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Proximal point algorithms involving fixed point of nonspreading-type multivalued mappings in Hilbert spaces

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Abstract

In this paper, a new modified proximal point algorithm involving fixed point of nonspreading-type multivalued mappings in Hilbert spaces is proposed. Under suitable conditions, some weak convergence and strong convergence to a common element of the set of minimizers of a convex function and the set of fixed points of the nonspreading-type multivalued mappings in Hilbert space are proved. The presented results in the paper are new.

Keywords: Convex minimization problem, resolvent identity, proximal point algorithm, weak and strong convergence theorem, nonspreading-type multivalued mapping.
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1. Introduction

Throughout this paper we always assume that H is a real Hilbert space and C is a nonempty closed and convex subsets of H. In the sequel we denote by $CB(C)$ and $K(C)$ the families of nonempty closed bounded subsets and nonempty compact subsets of C, respectively. The Hausdorff metric on $CB(C)$ is defined by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\},\ A,B \in CB(C),$$

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where $d(x, B) = \inf_{b\in B} d(x, b)$.

In what follows, we denote by $\mathrm{Fix}(T)$ the fixed point set of a mapping T. And write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

Recall that a single-valued mapping $T : C \to C$ is said to be nonexpansive if

$$||Tx - Ty|| \le ||x - y||, \ \ \forall x, y \in C.$$

A multivalued mapping $T : C \to CB(C)$ is said to be nonexpansive if

$$H(Tx, Ty) \le ||x - y||, \ \ \forall x, y \in C,$$

and $T : C \to CB(C)$ is said to be *quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and

$$H(Tx, Tp) \le ||x - p||, \ \ \forall x \in C, \ p \in \mathrm{Fix}(T).$$

Recall that a single-valued mapping $T : C \to C$ is said to be *nonspreading mappings* [12] if

$$2||Tx - Ty||^2 \le ||x - Ty||^2 + ||y - Tx||^2, \ \ \forall x, y \in C.$$

It is easy to prove that $T : C \to C$ is nonspreading if and only if

$$||Tx - Ty||^2 \le ||x - y||^2 + 2\langle x - Tx, \ y - Ty\rangle, \ \ \forall x, y \in C.$$

A mapping $T : C \to CB(C)$ is said to be *nonspreading-type multi-valued mapping* [6] if

$$2H(Tx, Ty)^2 \le d(x, Ty)^2 + d(y, Tx)^2, \ \ \forall x, y \in C.$$

It is easy to see that, if T is a nonspreading-type multi-valued mapping, and $\mathrm{Fix}(T) \neq \emptyset$, then T is a *quasi-nonexpansive multi-valued mapping*, i.e.,

$$H(Tx, Tp) \le ||x - p|| \ \ \forall x \in C \text{ and } p \in \mathrm{Fix}(T). \tag{1.1}$$

Indeed, for all $x \in C$ and $p \in \mathrm{Fix}(T)$, we have

$$\begin{aligned} 2H(Tx, Tp)^2 &\le d(p, Tx)^2 + d(x, Tp)^2 \\ &\le H(Tx, Tp)^2 + ||x - p||^2. \end{aligned}$$

This implies that

$$H(Tx, Tp) \le ||x - p||.$$

Example 1.1 (Example of nonspreading-type multi-valued mapping, [6])**.** Let $C = [-3, 0]$ with the usual norm. Define a multivalued mapping $T : C \to CB(C)$ by

$$Tx = \begin{cases} \{0\}, \ if \ x \in [-2, 0]; \\ [-\exp\{x + 2\}, \ 0], \text{ if } x \notin [-2, 0]. \end{cases}$$

It is easy to prove that T is a nonspreading-type multi-valued mapping but it is not a multi-valued nonexpansive mapping.

Recall that a multivalued mapping $T : C \to CB(C)$ is said to be *demi-closed* at 0, if $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ imply $x \in Tx$.

Let $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. One of the major problems in optimization in Hilbert space H is to find $x \in H$ such that

$$f(x) = \min_{y\in H} f(y).$$

We denote by $\text{argmin}_{y\in H} f(y)$ the set of minimizers of f in H.

