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Viscosity approximation methods for hierarchical optimization problems of multivalued nonexpansive mappings in CAT(0) spaces

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Abstract

The purpose of this paper is to prove some strong convergence theorems for hierarchical optimization problems of multivalued nonexpansive mappings in CAT(0) spaces by using the viscosity approximation method. Our results generalize the results of [X.-D. Liu, S.-S. Chang, J. Inequal. Appl., **2013** (2013), 14 pages], [R. Wangkeeree, P. Preechasilp, J. Inequal. Appl., **2013** (2013), 15 pages], and many others. Some related results in $\mathbb{R}$-trees are also given.

Keywords: Viscosity approximation method, fixed point, variational inequality, hierarchical optimization problems, multivalued nonexpansive mapping, CAT(0) space.
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1. Introduction

One of the successful approximation methods for finding fixed points of nonexpansive mappings was given by Moudafi [20]. Let E be a nonempty closed convex subset of a Hilbert space H and $T : E \to E$ be a nonexpansive mapping with a nonempty fixed point set $F(T)$. The following scheme is known as the *viscosity approximation method* or *Moudafi's viscosity approximation method*:

$$\begin{aligned} &x_1 \in E \text{ arbitrarily chosen,} \\ &x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(x_n), \quad n \in \mathbb{N}, \end{aligned} \tag{1.1}$$

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where $f : E \to E$ is a contraction and $\{\alpha_n\}$ is a sequence in (0,1). In [20], under some suitable assumptions, the author proved that the sequence $\{x_n\}$ defined by (1.1) converges strongly to a point z in $F(T)$ which satisfies the following variational inequality:

$$\langle f(z) - z, z - x\rangle \geq 0, \quad x \in F(T).$$

We note that the Halpern approximation method [13],

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T(x_n), \quad n \in \mathbb{N},$$

where u is a fixed element in E, is a special case of the Moudafi's viscosity approximation method. Notice also that the Moudafi's viscosity approximation method can be applied to convex optimization, linear programming, monotone inclusions, and elliptic differential equations.

In 2013, Wangkeeree and Preechasilp [25] by using the concept of quasilinearization, studied the convergence problem of the following viscosity iterations in CAT(0) space:

$$x_t = tf(x_t) \oplus (1 - t)Tx_t \tag{1.2}$$

and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.3}$$

where $T : C \to C$ is a nonexpansive mapping, $f : C \to C$ is a contraction, $t \in (0, 1)$, and $\{\alpha_n\}$ is a sequence in $\subset (0, 1)$. They proved that $\{x_t\}$ defined by (1.2) converges strongly to $x^* \in F(T)$ (as $t \to 0$) such that $x^* = P_{F(T)}f(x^*)$ in the framework of a CAT(0) space. Furthermore, they also proved that $\{x_n\}$ defined by (1.3) converges strongly as $n \to \infty$ to $x^* \in F(T)$ under certain appropriate conditions imposed on $\{\alpha_n\}$.

Recently, Liu and Chang [18] introduced and studied the following *hierarchical optimization problems* (HOP) in CAT(0) space.

Let $f, g : C \to C$ be two contractions with contractive constant $k \in [0, 1)$, and $T_1, T_2 : C \to C$ be two nonexpansive mappings such that $F(T_1)$ and $F(T_2)$ are nonempty. The " so-called" *hierarchical optimization problem in CAT(0) space* is to find $(x^*, y^*) \in F(T_1) \times F(T_2)$ satisfying the following:

$$\begin{cases} \langle \overrightarrow{x^* f(y^*)}, \overrightarrow{xx^*}\rangle \geq 0, & x \in F(T_1), \\ \langle \overrightarrow{y^* g(x^*)}, \overrightarrow{yy^*}\rangle \geq 0, & y \in F(T_2). \end{cases} \tag{1.4}$$

They proved the following theorems.

Theorem 1.1 ([18]). *Let C be a closed convex subset of a complete CAT(0) space X, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings such that $F(T_1)$ and $F(T_2)$ are nonempty. Let f, g be two contractions on C with contractive constant $k \in (0, 1)$. For each $t \in (0, 1]$, let $\{x_t\}$ and $\{y_t\}$ be given by*

$$\begin{cases} x_t = tf(T_2 y_t) \oplus (1 - t)T_1 x_t, \\ y_t = tg(T_1 x_t) \oplus (1 - t)T_2 y_t. \end{cases}$$

Then $x_t \to x^*$ and $y_t \to y^*$ as $t \to 0$ such that $x^* = P_{F(T_1)}f(y^*), y^* = P_{F(T_2)}g(x^*)$ which solves HOP (1.4).

Theorem 1.2 ([18]). *Let C be a closed convex subset of a complete CAT(0) space X, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings such that $F(T_1)$ and $F(T_2)$ are nonempty. Let f, g be two contractions on C with contractive constant $k \in (0, 1)$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$\begin{cases} x_0, y_0 \in C, \\ x_{n+1} = \alpha_n f(T_2 y_n) \oplus (1 - \alpha_n)T_1 x_n, \\ y_{n+1} = \alpha_n g(T_1 x_n) \oplus (1 - \alpha_n)T_2 y_n, \ n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the following:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Then $x_n \to x^*$ *and* $y_n \to y^*$ *as* $n \to \infty$ *such that* $x^* = P_{F(T_1)} f(y^*), y^* = P_{F(T_2)} g(x^*)$ *which solves HOP* (1.4).

Fixed point theory for multivalued mappings has many useful applications in applied sciences, in particular, in game theory and optimization theory. It is naturally to put forward the following:

Open question: Can we extend the above Theorems 1.1 and 1.2 to multi-valued nonexpansive mappings in CAT(0) spaces?

The purpose of this paper is by using the viscosity approximation method to prove some strong convergence theorems for hierarchical optimization problems of multivalued nonexpansive mappings in CAT(0) spaces. Our results not only give an affirmative answer to the above open question but also generalize the results of Wangkeeree and Preechasilp [25], Liu and Chang [18], Kumam et al. [17], Saipara et al. [22], and many others. Some related results in $\mathbb{R}$-trees are also given.

2. preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers. Let $[0, l]$ be a closed interval in $\mathbb{R}$ and x, y be two points in a metric space (X, d). A geodesic joining x to y is a map $\xi : [0, l] \to X$ such that $\xi(0) = x, \xi[l] = y$, and $d(\xi(s), \xi(t)) = |s - t|$ for all $s, t \in [0, l]$. The image of ξ is called a geodesic segment joining x and y, which is denoted by $[x, y]$ whenever it is unique. The space (X, d) is said to be a geodesic space if every two points in X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset E of X is said to be convex if every pair of points $x, y \in E$ can be joined by a geodesic in X and the image of such a geodesic is contained in E.

Definition 2.1. A geodesic space X is said to be a CAT(0) space if for each $x, y, z \in X$ and $\lambda \in [0, 1]$, we have

$$d^2(\lambda x \oplus (1 - \lambda)y, z) \le \lambda d^2(x, z) + (1 - \lambda)d^2(y, z) - \lambda(1 - \lambda)d^2(x, y). \tag{2.1}$$

For other equivalent definitions and basic properties of CAT(0) space, we refer the reader to standard texts, such as [4, 6].

