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Schur-convexity for Lehmer mean of n variables

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Abstract

Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity for Lehmer mean of n variables are investigated, and some mean value inequalities of n variables are established.

Keywords: Schur convexity, Schur geometric convexity, Schur harmonic convexity, n variables Lehmer mean, majorization, inequalities.
2010 MSC: 26E60, 26D15, 26A51.

1. Introduction and preliminaries

Throughout the paper we denote the set of n-dimensional row vector on the real number field by $\mathbb{R}^n$. Also,

$$\mathbb{R}^n_+ = \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0,\, i = 1, \dots, n\}.$$

In particular, $\mathbb{R}^1$ and $\mathbb{R}^1_+$ denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

For $x, y > 0$ and $p \in \mathbb{R}$, the Lehmer mean values $L_p(x, y)$ were introduced by Lehmer [13] as follows:

$$L_p(x, y) = \frac{x^p + y^p}{x^{p-1} + y^{p-1}}.$$

Many mean values are special cases of the Lehmer mean values, for example

$$A(x, y) = \frac{x + y}{2} = L_1(x, y)$$

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is the arithmetic mean,

$$G(x,y)=\sqrt{xy}=L_{\frac{1}{2}}(x,y)$$

is the geometric mean,

$$H(x,y)=\frac{2xy}{x+y}=L_0(x,y)$$

is the harmonic mean,

$$\tilde{H}(x,y)=\frac{x^2+y^2}{x+y}=L_2(x,y)$$

is the anti-harmonic mean.

Investigation of the elementary properties and inequalities for $L_p(x,y)$ has attracted the attention of a considerable number of mathematicians (see [1–3, 10–12, 14, 21, 23, 26, 28–31]).

In 2009, Gu and Shi [11] discussed the Schur convexity and Schur geometric convexity of the Lehmer means $L_p(x,y)$ with respect to $(x,y)\in\mathbb{R}_+^2$ for fixed p. Subsequently, Xia and Chu [36] researched the Schur harmonic convexity of the Lehmer means $L_p(x,y)$ with respect to $(x,y)\in\mathbb{R}_+^2$ for fixed p.

Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}_+^n$. For Schur-convexity and Schur-geometric convexity of n variables Lehmer mean,

$$L_p(\mathbf{x})=L_p(x_1,x_2,\ldots,x_n)=\frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n x_i^{p-1}},$$

Gu and Shi [11] obtained the following results.

Theorem 1.1. *Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}_+^n$ and $p\in\mathbb{R}$. If $1\le p\le 2$, then $L_p(\mathbf{x})$ is Schur-convex with $\mathbf{x}\in\mathbb{R}_+^n$, if $0\le p\le 1$, then $L_p(\mathbf{x})$ is Schur-concave with $\mathbf{x}\in\mathbb{R}_+^n$.*

Furthermore, Gu and Shi [11] proposed the following conjecture.

Conjecture 1.2. *If $p\ge 2$, then $L_p(\mathbf{x})$ is Schur-convex with $\mathbf{x}\in\mathbb{R}_+^n$, if $p\le 0$, then $L_p(\mathbf{x})$ is Schur-concave with $\mathbf{x}\in\mathbb{R}_+^n$.*

We first point out that this conjecture does not hold.

In fact, for $n=3, p=3$, by computing, we have

$$\Delta := (x_1-x_2)\left(\frac{\partial L_3(\mathbf{x})}{\partial x_1}-\frac{\partial L_3(\mathbf{x})}{\partial x_2}\right)=\frac{(x_1-x_2)^2\lambda(\mathbf{x})}{(x_1^2+x_2^2+x_3^2)^2},$$

where

$$\lambda(\mathbf{x})=\lambda(x_1,x_2,x_3)=3(x_1+x_2)(x_1^2+x_2^2+x_3^2)-2(x_1^3+x_2^3+x_3^3),$$

if $\mathbf{x}=(1,3,7)$, then $\lambda(\mathbf{x})=-34$, so that $\Delta<0$, but by taking $\mathbf{y}=(1,2,3)$, then $\lambda(\mathbf{y})=54$, so that $\Delta>0$. According to Lemma 2.4 in second section, we assert that the Schur-convexity of $L_3(x_1,x_2,x_3)$ is not determined on the whole $\mathbb{R}_+^3$.