A successful and powerful tool for solving this problem is the well-known proximal point algorithm (shortly, the PPA) which was initiated by Martinet [14] in 1970. In 1976, Rockafellar [15] generally studied, by the PPA, the convergence to a solution of the convex minimization problem in the framework of Hilbert spaces.

Indeed, let f be a proper, convex, and lower semi-continuous function on a Hilbert space H which attains its minimum. The PPA is defined by

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \text{argmin}_{y\in H}(f(y) + \frac{1}{2\lambda_n}||y - x_n||^2) \quad \forall n \geq 1, \end{cases}$$

where $\lambda_n > 0$ for all $n \geq 1$. It was proved that the sequence $\{x_n\}$ converges weakly to a minimizer of f provided $\Sigma_{n=1}^{\infty}\lambda_n = \infty$.

However, as shown by Güler [8], the PPA does not necessarily converge strongly in general. In 2000, Kamimura-Takahashi [11] combined the PPA with Halpern's algorithm [9] so that the strong convergence is guaranteed.

In the recent years, the problem of finding a common element of the set of solutions of various convex minimization problems and the set of fixed points for a single-valued mapping in the framework of Hilbert spaces and Banach spaces have been intensively studied by many authors, for instance, see [4, 5, 7, 13] and the references therein.

The purpose of this paper is to introduce and study the following modified proximal point algorithm involving fixed point for nonspreading-type multivalued mappings in Hilbert spaces.

$$\begin{cases} u_n = \text{argmin}_{y\in C}[f(y) + \frac{1}{2\lambda_n}||y - x_n||^2], \\ y_n = (1 - \beta_n)x_n + \beta_n w_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n v_n, \end{cases} \quad \forall n \geq 1,$$

where $T : C \to CB(C)$ is a nonspreading-type multivalued mapping, $w_n \in Tu_n$ and $v_n \in Ty_n$ for all $n \geq 1$. Under suitable conditions, some weak convergence and strong convergence to a common element of the set of minimizers of a convex function and the set of fixed points of the nonspreading-type multivalued mappings in Hilbert space are proved. The presented results in the paper are new.

2. Preliminaries

In order to prove the main results of the paper, we need the following notations and lemmas.

Let $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in H by

$$J_\lambda(x) = \text{argmin}_{y\in H}[f(y) + \frac{1}{2\lambda}||y - x||^2] \;\; \forall x \in H. \tag{2.1}$$

It was shown in [3] that the fixed point set $\text{Fix}(J_\lambda)$ of the resolvent associated of f coincides with the set $\text{argmin}_{y\in H} f(y)$ of minimizers of f.

Lemma 2.1 ([10])**.** *Let $f : H \to (-\infty, \infty]$ be proper convex and lower semi-continuous. For any $\lambda > 0$, the resolvent J_λ of f is nonexpansive.*

Lemma 2.2 (Sub-differential inequality, [2])**.** *Let $f : H \to (-\infty, \infty]$ be proper convex and lower semi-continuous. Then, for all $x, y \in H$ and $\lambda > 0$, the following sub-differential inequality holds:*

$$\frac{1}{2\lambda}||J_\lambda x - y||^2 - \frac{1}{2\lambda}||x - y||^2 + \frac{1}{2\lambda}||x - J_\lambda x||^2 + f(J_\lambda x) \leq f(y). \tag{2.2}$$

Lemma 2.3 ([6]). *Let C be a nonempty closed and convex subset of H.*

(I) *If $T : C \to CB(C)$ is a nonspreading-type multivalued mapping and $\mathrm{Fix}(T) \neq \emptyset$, then the following conclusions hold.*
   (i) $\mathrm{Fix}(T)$ *is closed;*
   (ii) *if, in addition, T satisfies the condition: $Tp = \{p\}$ for all $p \in \mathrm{Fix}(T)$, then $\mathrm{Fix}(T)$ is convex.*

