H. X. Liu, *Solving a class of constrained 'black-box' inverse variational inequalities*, European J. Oper. Res., **204** (2010), 391–401. 1

[8] B. S. He, H. X. Liu, *Inverse variational inequalities in economics-applications and algorithms*, Sciencepaper Online, (2006). 1

[9] B. S. He, H. X. Liu, M. Li, X.-Z. He, *PPA-based methods for monotone inverse variational inequalities*, Sciencepaper Online, (2006). 1

[10] R. Hu, Y.-P. Fang, *Well-posedness of inverse variational inequalities*, J. Convex Anal., **15** (2008), 427–437. 1

[11] R. Hu, Y.-P. Fang, *Levitin-Polyak well-posedness by perturbations of inverse variational inequalities*, Optim. Lett., **7** (2013), 343–359. 1

[12] R. Hu, Y.-P. Fang, *Well-posedness of the split inverse variational inequality problem*, Bull. Malays. Math. Sci. Soc., **2015** (2015), 12 pages. 1, 3.3, 3, 3.7, 3.8, 4.7, 4.8

[13] L. L. Huang, *Matrix Lagrange multipliers*, J. Comput. Complex. Appl., **2** (2016), 86–88. 1

[14] N.-J. Huang, J. Li, B. H. Thompson, *Stability for parametric implicit vector equilibrium problems*, Math. Comput. Modelling, **43** (2006), 1267–1274. 1

[15] K. Kimura, Y.-C. Liou, S.-Y. Wu, J.-C. Yao, *Well-posedness for parametric vector equilibrium problems with applications*, J. Ind. Manag. Optim., **4** (2008), 313–327. 1

[16] K. Kuratowski, *Topology*, Vol. I, New edition, revised and augmented, Translated from the French by J. Jaworowski, Academic Press, New York-London; PańStwowe Wydawnictwo Naukowe, Warsaw, (1966). 4

[17] B. Lemaire, *Well-posedness, conditioning and regularization of minimization, inclusion and fixed-point problems*, Proceedings of the 4th International Conference on Mathematical Methods in Operations Research and 6th Workshop on Well-posedness and Stability of Optimization Problems, Sozopol, (1997), Pliska Stud. Math. Bulgar., **12** (1998), 71–84. 1

[18] X. Li, X.-S. Li, N.-J. Huang, *A generalized f-projection algorithm for inverse mixed variational inequalities*, Optim. Lett., **8** (2014), 1063–1076. 1

[19] X.-J. Long, N.-J. Huang, *Metric characterizations of α-well-posedness for symmetric quasi-equilibrium problems*, J. Global Optim., **45** (2009), 459–471. 1

[20] N. K. Mahato, C. Nahak, *Weakly relaxed α-pseudomonotonicity and equilibrium problem in Banach spaces*, J. Appl. Math. Comput., **40** (2012), 499–509. 1

[21] M. A. Noor, K. I. Noor, V. Gupta, *On equilibrium-like problems*, Appl. Anal., **86** (2007), 807–818.

[22] M. A. Noor, K. I. Noor, S. Zainab, *On a predictor-corrector method for solving invex equilibrium problems*, Nonlinear Anal., **71** (2009), 3333–3338. 1

[23] L. Scrimali, *An inverse variational inequality approach to the evolutionary spatial price equilibrium problem*, Optim. Eng., **13** (2012), 375–387. 1

[24] A. N. Tykhonov, *On the stability of the functional optimization problem*, USSR J. Comput. Math. Math. Phys., **6** (1966), 631–634. 1

[25] G. C. Wu, *New correction functional and Lagrange multipliers for two point value problems*, J. Comput. Complex. Appl., **2** (2016), 46–48. 1

[26] J. F. Yang, *Dynamic power price problem: an inverse variational inequality approach*, J. Ind. Manag. Optim., **4** (2008), 673–684. 1