It can easily be shown that $L_{-2}(x_1,x_2,x_3)=\frac{1}{L_3(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3})}$, since the Schur-convexity of $L_3(x_1,x_2,x_3)$ is not determined on the whole $\mathbb{R}_+^3$, $L_{-2}(x_1,x_2,x_3)$ so does.

In this paper, we study Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of $L_p(\mathbf{x})$ on certain subsets of $\mathbb{R}_+^n$. As consequences, some interesting inequalities are obtained.

Our main results are as follows:

Theorem 1.3. *Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}_+^n, n\ge 2$ and $p\in\mathbb{R}$.*

(I) *If $p\ge 2$, then for any $a>0$, $L_p(\mathbf{x})$ is Schur-convex with $\mathbf{x}\in\left[\frac{(p-2)a}{p},a\right]^n$.*

(II) *If $p<0$, then for any $a>0$, $L_p(\mathbf{x})$ is Schur-concave with $\mathbf{x}\in\left[a,\frac{(p-2)a}{p}\right]^n$.*

Theorem 1.4. *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n, n \geq 2$ *and* $p \in \mathbb{R}$.

(I) *If* $p < \frac{1}{2}$ *and* $p \neq 0$, *then for any* $a > 0$, $L_p(\mathbf{x})$ *is Schur-geometrically concave with* $\mathbf{x} \in \left[a, (\frac{p-1}{p})^2 a\right]^n$.

(II) *If* $p > \frac{1}{2}$, *then for any* $a > 0$, $L_p(\mathbf{x})$ *is Schur-geometrically convex with* $\mathbf{x} \in \left[(\frac{p-1}{p})^2 a, a\right]^n$.

(III) *If* $p = 0$, *then* $L_p(\mathbf{x})$ *is Schur-geometrically convex with* $\mathbf{x} \in \mathbb{R}_+^n$.

Theorem 1.5. *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n, n \geq 2$ *and* $p \in \mathbb{R}$.

(I) *If* $0 \leq p \leq 1$, *then* $L_p(\mathbf{x})$ *is Schur-harmonically convex with* $\mathbf{x} \in \mathbb{R}_+^n$, *if* $-1 \leq p \leq 0$, *then* $L_p(\mathbf{x})$ *is Schur-harmonically concave with* $\mathbf{x} \in \mathbb{R}_+^n$.

(II) *If* $p > 1$, *then for any* $a > 0$, $L_p(\mathbf{x})$ *is Schur-harmonically convex with* $\mathbf{x} \in \left[\frac{(p-1)a}{p+1}, a\right]^n$.

(III) *If* $p < -1$, *then for any* $a > 0$, $L_p(\mathbf{x})$ *is Schur-harmonically concave with* $\mathbf{x} \in \left[a, \frac{(p-1)a}{p+1}\right]^n$.

2. Definitions and lemmas

We need the following definitions and lemmas.

Definition 2.1 ([17, 27])**.** Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} \prec \mathbf{y}$), if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$, for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(ii) $\Omega \subset \mathbb{R}^n$ is called a convex set, if $(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \ldots, \alpha x_n + \beta y_n) \in \Omega$, for any $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(iii) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω, if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. φ is said to be a Schur-concave function on Ω, if and only if $-\varphi$ is Schur-convex function.

Definition 2.2 ([20, 44])**.** Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n$.

(i) $\Omega \subset \mathbb{R}_+^n$ is called a geometrically convex set, if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$, for any $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(ii) Let $\Omega \subset \mathbb{R}_+^n$, φ: $\Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω, if

$$(\ln x_1, \ln x_2, \ldots, \ln x_n) \prec (\ln y_1, \ln y_2, \ldots, \ln y_n)$$

on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. φ is said to be a Schur-geometrically concave function on Ω, if and only if $-\varphi$ is Schur-geometrically convex function.

Definition 2.3 ([4, 18])**.** Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n$.

(i) A set $\Omega \subset \mathbb{R}_+^n$ is said to be a harmonically convex set, if

$$\left(\frac{x_1 y_1}{\lambda x_1 + (1-\lambda) y_1}, \frac{x_2 y_2}{\lambda x_2 + (1-\lambda) y_2}, \ldots, \frac{x_n y_n}{\lambda x_n + (1-\lambda) y_n}\right) \in \Omega,$$

for every $\mathbf{x}, \mathbf{y} \in \Omega$ and $\lambda \in [0, 1]$.