It is well-known that every CAT(0) space is uniquely geodesic. Notice also that Pre-Hilbert spaces, $\mathbb{R}$-trees, and Euclidean buildings are examples of CAT(0) spaces (see [4, 5]).

Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d). It follows from Proposition 2.4 of [4] that for each $x \in X$, there exists a unique point $x_0 \in E$ such that

$$d(x, x_0) = \inf\{d(x, y) : y \in E\}.$$

In this case, x_0 is called the unique nearest point of x in E. The metric projection of X onto E is the mapping $P_E : X \to E$ defined by

$$P_E(x) := \text{ the unique nearest point of } x \text{ in } E.$$

By Lemma 2.1 of [12], for each $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = (1 - t)d(x, y) \quad \text{and} \quad d(y, z) = td(x, y). \tag{2.2}$$

We denote by $tx \oplus (1 - t)y$ the unique point z satisfying (2.2).

Now, we collect some elementary facts about CAT(0) spaces.

Lemma 2.2. *Let X be a CAT(0) space. Then*

(i) *(see lemma 2.4 of [12]) for each $x, y, z \in X$ and $\lambda \in [0, 1]$,*

$$d(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d(x, z) + (1-\lambda)d(y, z);$$

(ii) *(see [7]) for each $x, y, z \in X$ and $s, t \in [0, 1]$,*

$$d((1-t)x \oplus ty, (1-s)x \oplus sy) \leq |s-t|d(x, y);$$

(iii) *(see [4]) for each $x, y, z, w \in X$ and $t \in [0, 1]$,*

$$d((1-t)x \oplus ty, (1-t)z \oplus tw) \leq (1-t)d(x, z) + td(y, w);$$

(iv) *(see lemma 3 of [15]) for each $x, y, z \in X$ and $t \in [0, 1]$,*

$$d(tx \oplus (1-t)z, ty \oplus (1-t)z) \leq td(x, y).$$

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It follows from Proposition 7 of [11] that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if $A(\{x_n\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the $\triangle-$limit of $\{x_n\}$.

Lemma 2.3 ([16])**.** *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.4 ([10])**.** *If E is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in E, then the asymptotic center of $\{x_n\}$ is in E.*

The concept of quasi-linearization was introduced by Berg and Nikolaev [3]. Let (X, d) be a metric space. We denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. The quasilinearization is a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)) \quad \text{for all } a, b, c, d \in X.$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle, \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that (X, d) satisfies the Cauchy-Schwarz inequality if

$$|\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle| \leq d(a, b)d(c, d) \quad \text{for all } a, b, c, d \in X.$$

It is well-known from Corollary 3 of [3] that a geodesic space X is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality. Some other properties of quasi-linearization are included as follows.

Lemma 2.5 ([3]). *Let E be a nonempty closed convex subset of a complete CAT(0) space X. Then*

$$v = P_E(u) \text{ if and only if } \langle \overrightarrow{vu}, \overrightarrow{wv} \rangle \geq 0 \text{ for all } w \in E.$$

Lemma 2.6 ([25]). *Let X be a complete CAT(0) space. Then, for all $u, x, y \in X$, the following inequality holds:*

$$d^2(x,u) \leq d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Lemma 2.7 ([25]). *Let X be a complete CAT(0) space. For any $t \in [0,1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ *and* $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Lemma 2.8 ([1]). *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X, and $x \in X$. Then $\{x_n\}$ $\Delta-$converges to x if and only if $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0$ for all $y \in X$.*

Recall that a continuous linear functional μ on ℓ_∞, the Banach space of bounded real sequences, is called a Banach limit, if $||\mu|| = \mu(1, 1, \cdots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in \ell_\infty$.

Lemma 2.9 ([24]). *Let α be a real number and let $(a_1, a_2, \cdots) \in \ell_\infty$ be such that $\mu_n(a_n) \leq \alpha$ for all Banach limits μ and $\limsup_n (a_{n+1} - a_n) \leq 0$. Then $\limsup_n a_n \leq \alpha$.*

Lemma 2.10 ([26]). *Let $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\delta_n, \quad n \geq 0,$$

where $\{\gamma_n\} \subseteq (0,1)$ and $\{\delta_n\} \subseteq \mathbb{R}$ such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Let E be a nonempty subset of a CAT(0) space (X, d). We denote the family of nonempty bounded closed subsets of E by $BC(E)$, the family of nonempty bounded closed convex subsets of E by $BCC(E)$, and the family of nonempty compact subsets of E by $K(E)$. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $BC(X)$, i.e.,

$$H(A,B) = \max\{\sup_{a\in A} \text{dist}(a,B), \sup_{b\in B} \text{dist}(b,A)\}, \ A, B \in BC(X),$$

where $\text{dist}(a,B) := \inf\{d(a,b) : b \in B\}$ is the distance from the point a to the set B.

Definition 2.11. A multivalued mapping $T : E \to BC(X)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that

$$H(T(x), T(y)) \leq kd(x,y), \ x, y \in E. \tag{2.3}$$

If (2.3) is valid when $k = 1$, then T is called nonexpansive. A point $x \in E$ is called a fixed point of T if $x \in T(x)$. We shall denote by $F(T)$ the set of all fixed points of T. A multivalued mapping T is said to satisfy the endpoint condition [8] if $F(T) \neq \emptyset$ and $T(x) = \{x\}$ for all $x \in F(T)$.

The following fact is a consequence of Lemma 3.2 in [9]. Notice also that it is an extension of Proposition 3.7 in [16].

Lemma 2.12. *If E is a closed convex subset of a complete CAT(0) space X and $T : E \to K(E)$ is a nonexpansive mapping, then the condition $\{x_n\}$ $\triangle-$ converges to x and $dist(x_n, T(x_n)) \to 0$ imply $x \in F(T)$.*

The following fact is also needed.

Lemma 2.13 ([9]). *Let E be a closed convex subset of a complete CAT(0) space X and $T : E \to BC(X)$ be a nonexpansive mapping. If T satisfies the endpoint condition, then $F(T)$ is closed and convex.*

3. Main results

Now we are ready to give our main results in this paper.

Let (X, d) be a metric space. Define a mapping $\hat{d} : (X \times X) \times (X \times X) \to \mathbb{R}^+$ by

$$\hat{d}((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$$

for all $x_1, x_2, y_1, y_2 \in X$. Then it is easy to verify that $(X \times X, \hat{d})$ is a metric space, and $(X \times X, \hat{d})$ is complete if and only if (X, d) is complete.

Lemma 3.1. *Let E be a closed convex subset of a complete CAT(0) space X. Let $f, g : E \to E$ be two contractions with contractive constant $k \in (0, 1)$, and let $T_1, T_2 : E \to K(E)$ be two nonexpansive mappings. For any $s \in (0, 1)$, define a multivalued mapping $G_s : E \times E \to E \times E$ by*

$$G_s(x, y) = (sf(T_2y) \oplus (1 - s)T_1x, sg(T_1x) \oplus (1 - s)T_2y).$$

Then G_s is a multivalued contraction on $E \times E$.