(II) *Let $T : C \to K(C)$ be a nonspreading-type multivalued mapping,*
   (iii) *if $x, y \in C$ and $u \in Tx$, then there exists $v \in Ty$ such that*

$$H(Tx, Ty)^2 \leq ||x - y||^2 + 2\langle x - u,\ y - v\rangle;$$

   (iv) *(demi-closed principle) if $\{x_n\}$ is a bounded sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} ||x_n - y_n|| = 0$ for some $y_n \in Tx_n$, then $p \in Tp$.*

Lemma 2.4 (The resolvent identity, [10]). *Let $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then the following identity holds:*

$$J_\lambda x = J_\mu\left(\frac{\lambda - \mu}{\lambda} J_\lambda x + \frac{\mu}{\lambda} x\right) \quad \forall x \in H \text{ and } \lambda > \mu > 0.$$

Definition 2.5. A Banach space X is said to satisfy *Opial condition*, if $x_n \rightharpoonup z$ (as $n \to \infty$) and $z \neq y$ imply that

$$\limsup_{n\to\infty} ||x_n - z|| < \limsup_{n\to\infty} ||x_n - y||.$$

It is well-known that each Hilbert space H satisfies the Opial condition.

3. Weak convergence theorems

We are now in a position to give the following main result.

Theorem 3.1. *Let $f : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function, and $T : C \to K(C)$ be a nonspreading-type multivalued mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$ with $0 < a \leq \alpha_n,\ \beta_n < b < 1$ for all $n \geq 1$. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some λ. For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by the following manner:*

$$\begin{cases} u_n = \mathrm{argmin}_{y\in C}[f(y) + \frac{1}{2\lambda_n}||y - x_n||^2], \\ y_n = (1 - \beta_n)x_n + \beta_n w_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n v_n, \end{cases} \quad \forall n \geq 1, \tag{3.1}$$

where $w_n \in Tu_n$ and $v_n \in Ty_n$ for all $n \geq 1$. If $\Omega := \mathrm{Fix}(T) \bigcap \mathrm{argmin}_{y\in C} f(y) \neq \emptyset$, then $\{x_n\}$ converges weakly to a point $x^* \in \Omega$ which is a minimizer of f in C as well as it is also a fixed point of T in C.

Proof. Let $q \in \Omega$. Then $q = Tq$ and $f(q) \leq f(y)$ for all $y \in C$. This implies that

$$f(q) + \frac{1}{2\lambda_n}||q - q||^2 \leq f(y) + \frac{1}{2\lambda_n}||y - q||^2, \quad \forall y \in C,$$

and hence $q = J_{\lambda_n} q$ for all $n \geq 1$, where J_{λ_n} is the Moreau-Yosida resolvent of f in H defined by (2.1).

(I) First, we prove that $\lim_{n\to\infty} ||x_n - q||$ exists for all $q \in \Omega$.

Indeed, since $u_n = J_{\lambda_n} x_n$, by Lemma 2.1, J_{λ_n} is nonexpansive. Hence we have

$$||u_n - q|| = ||J_{\lambda_n} x_n - J_{\lambda_n} q|| \leq ||x_n - q||. \tag{3.2}$$

It follows from (3.1), (3.2), and (1.1) that

$$\begin{aligned} ||y_n - q|| &\le ||(1-\beta_n)x_n + \beta_n w_n - q|| \\ &\le (1-\beta_n)||x_n - q|| + \beta_n||w_n - q|| \\ &\le (1-\beta_n)||x_n - q|| + \beta_n H(Tu_n, Tq) \\ &\le (1-\beta_n)||x_n - q|| + \beta_n||u_n - q|| \text{ (by (1.1)} \\ &\le ||x_n - q||. \end{aligned} \tag{3.3}$$

From (3.1), (3.3), and (1.1) we have

$$\begin{aligned} ||x_{n+1} - q|| &= ||(1-\alpha_n)x_n + \alpha_n v_n - q|| \\ &\le (1-\alpha_n)||x_n - q|| + \alpha_n||v_n - q|| \\ &\le (1-\alpha_n)||x_n - q|| + \alpha_n H(T(y_n), Tq) \\ &\le (1-\alpha_n)||x_n - q|| + \alpha_n||y_n - q|| \\ &\le ||x_n - q||, \quad \forall n \ge 1. \end{aligned} \tag{3.4}$$

This shows that $\{||x_n - q||\}$ is decreasing and bounded below. Hence $\lim_{n\to\infty}||x_n - q||$ exists. Without loss of generality, we can assume that

$$\lim_{n\to\infty}||x_n - q|| = c. \tag{3.5}$$

Therefore the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{w_n\}$, $\{Tz_n\}$, and $\{Ty_n\}$ all are bounded.