(ii) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur-harmonically convex function on Ω, if $\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right) \prec \left(\frac{1}{y_1}, \frac{1}{y_2}, \ldots, \frac{1}{y_n}\right)$ implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. A function φ is said to be a Schur-harmonically concave function on Ω, if and only if $-\varphi$ is a Schur-harmonically convex function.

Lemma 2.4 ([17, 27])**.** *Let* $\Omega \subset \mathbb{R}^n$ *is convex set, and has a nonempty interior set* Ω^0. *Let* $\varphi : \Omega \to \mathbb{R}$ *be continuous on* Ω *and differentiable in* Ω^0. *Then* φ *is the* $Schur-convex$ *(or* $Schur-concave$*, resp.) function, if and only if it is symmetric on* Ω *and if*

$$(x_1 - x_2)\left(\frac{\partial \varphi(\mathbf{x})}{\partial x_1} - \frac{\partial \varphi(\mathbf{x})}{\partial x_2}\right) \geq 0, \quad (or \ \leq 0 \ resp.), \tag{2.1}$$

holds for any $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$.

Remark 2.5 ([9, 19]). It is easy to see that the condition (2.1) is equivalent to

$$\frac{\partial\varphi(\mathbf{x})}{\partial x_i} \le \frac{\partial\varphi(\mathbf{x})}{\partial x_{i+1}}, \quad (\text{or } \ge \text{ resp.}), \quad i = 1, \dots, n-1, \quad \text{for all } \mathbf{x} \in \mathbb{D} \cap \Omega,$$

where $\mathbb{D} = \{\mathbf{x} : x_1 \le x_2 \le \cdots \le x_n\}$.

The condition (2.1) is also equivalent to

$$\frac{\partial\varphi(\mathbf{x})}{\partial x_i} \ge \frac{\partial\varphi(\mathbf{x})}{\partial x_{i+1}}, \quad (\text{or } \le \text{ resp.}), \quad i = 1, \dots, n-1, \quad \text{for all } \mathbf{x} \in \mathbb{E} \cap \Omega,$$

where $\mathbb{E} = \{\mathbf{x} : x_1 \ge x_2 \ge \cdots \ge x_n\}$.

Lemma 2.6 ([20, 44])**.** *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric geometrically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω^0. Then φ is a Schur-geometrically convex (or Schur-geometrically concave, resp.) function, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1\frac{\partial\varphi(\mathbf{x})}{\partial x_1} - x_2\frac{\partial\varphi(\mathbf{x})}{\partial x_2}\right) \ge 0, \quad (\textit{or } \le 0 \textit{ resp.}), \tag{2.2}$$

holds for any $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$.

Remark 2.7. It is easy to see that the condition (2.2) is equivalent to

$$x_i\frac{\partial\varphi(\mathbf{x})}{\partial x_i} \le x_{i+1}\frac{\partial\varphi(\mathbf{x})}{\partial x_{i+1}}, \quad (\text{or } \ge \text{ resp.}), \quad i = 1, \dots, n-1, \quad \text{for all } \mathbf{x} \in \mathbb{D} \cap \Omega,$$

where $\mathbb{D} = \{\mathbf{x} : x_1 \le x_2 \le \cdots \le x_n\}$.

The condition (2.2) is also equivalent to

$$x_i\frac{\partial\varphi(\mathbf{x})}{\partial x_i} \ge x_{i+1}\frac{\partial\varphi(\mathbf{x})}{\partial x_{i+1}}, \quad (\text{or } \le \text{ resp.}), \quad i = 1, \dots, n-1, \quad \text{for all } \mathbf{x} \in \mathbb{E} \cap \Omega,$$

where $\mathbb{E} = \{\mathbf{x} : x_1 \ge x_2 \ge \cdots \ge x_n\}$.

Lemma 2.8 ([4, 18])**.** *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric harmonically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω^0. Then φ is a Schur- harmonically convex (or Schur-harmonically concave, resp.) function, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1^2\frac{\partial\varphi(\mathbf{x})}{\partial x_1} - x_2^2\frac{\partial\varphi(\mathbf{x})}{\partial x_2}\right) \ge 0, \quad (\textit{or } \le 0 \textit{ resp.}), \tag{2.3}$$

holds for any $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$.