Proof. For all $(x_1, y_1), (x_2, y_2) \in E \times E$ and for all $(u_1, v_1) \in G_s(x_1, y_1)$, for all $(u_2, v_2) \in G_s(x_2, y_2)$, there exist $w_1 \in T_1x_1, w_2 \in T_1x_2, z_1 \in T_2y_1, z_2 \in T_2y_2$, such that

$$\begin{cases} u_1 = sf(z_1) \oplus (1 - s)w_1, \\ v_1 = sg(w_1) \oplus (1 - s)z_1, \end{cases}$$

and

$$\begin{cases} u_2 = sf(z_2) \oplus (1 - s)w_2, \\ v_2 = sg(w_2) \oplus (1 - s)z_2, \end{cases}$$

then we have

$$\begin{aligned} \hat{d}((u_1, v_1), (u_2, v_2)) =& d(u_1, u_2) + d(v_1, v_2) \\ =& d(sf(z_1) \oplus (1 - s)w_1, sf(z_2) \oplus (1 - s)w_2) \\ &+ d(sg(w_1) \oplus (1 - s)z_1, sg(w_2) \oplus (1 - s)z_2) \\ \leq& sd(f(z_1), f(z_2)) + (1 - s)d(w_1, w_2) + sd(g(w_1), g(w_2)) + (1 - s)d(z_1, z_2) \\ \leq& skd(z_1, z_2) + (1 - s)d(w_1, w_2) + skd(w_1, w_2) + (1 - s)d(z_1, z_2) \\ \leq& (1 - s(1 - k))(d(z_1, z_2) + d(w_1, w_2)) \\ \leq& (1 - s(1 - k))(H(T_2y_1, T_2y_2) + H(T_1x_1, T_1x_2)) \\ \leq& (1 - s(1 - k))(d(x_1, x_2) + (y_1, y_2)) \\ =& (1 - s(1 - k))\hat{d}((x_1, y_1), (x_2, y_2)). \end{aligned}$$

Again since $(u_1, v_1) \in G_s(x_1, y_1)$ and $(u_2, v_2) \in G_s(x_2, y_2)$, we have

$$\hat{d}((u_1, v_1), G_s(x_2, y_2)) \leq \hat{d}((u_1, v_1), (u_2, v_2)), \quad \hat{d}((u_2, v_2), G_s(x_1, y_1)) \leq \hat{d}((u_2, v_2), (u_1, v_1)).$$

These imply that

$$\hat{d}((u_1, v_1), (u_2, v_2)) \geq \max\{\hat{d}((u_1, v_1), G_s(x_2, y_2)), \hat{d}(G_s(x_1, y_1), (u_2, v_2))\}$$

for all $(u_1, v_1) \in G_s(x_1, y_1)$ and for all $(u_2, v_2) \in G_s(x_2, y_2)$. Hence we have

$$\max\{\hat{d}((u_1, v_1), G_s(x_2, y_2)), \hat{d}(G_s(x_1, y_1), (u_2, v_2))\} \leq (1 - s(1 - k))\hat{d}((x_1, y_1), (x_2, y_2)). \tag{3.1}$$

Taking supremum limit on both sides in (3.1), we have

$$\max\{\sup_{(u_1,v_1)\in G_s(x_1,y_1)} \hat{d}((u_1,v_1),G_s(x_2,y_2)), \sup_{(u_2,v_2)\in G_s(x_2,y_2)} \hat{d}(G_s(x_1,y_1),(u_2,v_2))\}$$
$$\leq (1-s(1-k))\hat{d}((x_1,y_1),(x_2,y_2)),$$

i.e.,

$$H(G_s(x_1,y_1),G_s(x_2,y_2)) \leq (1-s(1-k))\hat{d}((x_1,y_1),(x_2,y_2)).$$

This implies that G_s is a multivalued contraction mapping. Applying Nadler's theorem [21], G_s has a (not necessarily unique) fixed point $(x_s,y_s)\in E\times E$ such that

$$\begin{cases} x_s \in sf(T_2y_s)\oplus(1-s)T_1x_s, \\ y_s \in sg(T_1x_s)\oplus(1-s)T_2y_s. \end{cases} \tag{3.2}$$

□

Theorem 3.2. *Let E be a closed convex subset of a complete CAT(0) space X, and let $T_1,T_2 : E \to K(E)$ be two nonexpansive mappings satisfying the endpoint condition. Let f,g be two contractions on E with contractive constant $k\in(0,1)$. For each $s\in(0,1]$, let $\{x_s\}$ and $\{y_s\}$ be the nets defined by (3.2). Then $x_s\to x^*$ and $y_s\to y^*$ as $s\to 0$ such that $x^* = P_{F(T_1)}f(y^*), y^* = P_{F(T_2)}g(x^*)$ which is a solution of HOP (1.4).*

Proof. We first show that both $\{x_s\}$ and $\{y_s\}$ are bounded. In fact, it follows from (3.2) that for each pair $x_s,\ y_s$, there exist $z_s\in T_1x_s, u_s\in T_2y_s$ such that

$$\begin{cases} x_s = sf(u_s)\oplus(1-s)z_s, \\ y_s = sg(z_s)\oplus(1-s)u_s. \end{cases}$$

By the endpoint condition, for each $(p,q)\in F(T_1)\times F(T_2)$, we have

$$\begin{aligned} d(x_s,p)+d(y_s,q) =& d(sf(u_s)\oplus(1-s)z_s,p)+d(sg(z_s)\oplus(1-s)u_s,q) \\ \leq& sd(f(u_s),p)+(1-s)d(z_s,p)+sd(g(z_s),q)+(1-s)d(u_s,q) \\ \leq& s(d(f(u_s),f(q))+d(f(q),p))+(1-s)d(z_s,p) \\ &+s(d(g(z_s),g(p))+d(g(p),q))+(1-s)d(u_s,q) \\ \leq& skd(u_s,q)+sd(f(q),p)+(1-s)d(z_s,p)+skd(z_s,p)+sd(g(p),q)+(1-s)d(u_s,q) \\ =& sk\mathrm{dist}(u_s,T_2q)+sd(f(q),p)+(1-s)\mathrm{dist}(z_s,T_1p)+sk\mathrm{dist}(z_s,T_1p) \\ &+sd(g(p),q)+(1-s)\mathrm{dist}(u_s,T_2q) \\ \leq& skH(T_2y_s,T_2q)+sd(f(q),p)+(1-s)H(T_1x_s,T_1p) \\ &+skH(T_1x_s,T_1p)+sd(g(p),q)+(1-s)H(T_2y_s,T_2q) \\ \leq& skd(y_s,q)+sd(f(q),p)+(1-s)d(x_s,p)+skd(x_s,p)+sd(g(p),q)+(1-s)d(y_s,q). \end{aligned}$$

After simplifying, we have

$$d(x_s,p)+d(y_s,q) \leq \frac{1}{1-k}(d(f(q),p)+d(g(p),q)).$$

Hence both $\{x_s\}$ and $\{y_s\}$ are bounded, so are $\{z_s\},\{u_s\}$ and $\{f(u_s)\}$ and $\{g(z_s)\}$. We note that,

$$\begin{aligned} \mathrm{dist}(x_s,T_1x_s)+\mathrm{dist}(y_s,T_2y_s) &\leq d(x_s,z_s)+d(y_s,u_s) \\ &\leq sd(f(u_s),u_s)+sd(g(z_s),z_s)\to 0 \text{ (as } s\to 0). \end{aligned}$$

Next, we show that $\{(x_s, y_s)\}$ converges strongly to (x^*, y^*) where $x^* = P_{F(T_1)}f(y^*), y^* = P_{F(T_2)}g(x^*)$ and it is a solution of HOP (1.4).