(II) Next, we prove that

$$\lim_{n\to\infty}||x_n - u_n|| = 0. \tag{3.6}$$

Indeed, by the sub-differential inequality (2.2) we have

$$\frac{1}{2\lambda_n}\{||u_n - q||^2 - ||x_n - q||^2 + ||x_n - u_n||^2\} \le f(q) - f(u_n).$$

Since $f(q) \le f(u_n)$ for all $n \ge 1$, it follows that

$$||x_n - u_n||^2 \le ||x_n - q||^2 - ||u_n - q||^2. \tag{3.7}$$

Therefore in order to prove $\lim_{n\to\infty}||x_n - u_n|| = 0$, it suffices to prove $||u_n - q|| \to c$.

In fact, it follows from (3.4) that

$$||x_{n+1} - q|| \le (1-\alpha_n)||x_n - q|| + \alpha_n||y_n - q||.$$

By simplifying we have

$$\begin{aligned} ||x_n - q|| &\le \frac{1}{\alpha_n}[||x_n - q|| - ||x_{n+1} - q||] + ||y_n - q|| \\ &\le \frac{1}{a}[||x_n - q|| - ||x_{n+1} - q||] + ||y_n - q||. \end{aligned}$$

This together with (3.5) shows that

$$c = \liminf_{n\to\infty}||x_n - q|| \le \liminf_{n\to\infty}||y_n - q||. \tag{3.8}$$

On the other hand, it follows from (3.3) and (3.5) that

$$\limsup_{n\to\infty}||y_n - q|| \le \limsup_{n\to\infty}||x_n - q|| = c.$$

This together with (3.8) implies that

$$\lim_{n\to\infty}||y_n - q|| = c. \tag{3.9}$$

Also, by (3.3),

$$||y_n - q|| \le (1-\beta_n)||x_n - q|| + \beta_n||u_n - q||,$$

which can be rewritten as

$$\begin{aligned}||x_n - q|| &\le \frac{1}{\beta_n}[||x_n - q|| - ||y_n - q||] + ||u_n - q||\\ &\le \frac{1}{a}[||x_n - q|| - ||y_n - q||] + ||u_n - q||.\end{aligned}$$

This together with (3.9) shows that

$$c = \liminf_{n\to\infty} ||x_n - q|| \le \liminf_{n\to\infty} ||u_n - q||. \tag{3.10}$$

From (3.2), it follows that

$$\limsup_{n\to\infty} ||u_n - q|| \le \limsup_{n\to\infty} ||x_n - q|| = c.$$

This together with (3.10) shows that $\lim_{n\to\infty} ||u_n - q|| = c$. Therefore it follows from (3.7) that

$$\lim_{n\to\infty} ||x_n - u_n|| = 0.$$

(III) Now we prove that

$$\lim_{n\to\infty} ||x_n - w_n|| = 0, \quad \lim_{n\to\infty} ||x_n - y_n|| = 0, \text{ and } \lim_{n\to\infty} ||u_n - w_n|| = 0. \tag{3.11}$$