Remark 2.9. It is easy to see that the condition (2.3) is equivalent to

$$x_i^2\frac{\partial\varphi(\mathbf{x})}{\partial x_i} \le x_{i+1}^2\frac{\partial\varphi(\mathbf{x})}{\partial x_{i+1}}, \quad (\text{or} \ge \text{ resp.}), \quad i = 1, \dots, n-1, \quad \text{for all } \mathbf{x} \in \mathbb{D} \cap \Omega,$$

where $\mathbb{D} = \{\mathbf{x} : x_1 \le x_2 \le \cdots \le x_n\}$.

The condition (2.3) is also equivalent to

$$x_i^2\frac{\partial\varphi(\mathbf{x})}{\partial x_i} \ge x_{i+1}^2\frac{\partial\varphi(\mathbf{x})}{\partial x_{i+1}}, \quad (\text{or} \le \text{ resp.}), \quad i = 1, \dots, n-1, \quad \text{for all } \mathbf{x} \in \mathbb{E} \cap \Omega,$$

where $\mathbb{E} = \{\mathbf{x} : x_1 \ge x_2 \ge \cdots \ge x_n\}$.

Lemma 2.10. *Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, $m \in \mathbb{R}$. Then*

$$x_1 \geq \frac{x_1^m + x_2^m + \cdots + x_n^m}{x_1^{m-1} + x_2^{m-1} + \cdots + x_n^{m-1}} \geq x_n.$$

Proof.

$$\begin{aligned} &x_1(x_1^{m-1} + x_2^{m-1} + \cdots + x_n^{m-1}) - (x_1^m + x_2^m + \cdots + x_n^m) \\ &\qquad = x_1^{m-1}(x_1 - x_1) + x_2^{m-1}(x_1 - x_2) + \cdots + x_n^{m-1}(x_1 - x_n) \geq 0, \end{aligned}$$

$$\begin{aligned} &x_n(x_1^{m-1} + x_2^{m-1} + \cdots + x_n^{m-1}) - (x_1^m + x_2^m + \cdots + x_n^m) \\ &\qquad = x_1^{m-1}(x_n - x_1) + x_2^{m-1}(x_n - x_2) + \cdots + x_n^{m-1}(x_n - x_n) \leq 0. \end{aligned}$$

We have thus proved the Lemma 2.10. □

Lemma 2.11 ([17]). *Let $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $A_n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n x_i$. Then*

$$\mathbf{u} = \Bigg(\underbrace{A_n(\mathbf{x}), A_n(\mathbf{x}), \cdots, A_n(\mathbf{x})}_{n}\Bigg) \prec (x_1, x_2, \cdots, x_n) = \mathbf{x}.$$

3. Proofs of theorems

3.1. Proof of Theorem 1.3

Proof. Straightforward computation gives

$$\frac{\partial L_p(\mathbf{x})}{\partial x_i} = \frac{p x_i^{p-1}\sum_{j=1}^n x_j^{p-1} - (p-1)x_i^{p-2}\sum_{j=1}^n x_j^p}{(\sum_{j=1}^n x_j^{p-1})^2}, \quad i = 1, 2, \ldots, n, \tag{3.1}$$

and then

$$\frac{\partial L_p(\mathbf{x})}{\partial x_i} - \frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}} = \frac{f_i(\mathbf{x})}{(\sum_{i=1}^n x_i^{p-1})^2}, i = 1, 2, \ldots, n-1,$$

where

$$f_i(\mathbf{x}) = p(x_i^{p-1} - x_{i+1}^{p-1})\sum_{j=1}^n x_j^{p-1} - (p-1)(x_i^{p-2} - x_{i+1}^{p-2})\sum_{j=1}^n x_j^p.$$

It is clear that $L_p(\mathbf{x})$ is symmetric with $\mathbf{x} \in \mathbb{R}_+^n$. Without loss of generality, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n > 0$.