In fact, let $\{s_n\}$ be a sequence in $(0, 1)$ converging to 0 and put $x_n := x_{s_n}$ and $y_n := y_{s_n}$. Now we show that there exists a subsequence of $\{(x_n, y_n)\}$ converging to (x^*, y^*) where $x^* = P_{F(T_1)}f(y^*), y^* = P_{F(T_2)}g(x^*)$. By Lemmas 2.3 and 2.12, there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ and $(x^*, y^*) \in F(T_1) \times F(T_2)$ such that

$$\triangle - \lim_{k\to\infty} x_{n_k} = x^*,\ \triangle - \lim_{k\to\infty} y_{n_k} = y^*.$$

It follows from the endpoint condition and Lemma 2.7 (i) that

$$\begin{aligned}
d^2(x_{n_k}, x^*) + d^2(y_{n_k}, y^*) =& \langle \overrightarrow{x_{n_k}x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{y_{n_k}y^*}, \overrightarrow{y_{n_k}y^*}\rangle \\
\le& s_{n_k}\langle \overrightarrow{f(u_{n_k})x^*}, \overrightarrow{x_{n_k}x^*}\rangle + (1 - s_{n_k})\langle \overrightarrow{z_{n_k}x^*}, \overrightarrow{x_{n_k}x^*}\rangle \\
&+ s_{n_k}\langle \overrightarrow{g(z_{n_k})y^*}, \overrightarrow{y_{n_k}y^*}\rangle + (1 - s_{n_k})\langle \overrightarrow{u_{n_k}y^*}, \overrightarrow{y_{n_k}y^*}\rangle \\
\le& s_{n_k}\langle \overrightarrow{f(u_{n_k})x^*}, \overrightarrow{x_{n_k}x^*}\rangle + (1 - s_{n_k})d(z_{n_k}, x^*)d(x_{n_k}, x^*) \\
&+ s_{n_k}\langle \overrightarrow{g(z_{n_k})y^*}, \overrightarrow{y_{n_k}y^*}\rangle + (1 - s_{n_k})d(u_{n_k}, y^*)d(y_{n_k}, y^*) \\
\le& s_{n_k}\langle \overrightarrow{f(u_{n_k})x^*}, \overrightarrow{x_{n_k}x^*}\rangle + (1 - s_{n_k})\mathrm{dist}(z_{n_k}, T_1x^*)d(x_{n_k}, x^*) \\
&+ s_{n_k}\langle \overrightarrow{g(z_{n_k})y^*}, \overrightarrow{y_{n_k}y^*}\rangle + (1 - s_{n_k})\mathrm{dist}(u_{n_k}, T_2y^*)d(y_{n_k}, y^*) \\
\le& s_{n_k}\langle \overrightarrow{f(u_{n_k})x^*}, \overrightarrow{x_{n_k}x^*}\rangle + (1 - s_{n_k})H(T_1x_{n_k}, T_1x^*)d(x_{n_k}, x^*) \\
&+ s_{n_k}\langle \overrightarrow{g(z_{n_k})y^*}, \overrightarrow{y_{n_k}y^*}\rangle + (1 - s_{n_k})H(T_2y_{n_k}, T_2y^*)d(y_{n_k}, y^*) \\
\le& s_{n_k}\langle \overrightarrow{f(u_{n_k})x^*}, \overrightarrow{x_{n_k}x^*}\rangle + (1 - s_{n_k})d^2(x_{n_k}, x^*) \\
&+ s_{n_k}\langle \overrightarrow{g(z_{n_k})y^*}, \overrightarrow{y_{n_k}y^*}\rangle + (1 - s_{n_k})d^2(y_{n_k}, y^*).
\end{aligned}$$

Simplifying it we have

$$\begin{aligned}
&d^2(x_{n_k}, x^*) + d^2(y_{n_k}, y^*) \\
&\le \langle \overrightarrow{f(u_{n_k})x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{g(z_{n_k})y^*}, \overrightarrow{y_{n_k}y^*}\rangle \\
&= \langle \overrightarrow{f(u_{n_k})f(y^*)}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{f(y^*)x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{g(z_{n_k})g(x^*)}, \overrightarrow{y_{n_k}y^*}\rangle + \langle \overrightarrow{g(x^*)y^*}, \overrightarrow{y_{n_k}y^*}\rangle \\
&\le d(f(u_{n_k}), f(y^*))d(x_{n_k}, x^*) + \langle \overrightarrow{f(y^*)x^*}, \overrightarrow{x_{n_k}x^*}\rangle + d(g(z_{n_k}), g(x^*))d(y_{n_k}, y^*) + \langle \overrightarrow{g(x^*)y^*}, \overrightarrow{y_{n_k}y^*}\rangle \\
&\le k(d(u_{n_k}, y^*)d(x_{n_k}, x^*) + d(z_{n_k}, x^*)d(y_{n_k}, y^*)) + \langle \overrightarrow{f(y^*)x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{g(x^*)y^*}, \overrightarrow{y_{n_k}y^*}\rangle \\
&\le k(\mathrm{dist}(u_{n_k}, T_2y^*)d(x_{n_k}, x^*) + \mathrm{dist}(z_{n_k}, T_1x^*)d(y_{n_k}, y^*)) + \langle \overrightarrow{f(y^*)x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{g(x^*)y^*}, \overrightarrow{y_{n_k}y^*}\rangle \\
&\le k(H(T_2(y_{n_k}), T_2y^*)d(x_{n_k}, x^*) + H(T_1(x_{n_k}), T_1x^*)d(y_{n_k}, y^*)) + \langle \overrightarrow{f(y^*)x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{g(x^*)y^*}, \overrightarrow{y_{n_k}y^*}\rangle \\
&\le 2kd(y_{n_k}, y^*)d(x_{n_k}, x^*) + \langle \overrightarrow{f(y^*)x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{g(x^*)y^*}, \overrightarrow{y_{n_k}y^*}\rangle \\
&\le k(d^2(x_{n_k}, x^*) + d^2(y_{n_k}, y^*)) + \langle \overrightarrow{f(y^*)x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{g(x^*)y^*}, \overrightarrow{y_{n_k}y^*}\rangle.
\end{aligned}$$

Thus

$$d^2(x_{n_k}, x^*) + d^2(y_{n_k}, y^*) \le \frac{1}{1-k}[\langle \overrightarrow{f(y^*)x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{g(x^*)y^*}, \overrightarrow{y_{n_k}y^*}\rangle]. \tag{3.3}$$

Since $\triangle\text{−}\lim_{k\to\infty} x_{n_k} = x^*, \triangle\text{−}\lim_{k\to\infty} y_{n_k} = y^*$, by Lemma 2.8, we have

$$\limsup_{k\to\infty}[\langle \overrightarrow{f(y^*)x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \langle \overrightarrow{g(x^*)y^*}, \overrightarrow{y_{n_k}y^*}\rangle] \le \limsup_{k\to\infty}\langle \overrightarrow{f(y^*)x^*}, \overrightarrow{x_{n_k}x^*}\rangle + \limsup_{k\to\infty}\langle \overrightarrow{g(x^*)y^*}, \overrightarrow{y_{n_k}y^*}\rangle \le 0.$$

It follows from (3.3) that $d^2(x_{n_k}, x^*) + d^2(y_{n_k}, y^*) \to 0$. Hence $x_{n_k} \to x^*$ and $y_{n_k} \to y^*$.