Indeed, it follows from (3.1), (3.2), and (1.1) that

$$\begin{aligned}||y_n - q||^2 &= ||(1-\beta_n)x_n + \beta_n w_n - q||^2\\ &\le (1-\beta_n)||x_n - q||^2 + \beta_n||w_n - q||^2 - \beta_n(1-\beta_n)||x_n - w_n||^2\\ &\le (1-\beta_n)||x_n - q||^2 + \beta_n H(Tu_n, Tq||^2 - \beta_n(1-\beta_n)||x_n - w_n||^2\\ &\le (1-\beta_n)||x_n - q||^2 + \beta_n||u_n - q||^2 - \beta_n(1-\beta_n)||x_n - w_n||^2\\ &\le ||x_n - q||^2 - \beta_n(1-\beta_n)||x_n - w_n||^2.\end{aligned} \tag{3.12}$$

After simplifying and by using the condition that $0 < a \le \alpha_n,\ \beta_n < b < 1$, it follows from (3.12) that

$$\begin{aligned}a(1-b)||x_n - w_n||^2 &\le \beta_n(1-\beta_n)||x_n - w_n||^2\\ &\le ||x_n - q||^2 - ||y_n - q||^2 \to 0 \text{ (as } n \to \infty).\end{aligned}$$

This implies that

$$\lim_{n\to\infty} ||x_n - w_n|| = 0. \tag{3.13}$$

Hence from (3.6) and (3.13) we have

$$||y_n - x_n|| = ||(1-\beta_n)x_n + \beta_n w_n - x_n|| = (1-\beta_n)||x_n - w_n|| \to 0 \text{ (as } n \to \infty).$$

So is

$$||u_n - w_n|| \le ||u_n - x_n|| + ||x_n - w_n|| \to 0 \text{ (as } n \to \infty). \tag{3.14}$$

(IV) Now we prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.15}$$

In fact, by Lemma 2.3 (iii), for each u_n, x_n, and $w_n \in Tu_n$ there exists a $k_n \in Tx_n$ such that

$$H(Tu_n, Tx_n)^2 \le ||u_n - x_n||^2 + 2\langle u_n - w_n,\ x_n - k_n\rangle \text{ for each } n \ge 1. \tag{3.16}$$

Therefore by (3.16), (3.13), (3.6), (3.14), and noting that the sequences $\{x_n\}$ and $\{x_n\}$ are bounded, we have

$$\begin{aligned} d(x_n, Tx_n) &\le ||x_n - w_n|| + d(w_n, Tx_n) \\ &\le ||x_n - w_n|| + H(Tu_n, Tx_n) \\ &\le ||x_n - w_n|| + \sqrt{||u_n - x_n||^2 + 2\langle u_n - w_n,\ x_n - k_n\rangle} \to 0. \end{aligned}$$

(V) Now we prove that

$$\lim_{n\to\infty} ||x_n - J_\lambda x_n|| = 0, \text{ where } \lambda_n \ge \lambda > 0. \tag{3.17}$$

In fact, it follows from (3.6) and Lemma 2.4 that

$$\begin{aligned} ||J_\lambda x_n - x_n|| &\le ||J_\lambda x_n - u_n|| + ||u_n - x_n|| = ||J_\lambda x_n - J_{\lambda_n} x_n|| + ||u_n - x_n|| \\ &= ||J_\lambda x_n - J_\lambda(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n + \frac{\lambda}{\lambda_n} x_n)) + ||u_n - x_n|| \\ &\le ||x_n - (1 - \frac{\lambda}{\lambda_n}) J_{\lambda_n} x_n - \frac{\lambda}{\lambda_n} x_n|| + ||u_n - x_n|| \\ &\le (1 - \frac{\lambda}{\lambda_n}) ||x_n - J_{\lambda_n} x_n|| + ||u_n - x_n|| \\ &= (1 - \frac{\lambda}{\lambda_n}) ||x_n - u_n|| + ||u_n - x_n|| \to 0. \end{aligned}$$

(VI) Finally we prove that $\{x_n\}$ converges weakly to p^* (some point in Ω).