For any $a > 0$, according to the integral mean value theorem, there is a ξ which lies between x_i and x_{i+1}, such that

$$\begin{aligned} p(x_i^{p-1} - x_{i+1}^{p-1}) - a(p-1)(x_i^{p-2} - x_{i+1}^{p-2}) &= (p-1)p\int_{x_{i+1}}^{x_i} x^{p-2}dx - a(p-2)(p-1)\int_{x_{i+1}}^{x_i} x^{p-3}dx \\ &= (p-1)\int_{x_{i+1}}^{x_i}[px^{p-2} - a(p-2)x^{p-3}]dx \\ &= (p-1)[p\xi^{p-2} - a(p-2)\xi^{p-3}](x_i - x_{i+1}) \\ &= (p-1)p\xi^{p-3}\left(\xi - \frac{(p-2)a}{p}\right)(x_i - x_{i+1}). \end{aligned} \tag{3.2}$$

Proof of (I): When $p \geq 2$ and $a \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq \frac{(p-2)a}{p} > 0$, from (3.2), we have

$$p(x_i^{p-1} - x_{i+1}^{p-1}) - a(p-1)(x_i^{p-2} - x_{i+1}^{p-2}) \geq 0,$$

that is,

$$\frac{p(x_i^{p-1} - x_{i+1}^{p-1})}{(p-1)(x_i^{p-2} - x_{i+1}^{p-2})} \geq a,$$

and then from Lemma 2.10, it follows that

$$\frac{p(x_i^{p-1} - x_{i+1}^{p-1})}{(p-1)(x_i^{p-2} - x_{i+1}^{p-2})} \geq x_1 \geq \frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}},$$

namely, $f_i(\mathbf{x}) \geq 0$, and then $\frac{\partial L_p(\mathbf{x})}{\partial x_i} \geq \frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}}$. By Lemma 2.4 and Remark 2.5, it follows that $L_p(\mathbf{x})$ is Schur-convex with $\mathbf{x} \in \left[\frac{p-2}{p}a, a\right]^n$.

Proof of (II): When $p < 0$ and $\frac{(p-2)a}{p} \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq a > 0$, from (3.2), we have

$$p(x_i^{p-1} - x_{i+1}^{p-1}) - a(p-1)(x_i^{p-2} - x_{i+1}^{p-2}) \leq 0,$$

that is,

$$\frac{p(x_i^{p-1} - x_{i+1}^{p-1})}{(p-1)(x_i^{p-2} - x_{i+1}^{p-2})} \leq a,$$

and then from Lemma 2.10, it follows that

$$\frac{p(x_i^{p-1} - x_{i+1}^{p-1})}{(p-1)(x_i^{p-2} - x_{i+1}^{p-2})} \leq x_n \leq \frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}},$$

namely, $f_i(\mathbf{x}) \leq 0$, and then $\frac{\partial L_p(\mathbf{x})}{\partial x_i} \leq \frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}}$. By Lemma 2.4 and Remark 2.5, it follows that $L_p(\mathbf{x})$ is Schur-concave with $\mathbf{x} \in \left[a, \frac{p-2}{p}a\right]^n$.

The proof of Theorem 1.3 is complete. □

3.2. Proof of Theorem 1.4

Proof. From (3.1), we have

$$x_i \frac{\partial L_p(\mathbf{x})}{\partial x_i} - x_{i+1} \frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}} = \frac{g_i(\mathbf{x})}{(\sum_{i=1}^n x_i^{p-1})^2}, \quad i = 1, 2, \ldots, n-1,$$

where

$$g_i(\mathbf{x}) = p(x_i^p - x_{i+1}^p) \sum_{j=1}^n x_j^{p-1} - (p-1)(x_i^{p-1} - x_{i+1}^{p-1}) \sum_{j=1}^n x_j^p.$$

It is clear that $L_p(\mathbf{x})$ is symmetric with $\mathbf{x} \in \mathbb{R}_+^n$. Without loss of generality, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n > 0$.

For any $a > 0$, according to the integral mean value theorem, there is a ξ which lies between x_i and x_{i+1}, such that

$$\begin{aligned} p(x_i^p - x_{i+1}^p) - a(p-1)(x_i^{p-1} - x_{i+1}^{p-1}) &= p^2 \int_{x_{i+1}}^{x_i} x^{p-1} dx - a(p-1)^2 \int_{x_{i+1}}^{x_i} x^{p-2} dx \\ &= \int_{x_{i+1}}^{x_i} [p^2 x^{p-1} - a(p-1)^2 x^{p-2}] dx \end{aligned} \tag{3.3}$$

$$= [p^2\xi^{p-1} - a(p-1)^2\xi^{p-2}](x_i - x_{i+1})$$
$$= p^2\xi^{p-2}\left[\xi - (\frac{p-1}{p})^2 a\right](x_i - x_{i+1}).$$