Next, we show that $(x^*, y^*) \in F(T_1) \times F(T_2)$, which solves HOP (1.4), where $x^* = P_{F(T_1)} f(y^*), y^* = P_{F(T_2)} g(x^*)$.

In fact, since T_1 satisfies the endpoint condition, we have

$$\begin{aligned}
\operatorname{dist}(f(u_{n_k}), T_1 x_{n_k}) &\le d(f(u_{n_k}), f(y^*)) + d(f(y^*), x^*) + \operatorname{dist}(x^*, T_1 x_{n_k}) \\
&\le k d(u_{n_k}, y^*) + d(f(y^*), x^*) + \operatorname{dist}(x^*, T_1 x_{n_k}) \\
&\le k \operatorname{dist}(u_{n_k}, T_2 y^*) + d(f(y^*), x^*) + H(T_1 x^*, T_1 x_{n_k}) \\
&\le k H(T_2 y_{n_k}, T_2 y^*) + d(f(y^*), x^*) + H(T_1 x^*, T_1 x_{n_k}) \\
&\le k d(y_{n_k}, y^*) + d(x_{n_k}, x^*) + d(f(y^*), x^*),
\end{aligned}$$

and

$$\begin{aligned}
d(f(y^*), x^*) &= \operatorname{dist}(f(y^*), T_1 x^*) \\
&\le d(f(y^*), f(u_{n_k})) + \operatorname{dist}(f(u_{n_k}), T_1 x_{n_k}) + H(T_1 x_{n_k}, T_1 x^*) \\
&\le k d(y^*, u_{n_k}) + \operatorname{dist}(f(u_{n_k}), T_1 x_{n_k}) + d(x_{n_k}, x^*) \\
&\le k \operatorname{dist}(u_{n_k}, T_2 y^*) + \operatorname{dist}(f(u_{n_k}), T_1 x_{n_k}) + d(x_{n_k}, x^*) \\
&\le k H(T_2 y_{n_k}, T_2 y^*) + \operatorname{dist}(f(u_{n_k}), T_1 x_{n_k}) + d(x_{n_k}, x^*) \\
&\le k d(y_{n_k}, y^*) + d(x_{n_k}, x^*) + \operatorname{dist}(f(u_{n_k}), T_1 x_{n_k}).
\end{aligned}$$

Thus

$$|\operatorname{dist}(f(u_{n_k}), T_1 x_{n_k}) - d(f(y^*), x^*)| \le d(x_{n_k}, x^*) + k d(y_{n_k}, y^*) \to 0 \text{ (as } n_k \to \infty). \tag{3.4}$$

It follows from (2.1) that for any $(p, q) \in F(T_1) \times F(T_2)$, we have

$$\begin{aligned}
d^2(x_{n_k}, p) &= d^2(s_{n_k} f(u_{n_k}) \oplus (1 - s_{n_k}) z_{n_k}, p) \\
&\le s_{n_k} d^2(f(u_{n_k}), p) + (1 - s_{n_k}) d^2(z_{n_k}, p) - s_{n_k}(1 - s_{n_k}) d^2(f(u_{n_k}), z_{n_k}) \\
&\le s_{n_k} d^2(f(u_{n_k}), p) + (1 - s_{n_k}) H^2(T_1 x_{n_k}, T_1 p) - s_{n_k}(1 - s_{n_k}) d^2(f(u_{n_k}), z_{n_k}) \\
&\le s_{n_k} d^2(f(u_{n_k}), p) + (1 - s_{n_k}) d^2(x_{n_k}, p) - s_{n_k}(1 - s_{n_k}) d^2(f(u_{n_k}), z_{n_k}).
\end{aligned}$$

This implies that

$$\begin{aligned}
d^2(x_{n_k}, p) &\le d^2(f(u_{n_k}), p) - (1 - s_{n_k}) d^2(f(u_{n_k}), z_{n_k}) \\
&\le d^2(f(u_{n_k}), p) - (1 - s_{n_k}) [\operatorname{dist}(f(u_{n_k}), T_1 x_{n_k})]^2.
\end{aligned}$$

Taking $k \to \infty$, this together with (3.4) shows that

$$d^2(x^*, p) \le d^2(f(y^*), p) - d^2(f(y^*), x^*).$$

Hence

$$0 \le \frac{1}{2}[d^2(x^*, x^*) + d^2(f(y^*), p) - d^2(x^*, p) - d^2(f(y^*), x^*)] = \langle \overrightarrow{x^* f(y^*)}, \overrightarrow{p x^*} \rangle, (\forall p \in F(T_1)).$$

It is similar to prove that

$$\langle \overrightarrow{y^* g(x^*)} \rangle, \overrightarrow{q y^*} \rangle \ge 0, (\forall q \in F(T_2)).$$

That is, (x^*, y^*) solves inequalities (1.4). By Lemma 2.5, $x^* = P_{F(T_1)} f(y^*)$ and $y^* = P_{F(T_2)} g(x^*)$ and this completes the proof. □

Now, we define an explicit iterative sequence for multivalued nonexpansive mappings.

Let $T_1, T_2 : E \to K(E)$ be two nonexpansive mappings, $f, g : E \to E$ be two contractions, and $\{\alpha_n\}$ be a sequence in (0,1). For given $x_1, y_1 \in E$ and $z_1 \in T_1 x_1, u_1 \in T_2 y_1$, let

$$\begin{cases} x_2 = \alpha_1 f(u_1) \oplus (1 - \alpha_1) z_1, \\ y_2 = \alpha_1 g(z_1) \oplus (1 - \alpha_1) u_1. \end{cases}$$

By the definition of Hausdorff distance and the nonexpansiveness of T_1, T_2, we can choose $z_2 \in T_1x_2, u_2 \in T_2y_2$ such that

$$d(z_1, z_2) \le d(x_1, x_2), \;\; d(u_1, u_2) \le d(y_1, y_2).$$

Inductively, we have

$$\begin{cases} x_{n+1} = \alpha_n f(u_n) \oplus (1-\alpha_n)z_n, \;\; u_n \in T_2y_n, \\ y_{n+1} = \alpha_n g(z_n) \oplus (1-\alpha_n)u_n, \;\; z_n \in T_1x_n, \\ d(z_n, z_{n+1}) \le d(x_n, x_{n+1}), \;\; d(u_n, u_{n+1}) \le d(y_n, y_{n+1}), \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.5}$$

Theorem 3.3. *Let E be a closed convex subset of a complete CAT(0) space X, and let $T_1, T_2 : E \to K(E)$ be two nonexpansive mappings satisfying the endpoint condition. Let $f, g : E \to E$ be two contractions with contractive constant $k \in [0, \frac{1}{2})$. Let $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2-k})$ satisfying*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Then the sequence $\{(x_n, y_n)\}$ defined by (3.5) *converges strongly to (x^*, y^*), where $x^* = P_{F(T_1)}f(y^*), y^* = P_{F(T_2)}g(x^*)$, which solves HOP* (1.4)*.*

Proof. We divide the proof into three steps.