In fact, since $\{u_n\}$ is bounded, there exists a subsequence $u_{n_i} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup p^* \in C$ (some point in C). By (3.11), $||u_{n_i} - w_{n_i}|| \to 0$. It follows from Lemma 2.3 (iv) that $p^* \in \text{Fix}(T)$. Again by (3.6), $||x_{n_i} - u_{n_i}|| \to 0$, hence $x_{n_i} \rightharpoonup p^*$. Therefore from (3.17) we have $||x_{n_i} - J_\lambda x_{n_i}|| \to 0$. Since J_λ is a single-valued nonexpansive mapping, it is demi-closed at 0. Hence $p* \in \text{Fix}(J_\lambda) = \text{argmin}_{y\in C} f(y)$. This shows that $p* \in \Omega$.

If there exists another subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup q^* \in C$ and $p^* \neq q^*$, by the same method as given above we can also prove that $q^* \in \Omega$. Since H has the Opial property, we have

$$\begin{aligned} \limsup_{n_i\to\infty} ||x_{n_i} - p^*|| &< \limsup_{n_i\to\infty} ||x_{n_i} - q^*|| = \lim_{n\to\infty} ||x_n - q^*|| \\ &= \limsup_{n_j\to\infty} ||x_{n_j} - q^*|| < \limsup_{n_j\to\infty} ||x_{n_j} - p^*|| \\ &= \lim_{n\to\infty} ||x_n - p^*|| = \limsup_{n_i\to\infty} ||x_{n_i} - p^*||. \end{aligned}$$

This is a contradiction. Therefore $p^* = q^*$ and $x_n \rightharpoonup p^* \in \Omega$.

This completes the proof of Theorem 3.1. □

If $T : C \to C$ is a single-valued nonspreading mapping, then the following theorem can be obtained from Theorem 3.1 immediately.

Theorem 3.2. *Let H, C, f be the same as in Theorem 3.1. Let $T : C \to C$ be a single-valued nonspreading mapping and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$ with $0 < a \le \alpha_n$, $\beta_n < b < 1$ for all $n \ge 1$. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \ge 1$ and some λ. For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated in the following manner:*

$$\begin{cases} u_n = \text{argmin}_{y\in C}[f(y) + \frac{1}{2\lambda_n}||y - x_n||^2], \\ y_n = (1 - \beta_n)x_n + \beta_n T u_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \end{cases} \quad \forall n \ge 1.$$

If $\Omega := \mathrm{Fix}(T) \bigcap \mathrm{argmin}_{y\in C} f(y) \neq \emptyset$*, then* $\{x_n\}$ *converges weakly to a point* $x^* \in \Omega$ *which is a minimizer of* f *in* C *as well as it is also a fixed point of* T *in* C*.*

Remark 3.3.

(1) Theorem 3.1 is a generalization of Agarwal et al. [1], Bačák [4], and the corresponding results in Ariza-Ruiz et al. [3]. In fact, we present a new modified proximal point algorithm for solving the convex minimization problem as well as the fixed point problem of nonspreading-type multivalued mappings.

(2) Theorem 3.2 is an improvement and generalization of the main result in Rockafellar [15] and Güler [8].

4. Some strong convergence theorems

Let (X, d) be a metric space, and C be a nonempty closed and convex subset of X.

Recall that a mapping $T : C \to CB(C)$ is said to be *demi-compact*, if for any bounded sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$ (as $n \to \infty$), then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly (i.e., in metric topology) to some point $p \in C$.

Theorem 4.1. *Under the assumptions of Theorem* 3.1*, if, in addition,* T *or* J_λ *is demi-compact, then the sequence* $\{x_n\}$ *defined by* (3.1) *converges strongly to a point* $p^* \in \Omega$*.*

Proof. In fact, it follows from (3.15), (3.17), (3.6), and (3.11) that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0, \tag{4.1}$$

$$\lim_{n\to\infty} ||x_n - J_\lambda(x_n)|| = 0, \tag{4.2}$$

and

$$\lim_{n\to\infty} ||x_n - u_n|| = 0, \quad \lim_{n\to\infty} ||u_n - w_n|| = 0. \tag{4.3}$$

By the assumption that T or J_λ is demi-compact, without loss of generality, we can assume T is demi-compact, it follows from (4.1) that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $p^* \in C$. Since J_λ is nonexpansive, it is demi-closed at 0. Hence it follows from (4.2) that $p^* \in \mathrm{Fix}(J_\lambda)$. Also it follows from (4.3) that $u_{n_i} \to p^*$ and T is also demi-closed at 0. This implies that $p^* \in \mathrm{Fix}(T)$. Hence $p^* \in \Omega$. Again by (3.5) $\lim_{n\to\infty} ||x_n - p^*||$ exists. Hence we have $\lim_{n\to\infty} ||x_n - p^*|| = 0$.