Proof of (I): When $p \geq \frac{1}{2}$ and $a \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq (\frac{p-1}{p})^2 a > 0$, from (3.3) we have

$$p(x_i^p - x_{i+1}^p) - a(p-1)(x_i^{p-1} - x_{i+1}^{p-1}) \geq 0,$$

that is,

$$\frac{p(x_i^p - x_{i+1}^p)}{(p-1)(x_i^{p-1} - x_{i+1}^{p-1})} \geq a,$$

and then from Lemma 2.10, it follows that

$$\frac{p(x_i^p - x_{i+1}^p)}{(p-1)(x_i^{p-1} - x_{i+1}^{p-1})} \geq x_1 \geq \frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}},$$

namely, $g_i(\mathbf{x}) \geq 0$, and then $x_i\frac{\partial L_p(\mathbf{x})}{\partial x_i} \geq x_{i+1}\frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}}$. By Lemma 2.6 and Remark 2.7, it follows that $L_p(\mathbf{x})$ is Schur-geometrically convex with $\mathbf{x} \in \left[(\frac{p-1}{p})^2 a, a\right]^n$.

Proof of (II): When $p < \frac{1}{2}, p \neq 0$ and $(\frac{p-1}{p})^2 a \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq a > 0$, from (3.3), we have

$$p(x_i^p - x_{i+1}^p) - a(p-1)(x_i^{p-1} - x_{i+1}^{p-1}) \leq 0,$$

that is,

$$\frac{p(x_i^p - x_{i+1}^p)}{(p-1)(x_i^{p-1} - x_{i+1}^{p-1})} \leq a,$$

and then from Lemma 2.10, it follows that

$$\frac{p(x_i^p - x_{i+1}^p)}{(p-1)(x_i^{p-1} - x_{i+1}^{p-1})} \leq x_n \leq \frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}},$$

namely, $g_i(\mathbf{x}) \leq 0$, and then $x_i\frac{\partial L_p(\mathbf{x})}{\partial x_i} \leq x_{i+1}\frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}}$. By Lemma 2.6 and Remark 2.7, it follows that $L_p(\mathbf{x})$ is Schur-geometrically concave with $\mathbf{x} \in \left[a, (\frac{p-1}{p})^2 a\right]^n$.

Proof of (III): When $p = 0$, $g_i(\mathbf{x}) \leq 0$, it follows that $L_p(\mathbf{x})$ is Schur-geometrically concave with $\mathbf{x} \in \mathbb{R}_+^n$.

The proof of Theorem 1.4 is complete. □

3.3. Proof of Theorem 1.5

Proof. From (3.1), we have

$$x_i^2\frac{\partial L_p(\mathbf{x})}{\partial x_i} - x_{i+1}^2\frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}} = \frac{h_i(\mathbf{x})}{(\sum_{i=1}^n x_i^{p-1})^2}, \quad i = 1, 2, \ldots, n-1, \tag{3.4}$$

where

$$h_i(\mathbf{x}) = p(x_i^{p+1} - x_{i+1}^{p+1})\sum_{j=1}^n x_j^{p-1} - (p-1)(x_i^p - x_{i+1}^p)\sum_{j=1}^n x_j^p.$$

It is clear that $L_p(\mathbf{x})$ is symmetric with $\mathbf{x} \in \mathbb{R}_+^n$. Without loss of generality, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n > 0$.

Proof of (I): According to the integral mean value theorem, there is a ξ which lies between x_i and x_{i+1}, such that

$$\begin{aligned} h_i(\mathbf{x}) &= \sum_{j=1}^{n} x_j^{p-1}\left[p(x_i^{p+1}-x_{i+1}^{p+1})-(p-1)(x_i^p-x_{i+1}^p)\frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}}\right] \\ &= \sum_{j=1}^{n} x_j^{p-1}\left[(p+1)p\int_{x_{i+1}}^{x_i} x^p dx - p(p-1)\frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}}\int_{x_{i+1}}^{x_i} x^{p-1}dx\right] \\ &= \sum_{j=1}^{n} x_j^{p-1}p\int_{x_{i+1}}^{x_i}\left[(p+1)x^p-(p-1)\frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}}x^{p-1}\right]dx \\ &= \sum_{j=1}^{n} x_j^{p-1}p\left[(p+1)\xi^p-(p-1)\frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}}\xi^{p-1}\right](x_i-x_{i+1}) \\ &= \sum_{j=1}^{n} x_j^{p-1}(p+1)p\xi^{p-1}\left[\xi-\frac{p-1}{p+1}\frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}}\right](x_i-x_{i+1}). \end{aligned} \tag{3.5}$$

Notice that for $-1 < p \le 1$, $\xi - \frac{p-1}{p+1}\frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}} \ge 0$.