Step 1. We show that $\{x_n\}, \{y_n\}, \{u_n\}, \{z_n\}$ and $\{f(u_n)\}, \{g(z_n)\}$ are bounded sequences. Let $(p, q) \in F(T_1) \times F(T_2)$. In fact, by Lemma 2.2 (i), we have

$$\begin{aligned} &d(x_{n+1}, p) + d(y_{n+1}, q) \\ &\quad \le \alpha_n d(f(u_n), p) + (1-\alpha_n)d(z_n, p) + \alpha_n d(g(z_n), q) + (1-\alpha_n)d(u_n, q) \\ &\quad \le \alpha_n(d(f(u_n), f(q)) + d(f(q), p)) + (1-\alpha_n)H(T_1x_n, T_1p) \\ &\qquad + \alpha_n(d(g(z_n), g(p)) + d(g(p), q)) + (1-\alpha_n)H(T_2y_n, T_2q) \\ &\quad \le \alpha_n(kd(u_n, q) + d(f(q), p)) + (1-\alpha_n)d(x_n, p) + \alpha_n(kd(z_n, p) + d(g(p), q)) + (1-\alpha_n)d(y_n, q) \\ &\quad \le \alpha_n(kH(T_2y_n, T_2q) + d(f(q), p)) + (1-\alpha_n)d(x_n, p) \\ &\qquad + \alpha_n(kH(T_1x_n, T_1p) + d(g(p), q)) + (1-\alpha_n)d(y_n, q) \\ &\quad \le (\alpha_n k + (1-\alpha_n))[d(y_n, q) + d(x_n, p)] + \alpha_n(d(f(q), p) + d(g(p), q)) \\ &\quad = (1-\alpha_n(1-k))(d(x_n, p) + d(y_n, q)) + \alpha_n(1-k)\frac{d(f(q), p) + d(g(p), q)}{1-k} \\ &\quad \le \max\{d(x_n, p) + d(y_n, q), \frac{d(f(q), p) + d(g(p), q)}{1-k}\}. \end{aligned}$$

By the induction, we can prove that

$$d(x_n, p) + d(y_n, q) \le \max\{d(x_1, p) + d(y_1, q), \frac{d(f(q), p) + d(g(p), q)}{1-k}\}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{y_n\}$ are bounded, so are $\{u_n\}, \{z_n\}, \{f(u_n)\}$ and $\{g(z_n)\}$.

Step 2. We show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ and $\lim_{n\to\infty} d(y_{n+1}, y_n) = 0$.

In fact, it follows from (3.5) that

$$\begin{aligned} d(x_{n+1}, x_n) + d(y_{n+1}, y_n) =& d(\alpha_n f(u_n) \oplus (1-\alpha_n)z_n, \alpha_{n-1}f(u_{n-1}) \oplus (1-\alpha_{n-1})z_{n-1}) \\ &+ d(\alpha_n g(z_n) \oplus (1-\alpha_n)u_n, \alpha_{n-1}g(z_{n-1}) \oplus (1-\alpha_{n-1})u_{n-1}) \end{aligned}$$

$$\begin{aligned}&\le d(\alpha_n f(u_n)\oplus(1-\alpha_n)z_n,\alpha_n f(u_{n-1})\oplus(1-\alpha_n)z_{n-1})\\&\quad+d(\alpha_n f(u_{n-1})\oplus(1-\alpha_n)z_{n-1},\alpha_{n-1}f(u_{n-1})\oplus(1-\alpha_{n-1})z_{n-1})\\&\quad+d(\alpha_n g(z_n)\oplus(1-\alpha_n)u_n,\alpha_n g(z_{n-1})\oplus(1-\alpha_n)u_{n-1})\\&\quad+d(\alpha_n g(z_{n-1})\oplus(1-\alpha_n)u_{n-1},\alpha_{n-1}g(z_{n-1})\oplus(1-\alpha_{n-1})u_{n-1})\\&\le(1-\alpha_n)d(z_n,z_{n-1})+\alpha_n d(f(u_n),f(u_{n-1}))+|\alpha_n-\alpha_{n-1}|d(f(u_{n-1}),z_{n-1})\\&\quad+(1-\alpha_n)d(u_n,u_{n-1})+\alpha_n d(g(z_n),g(z_{n-1}))+|\alpha_n-\alpha_{n-1}|d(g(z_{n-1}),u_{n-1})\\&\le(1-\alpha_n(1-k))[d(x_n,x_{n-1})+d(y_n,y_{n-1})]\\&\quad+|\alpha_n-\alpha_{n+1}|[d(f(u_{n-1}),z_{n-1})+d(g(z_{n-1}),u_{n-1})].\end{aligned}$$

Hence we have

$$c_{n+1}\le(1-\gamma_n)c_n+\gamma_n\delta_n,$$

where $c_n=d(x_n,x_{n-1})+d(y_n,y_{n-1}),\gamma_n=(1-k)\alpha_n$ and

$$\delta_n=\frac{1}{1-k}|1-\frac{\alpha_{n-1}}{\alpha_n}|[d(f(u_{n-1}),z_{n-1})+d(g(z_{n-1}),u_{n-1})].$$

By conditions (C2) and (C3) and Lemma 2.10, we obtain

$$\lim_{n\to\infty}d(x_{n+1},x_n)+d(y_{n+1},y_n)=0$$

and thus $\lim_{n\to\infty}d(x_{n+1},x_n)=0$ and $\lim_{n\to\infty}d(y_{n+1},y_n)=0$.

Step 3. We show that $\{(x_n,y_n)\}$ converges strongly to $(x^*,y^*)\in F(T_1)\times F(T_2)$, where $x^*=P_{F(T_1)}f(y^*)$, $y^*=P_{F(T_2)}g(x^*)$.