This completes the proof of Theorem 4.1. □

Theorem 4.2. *Under the assumptions of Theorem* 3.1*, if, in addition, there exists a nondecreasing function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0, g(r) > 0$ *for all* $r > 0$*, such that*

$$g(d(x, \Omega)) \leq d(x, J_\lambda x) + d(x, Tx) \quad \forall x \in C,$$

then the sequence $\{x_n\}$ *defined by* (3.1) *converges strongly to a point* $p^* \in \Omega$*.*

Proof. It follows from (4.1) and (4.2) that $\lim_{n\to\infty} g(d(x_n, \Omega)) = 0$. Since g is nondecreasing with $g(0) = 0$ and $g(r) > 0,\ r > 0$, we have

$$\lim_{n\to\infty} d(x_n, \Omega) = 0.$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in C. In fact, it follows from (3.4) that for any $q \in \Omega$

$$||x_{n+1} - q|| \leq ||x_n - q|| \quad \forall n \geq 1.$$

Hence for any positive integers n, m we have

$$||x_{n+m} - x_n|| \leq ||x_{n+m} - q|| + ||x_n - q|| \leq 2d(x_n, q) \quad \forall q \in \Omega.$$

This shows that

$$||x_{n+m} - x_n|| \leq 2d(x_n, \Omega) \to 0 \text{ (as } n, m \to \infty).$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset in H, it is complete. Without loss of generality, we can assume that $\{x_n\}$ converges strongly to some point $p^* \in C$. Since $\mathrm{Fix}(J_\lambda)$ and $\mathrm{Fix}(T)$ both are closed subsets in C, so is Ω. Hence $p^* \in \Omega$. This completes the proof of Theorem 4.2. □

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References

[1] R. P. Agarwal, D. O'Regan, D. R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal., **8** (2007), 61–79. 3.3

[2] L. Ambrosio, N. Gigli, G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Second edition, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, (2008). 2.2

[3] D. Ariza-Ruiz, L. Leuştean, G. Lóez, *Firmly nonexpansive mappings in classes of geodesic spaces*, Trans. Amer. Math. Soc., **366** (2014), 4299–4322. 2, 3.3

[4] M. Bačák, *The proximal point algorithm in metric spaces*, Israel J. Math., **194** (2013), 689–701. 1, 3.3

[5] O. A. Boikanyo, G. Moroşanu, *A proximal point algorithm converging strongly for general errors*, Optim. Lett., **4** (2010), 635–641. 1

[6] W. Cholamjiak, *Shrinking projection methods for a split equilibrium problem and a nonspreading-type multivalued mapping*, J. Nonlinear Sci. Appl., (in press). 1, 1.1, 2.3

[7] P. Cholamjiak, A. A. N. Abdou, Y. J. Cho, *Proximal point algorithms involving fixed points of nonexpansive mappings in* CAT(0) *spaces*, Fixed Point Theory Appl., **2015** (2015), 13 pages. 1

[8] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim., **29** (1991), 403–419. 1, 3.3

[9] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., **73** (1967), 957–961. 1

[10] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv., **70** (1995), 659–673. 2.1, 2.4

[11] S. Kamimura, W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory, **106** (2000), 226–240. 1

[12] F. Kohsaka, W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim., **19** (2008), 824–835. 1

[13] G. Marino, H.-K. Xu, *Convergence of generalized proximal point algorithms*, Commun. Pure Appl. Anal., **3** (2004), 791–808. 1

[14] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, (French) Rev. Franaise Informat. Recherche Opérationnelle, **4** (1970), 154–158. 1

[15] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optimization, **14** (1976), 877–898. 1, 3.3