When $0 < p \le 1$, from (3.5), we have $h_i(\mathbf{x}) \ge 0$, and then $x_i^2\frac{\partial L_p(\mathbf{x})}{\partial x_i} \ge x_{i+1}^2\frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}}$. By Lemma 2.8 and Remark 2.9, it follows that $L_p(\mathbf{x})$ is Schur-harmonically convex with $\mathbf{x} \in \mathbb{R}_+^n$.

When $-1 < p \le 0$, $h_i(\mathbf{x}) \le 0$, and then $x_i^2\frac{\partial L_p(\mathbf{x})}{\partial x_i} \le x_{i+1}^2\frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}}$. By Lemma 2.8 and Remark 2.9, it follows that $L_p(\mathbf{x})$ is Schur-harmonically concave with $\mathbf{x} \in \mathbb{R}_+^n$.

When $p = -1$, $h_i(\mathbf{x}) = 2\sum_{j=1}^n x_j^{-1}(x_i^{-1} - x_{i+1}^{-1}) \le 0$, it follows that $L_p(\mathbf{x})$ is Schur-harmonically concave with $\mathbf{x} \in \mathbb{R}_+^n$.

Proof of (II): For any $a > 0$, according to the integral mean value theorem, there is a ξ which lies between x_i and x_{i+1}, such that

$$\begin{aligned} p(x_i^{p+1}-x_{i+1}^{p+1})-a(p-1)(x_i^p-x_{i+1}^p) &= p(p+1)\int_{x_{i+1}}^{x_i} x^p dx - a(p-1)p\int_{x_{i+1}}^{x_i} x^{p-1}dx \\ &= p\int_{x_{i+1}}^{x_i}[(p+1)x^p - a(p-1)x^{p-1}]dx \\ &= p[(p+1)\xi^p - a(p-1)\xi^{p-1}](x_i-x_{i+1}) \\ &= p(p+1)\xi^{p-1}\left[\xi-\frac{(p-1)a}{p+1}\right](x_i-x_{i+1}). \end{aligned} \tag{3.6}$$

When $p \ge 1$ and $a \ge x_1 \ge x_2 \ge \cdots \ge x_n \ge \frac{p-1}{p+1}a > 0$, from (3.6) we have

$$p(x_i^{p+1}-x_{i+1}^{p+1})-a(p-1)(x_i^p-x_{i+1}^p) \ge 0,$$

that is,

$$\frac{p(x_i^{p+1}-x_{i+1}^{p+1})}{(p-1)(x_i^p-x_{i+1}^p)} \ge a,$$

and then from Lemma 2.10, it follows that

$$\frac{p(x_i^{p+1}-x_{i+1}^{p+1})}{(p-1)(x_i^p-x_{i+1}^p)} \ge x_1 \ge \frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}},$$

namely, $h_i(\mathbf{x}) \ge 0$, and then $x_i^2\frac{\partial L_p(\mathbf{x})}{\partial x_i} \ge x_{i+1}^2\frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}}$. By Lemma 2.8 and Remark 2.9, it follows that $L_p(\mathbf{x})$ is Schur-harmonically convex with $\mathbf{x} \in \left[\frac{p-1}{p+1}a, a\right]^n$.

Proof of (III): When $p < -1$ and $\frac{p-1}{p+1}a \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq a > 0$, from (3.6), we have

$$p(x_i^{p+1} - x_{i+1}^{p+1}) - a(p-1)(x_i^p - x_{i+1}^p) \leq 0,$$

that is,

$$\frac{p(x_i^{p+1} - x_{i+1}^{p+1})}{(p-1)(x_i^p - x_{i+1}^p)} \leq a,$$

and then from Lemma (2.10), it follows that

$$\frac{p(x_i^{p+1} - x_{i+1}^{p+1})}{(p-1)(x_i^p - x_{i+1}^p)} \leq x_n \leq \frac{\sum_{j=1}^n x_j^p}{\sum_{j=1}^n x_j^{p-1}},$$

namely, $h_i(\mathbf{x}) \leq 0$, and then $x_i^2 \frac{\partial L_p(\mathbf{x})}{\partial x_i} \leq x_{i+1}^2 \frac{\partial L_p(\mathbf{x})}{\partial x_{i+1}}$. By Lemma 2.8 and Remark 2.9, it follows that $L_p(\mathbf{x})$ is Schur-harmonically concave with $\mathbf{x} \in \left[a, \frac{p-1}{p+1}a\right]^n$.