Indeed, for each $s\in(0,1)$, let $\{x_s\}$ and $\{y_s\}$ be defined by (3.2). By Theorem 3.2, we have $x_s\to x^*$ and $y_s\to y^*$ as $s\to 0$ such that $x^*=P_{F(T_1)}f(y^*),y^*=P_{F(T_2)}g(x^*)$, which solves the variational inequalities (1.4). We note that

$$\begin{aligned}\mathrm{dist}(x_n,T_1x_n)+\mathrm{dist}(y_n,T_2y_n)&\le d(x_n,z_n)+d(y_n,u_n)\\&\le d(x_n,x_{n+1})+d(x_{n+1},z_n)+d(y_n,y_{n+1})+d(y_{n+1},u_n)\\&\le d(x_n,x_{n+1})+\alpha_n d(f(u_n),z_n)+d(y_n,y_{n+1})+\alpha_n d(g(z_n),u_n)\\&\to 0\quad\text{as }n\to\infty.\end{aligned}$$

This implies that

$$\mathrm{dist}(x_n,T_1x_n)\to 0,\quad \mathrm{dist}(y_n,T_2y_n)\to 0\ (\text{as }\ n\to\infty).$$

Since $\{x_n\}$ is a bounded sequence in E and μ is a Banach limit, if there exist some $\eta,\gamma\in\mathbb{R}$ such that

$$\mu_n(d^2(f(y^*),x_n))<\eta<\gamma<d^2(f(y^*),x^*),$$

then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d^2(f(y^*),x_{n_k})<\gamma \text{ for all } k\in\mathbb{N}. \tag{3.6}$$

Indeed, suppose to the contrary that

$$d^2(f(y^*),x_n)\ge\gamma\ \text{ for all large } n,$$

which implies that $\mu_n d^2(f(y^*),x_n)\ge\gamma>\eta$, a contradiction, and therefore (3.6) holds. By Lemmas 2.3 and 2.12, we assume that $\triangle-\lim_{n_k\to\infty}x_{n_k}=p\in F(T_1)$. Then by (3.6) and Lemma 2.4, p is contained in the closed ball centered at $f(y^*)$ of radius $\sqrt{\gamma}$. This contradicts the fact that x^* is the unique nearest point of $f(y^*)$ in $F(T_1)$, hence we have

$$\mu_n(d^2(f(y^*),x^*) - d^2(f(y^*),x_n)) \le 0 \ \ \forall \text{ Banach limits } \ \mu.$$

Similarly we can also prove that

$$\mu_n(d^2(g(x^*),y^*) - d^2(g(x^*),y_n)) \le 0 \ \ \forall \text{ Banach limits } \ \mu.$$

Moreover, since $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$, $\lim_{n\to\infty} d(y_{n+1},y_n) = 0$, we have

$$\limsup_{n\to\infty}[d^2(f(y^*),x^*) - d^2(f(y^*),x_{n+1}) - (d^2(f(y^*),x^*) - d^2(f(y^*),x_n))] = 0,$$
$$\limsup_{n\to\infty}[d^2(g(x^*),y^*) - d^2(g(x^*),y_{n+1}) - (d^2(g(x^*),y^*) - d^2(g(x^*),y_n))] = 0.$$

Therefore it follows from Lemma 2.9 that

$$\limsup_{n\to\infty}(d^2(f(y^*),x^*) - d^2(f(y^*),x_n)) \le 0, \ \ \limsup_{n\to\infty}(d^2(g(x^*),y^*) - d^2(g(x^*),y_n)) \le 0. \tag{3.7}$$

For each $n \in \mathbb{N}$, we set $w_n = \alpha_n x^* \oplus (1-\alpha_n)z_n$ and $v_n = \alpha_n y^* \oplus (1-\alpha_n)u_n$. It follows from Lemmas 2.6 and 2.7 that

$$\begin{aligned}
&d^2(x_{n+1},x^*) + d^2(y_{n+1},y^*)\\
&\le d^2(w_n,x^*) + d^2(v_n,y^*) + 2\langle\overrightarrow{x_{n+1}w_n},\overrightarrow{x_{n+1}x^*}\rangle + 2\langle\overrightarrow{y_{n+1}v_n},\overrightarrow{y_{n+1}y^*}\rangle\\
&\le (1-\alpha_n)^2[d^2(z_n,x^*) + d^2(u_n,y^*)] + 2[\alpha_n\langle\overrightarrow{f(u_n)w_n},\overrightarrow{x_{n+1}x^*}\rangle + (1-\alpha_n)\langle\overrightarrow{z_nw_n},\overrightarrow{x_{n+1}x^*}\rangle]\\
&\quad + 2[\alpha_n\langle\overrightarrow{g(z_n)v_n},\overrightarrow{y_{n+1}y^*}\rangle + (1-\alpha_n)\langle\overrightarrow{u_nv_n},\overrightarrow{y_{n+1}y^*}\rangle]\\
&\le (1-\alpha_n)^2[H^2(T_1x_n,T_1x^*) + H^2(T_2y_n,T_2y^*)]\\
&\quad + 2[\alpha_n^2\langle\overrightarrow{f(u_n)x^*},\overrightarrow{x_{n+1}x^*}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{f(u_n)z_n},\overrightarrow{x_{n+1}x^*}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{z_nx^*},\overrightarrow{x_{n+1}x^*}\rangle]\\
&\quad + 2[\alpha_n^2\langle\overrightarrow{g(z_n)y^*},\overrightarrow{y_{n+1}y^*}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{g(z_n)u_n},\overrightarrow{y_{n+1}y^*}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{u_ny^*},\overrightarrow{y_{n+1}y^*}\rangle]\\
&\le (1-\alpha_n)^2[d^2(x_n,x^*) + d^2(y_n,y^*)] + 2[\alpha_n^2\langle\overrightarrow{f(u_n)x^*},\overrightarrow{x_{n+1}x^*}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{f(u_n)x^*},\overrightarrow{x_{n+1}x^*}\rangle]\\
&\quad + 2[\alpha_n^2\langle\overrightarrow{g(z_n)y^*},\overrightarrow{y_{n+1}y^*}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{g(z_n)y^*},\overrightarrow{y_{n+1}y^*}\rangle]\\
&= (1-\alpha_n)^2[d^2(x_n,x^*) + d^2(y_n,y^*)] + 2\alpha_n\langle\overrightarrow{f(u_n)x^*},\overrightarrow{x_{n+1}x^*}\rangle + 2\alpha_n\langle\overrightarrow{g(z_n)y^*},\overrightarrow{y_{n+1}y^*}\rangle\\
&= (1-\alpha_n)^2[d^2(x_n,x^*) + d^2(y_n,y^*)] + 2\alpha_n\langle\overrightarrow{f(u_n)f(y^*)},\overrightarrow{x_{n+1}x^*}\rangle + 2\alpha_n\langle\overrightarrow{f(y^*)x^*},\overrightarrow{x_{n+1}x^*}\rangle\\
&\quad + 2\alpha_n\langle\overrightarrow{g(z_n)g(x^*)},\overrightarrow{y_{n+1}y^*}\rangle + 2\alpha_n\langle\overrightarrow{g(x^*)y^*},\overrightarrow{y_{n+1}y^*}\rangle\\
&\le (1-\alpha_n)^2[d^2(x_n,x^*) + d^2(y_n,y^*)] + 2k\alpha_n d(u_n,y^*)d(x_{n+1},x^*) + 2\alpha_n\langle\overrightarrow{f(y^*)x^*},\overrightarrow{x_{n+1}x^*}\rangle\\
&\quad + 2k\alpha_n d(z_n,x^*)d(y_{n+1},y^*) + 2\alpha_n\langle\overrightarrow{g(x^*)y^*},\overrightarrow{y_{n+1}y^*}\rangle\\
&\le (1-\alpha_n)^2[d^2(x_n,x^*) + d^2(y_n,y^*)] + 2k\alpha_n H(T_2y_n,T_2y^*)d(x_{n+1},x^*) + 2\alpha_n\langle\overrightarrow{f(y^*)x^*},\overrightarrow{x_{n+1}x^*}\rangle\\
&\quad + 2k\alpha_n d(T_1x_n,T_1x^*)d(y_{n+1},y^*) + 2\alpha_n\langle\overrightarrow{g(x^*)y^*},\overrightarrow{y_{n+1}y^*}\rangle\\
&\le (1-\alpha_n)^2[d^2(x_n,x^*) + d^2(y_n,y^*)] + 2k\alpha_n d(y_n,y^*)d(x_{n+1},x^*) + 2\alpha_n\langle\overrightarrow{f(y^*)x^*},\overrightarrow{x_{n+1}x^*}\rangle\\
&\quad + 2k\alpha_n d(x_n,x^*)d(y_{n+1},y^*) + 2\alpha_n\langle\overrightarrow{g(x^*)y^*},\overrightarrow{y_{n+1}y^*}\rangle\\
&\le (1-\alpha_n)^2[d^2(x_n,x^*) + d^2(y_n,y^*)] + k\alpha_n[d^2(y_n,y^*) + d^2(x_{n+1},x^*)]\\
&\quad + \alpha_n[d^2(f(y^*),x^*) + d^2(x_{n+1},x^*) - d^2(f(y^*),x_{n+1})] + k\alpha_n[d^2(x_n,x^*) + d^2(y_{n+1},y^*)]\\
&\quad + \alpha_n[d^2(g(x^*),y^*) + d^2(y_{n+1},y^*) - d^2(g(x^*),y_{n+1})].
\end{aligned}$$