The proof of Theorem 1.5 is complete. □

4. Applications

Theorem 4.1. *For any $a > 0$, if $p \geq 2$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \left[\frac{p-2}{p}a, a\right]^n$, then we have*

$$A_n(\mathbf{x}) \geq L_p(\mathbf{x}). \tag{4.1}$$

If $p < 0$ and $\mathbf{x} \in \left[a, \frac{p-2}{p}a\right]^n$, then the inequality (4.1) is reversed.

Proof. If $p \geq 2$ and $\mathbf{x} \in \left[\frac{p-2}{p}a, a\right]^n$, then by Theorem 1.3, from Lemma 2.11, we have

$$L_p(\mathbf{u}) \geq L_p(\mathbf{x}),$$

rearranging gives (4.1), if $p < 0$ and $\mathbf{x} \in \left[a, \frac{p-2}{p}a\right]^n$, then the inequality (4.1) is reversed.

The proof is complete. □

Theorem 4.2. *For any $a > 0$, if $p > \frac{1}{2}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \left[(\frac{p-1}{p})^2 a, a\right]^n$, then we have*

$$G_n(\mathbf{x}) \leq L_p(\mathbf{x}). \tag{4.2}$$

where $G_n(\mathbf{x}) = \sqrt[n]{x_1 x_2 \cdots x_n}$ is the geometric mean of $\mathbf{x}$. If $p < \frac{1}{2}, p \neq 0$ and $\mathbf{x} \in \left[a, (\frac{p-1}{p})^2 a\right]^n$, then the inequality (4.2) is reversed.

Proof. By Lemma 2.11, we have

$$\left(\underbrace{\log G_n(\mathbf{x}), \cdots, \log G_n(\mathbf{x})}_{n}\right) \prec (\log x_1, \log x_2, \cdots, \log x_n),$$

if $p > \frac{1}{2}$ and $\mathbf{x} \in \left[(\frac{p-1}{p})^2 a, a\right]^n$, by Theorem 1.4, it follows

$$L_p\left(\underbrace{G_n(\mathbf{x}), \cdots, G_n(\mathbf{x})}_{n}\right) \leq L_p(x_1, x_2, \cdots, x_n),$$

rearranging gives (4.2). If $p < \frac{1}{2}, p \neq 0$ and $\mathbf{x} \in \left[a, (\frac{p-1}{p})^2 a\right]^n$, then the inequality (4.2) is reversed.

The proof is complete. □

Theorem 4.3. *For any $a > 0$, if $p > 1$ and $\mathbf{x} \in \left[\frac{p-1}{p+1}a, a\right]^n$, then we have*

$$H_n(\mathbf{x}) \leq L_p(\mathbf{x}), \tag{4.3}$$

where $H_n(\mathbf{x}) = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}$ is the harmonic mean of $\mathbf{x}$. If $p < -1$ and $\mathbf{x} \in \left[a, \frac{p-1}{p+1}a\right]^n$, then the inequality (4.3) is reversed.

Proof. By Lemma 2.11, we have

$$\Bigg(\underbrace{\frac{1}{H_n(\mathbf{x})}, \cdots, \frac{1}{H_n(\mathbf{x})}}_{n}\Bigg) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right).$$

If $p > 1$ and $\mathbf{x} \in \left[\frac{p-1}{p+1}a, a\right]^n$, by Theorem 1.5, it follows

$$L_p\Bigg(\underbrace{H_n(\mathbf{x}), \cdots, H_n(\mathbf{x})}_{n}\Bigg) \leq L_p(x_1, x_2, \cdots, x_n),$$

rearranging gives (4.3), if $p < -1$ and $\mathbf{x} \in \left[a, \frac{p-1}{p+1}a\right]^n$, then the inequality (4.3) is reversed.

The proof is complete. □

In recent years, the study on the properties of the mean by using theory of majorization is unusually active, interested readers may refer to the literature [5–9, 15, 16, 19, 22, 24, 25, 32–35, 37–43, 45].

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