After simplifying, it yields that

$$\begin{aligned}
&d^2(x_{n+1},x^*)+d^2(y_{n+1},y^*)\\
&\quad\le\frac{1-(2-k)\alpha_n+\alpha_n^2}{1-(1+k)\alpha_n}[d^2(x_n,x^*)+d^2(y_n,y^*)]+\frac{\alpha_n}{1-(1+k)\alpha_n}[d^2(f(y^*),x^*)-d^2(f(y^*),x_{n+1})]\\
&\qquad+\frac{\alpha_n}{1-(1+k)\alpha_n}[d^2(g(x^*),y^*)-d^2(g(x^*),y_{n+1})]\\
&\quad\le\frac{1-(2-k)\alpha_n}{1-(1+k)\alpha_n}[d^2(x_n,x^*)+d^2(y_n,y^*)]+\frac{\alpha_n^2}{1-(1+k)\alpha_n}M\\
&\qquad+\frac{\alpha_n}{1-(1+k)\alpha_n}[d^2(f(y^*),x^*)-d^2(f(y^*),x_{n+1})+d^2(g(x^*),y^*)-d^2(g(x^*),y_{n+1})],
\end{aligned}$$

where $M=\sup_{n\in\mathbb{N}}\{d(x_n,x^*)+d(y_n,y^*)\}<\infty$. It follows that

$$d^2(x_{n+1},x^*)+d^2(y_{n+1},y^*)\le(1-\gamma_n)[d^2(x_n,x^*)+d^2(y_n,y^*)]+\gamma_n\eta_n, \tag{3.8}$$

where

$$\gamma_n=\frac{(1-2k)\alpha_n}{1-(1+k)\alpha_n}$$

and

$$\eta_n=\frac{\alpha_n}{(1-2k)}M+\frac{1}{(1-2k)}[d^2(f(y^*),x^*)-d^2(f(y^*),x_{n+1})+d^2(g(x^*),y^*)-d^2(g(x^*),y_{n+1})].$$

Since $\alpha_n\in(0,\frac{1}{2-k})$ and $k\in[0,\frac{1}{2})$, we have $\gamma_n\in(0,1)$. By (C1) and (3.7), $\limsup_n\eta_n\le 0$. Applying Lemma 2.10 to the inequality(3.8), we have $d^2(x_n,x^*)+d^2(y_n,y^*)\to 0$. Hence $x_n\to x^*$ and $y_n\to y^*$ as $n\to\infty$. This completes the proof of Theorem 3.3. □

4. ℝ-Trees

Definition 4.1. An ℝ-tree is a geodesic space X such that:

(i) there is a unique geodesic segment $[x,y]$ joining each pair of points $x,y\in X$;

(ii) if $[y,x]\cap[x,z]=\{x\}$, then $[y,x]\cup[x,z]=[y,z]$.

By (i) and (ii) we have

(iii) if $u,v,w\in X$, then $[u,v]\cap[u,w]=[u,z]$ for some $z\in X$.

It is well-known that every ℝ-tree is a CAT(0) space which does not contain the Euclidean plan. To avoid the endpoint condition, we prefer to work on ℝ-trees. Although an ℝ-tree is not strong enough to make all nonexpansive mappings having the endpoint condition (see Example 5.3 in [23]), but it is strong enough to make our theorems hold without this condition.

Let E be closed convex subset of a complete ℝ-tree (X,d) and $T_1,T_2:E\to BCC(E)$ be two multivalued mappings. Then, by Theorem 4.1 of [2], there exists a single-valued mapping $t_i:E\to E$ $(i=1,2)$ such that $t_i(x)\in T_i(x)$ and

$$d(t_i(x),t_i(y))\le H(T_i(x),T_i(y))\ \text{ for all } x,y\in E. \tag{4.1}$$

In this case, we call t_i a nonexpansive selection of T_i $(i=1,2)$.

Let f,g be two contractions on E, and let $T_1,T_2:E\to BCC(E)$ be two multivalued mappings and fix $x_1,y_1\in E$. We define a sequence $\{(x_n,y_n)\}$ in $E\times E$ by

$$\begin{cases} x_{n+1}=\alpha_n f(u_n)\oplus(1-\alpha_n)z_n,\\ y_{n+1}=\alpha_n g(z_n)\oplus(1-\alpha_n)u_n,\end{cases} \tag{4.2}$$

where $u_n=t_2(y_n)\in T_2(y_n)$, $z_n=t_1(x_n)\in T_1(x_n)$ for all $n\in\mathbb{N}$.

Theorem 4.2. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}-$tree (X, d), and let $T_1, T_2 : E \to BCC(E)$ be two nonexpansive mappings with $F(T_i) \neq \emptyset$ $(i = 1, 2)$. Let f, g be two contractions on E with contractive constant $k \in [0, \frac{1}{2})$ and $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2-k})$ satisfying:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Then the sequence $\{(x_n, y_n)\}$ defined by (4.2) converges strongly to (x^*, y^*), where $x^* = P_{F(T_1)} f(y^*), y^* = P_{F(T_2)} g(x^*)$, which solves HOP (1.4).

Proof. By Theorem 4.2 of [2] (see also Theorem 2 of [14]), $F(t_i) = F(T_i), i = 1, 2$, and the set $F(t_i)$ is closed and convex by Proposition 1 of [19] and t_i $(i = 1, 2)$ are nonexpansive by (4.1). The conclusion follows from Theorem 4.2 immediately. □

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