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Common fixed point theorems for generalized ordered contractive mappings on cone b-metric spaces over Banach algebras

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Abstract

In this paper, we introduce the concept of generalized ordered contractive mappings on cone b-metric spaces over Banach algebras and establish a new common some fixed point theorems of such mappings under some natural conditions. The results extend and improve recent related results.

Keywords: Cone b-metric spaces over Banach algebras, common fixed point, ordered contractive mappings.
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1. Introduction and preliminaries

In 2007, Huang and Zhang [3] replaced real numbers by ordering Banach space and defined a cone metric space, and some fixed point theorems of contractive mapping on cone metric spaces are established in this paper. In 2013, in order to generalize the Banach contraction principle to more general form, Liu and Xu [4] introduced the concept of cone metric spaces over Banach algebras by replacing Banach spaces with Banach algebras. In 1989, Bakhtin [2] introduced b-metric space as a generalization of metric space. Since then, more other generalized b-metric spaces such as quasi-b-metric spaces [7], b-metric-like spaces [1] and quasi-b-metric-like spaces [10] were introduced.

Recently, Yin et al. [9] introduced the concept of ordered contractive mappings on cone metric spaces over Banach algebras. They have proved some fixed point theorems of ordered contractive mappings.

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Motivated by the above works, in this paper, we introduce the concept of cone b-metric space over the Banach algebra $\mathcal{A}$ and the generalized ordered contractive mappings on such spaces. In this way, we prove some fixed point theorems of generalized ordered contractive mappings on cone b-metric spaces over Banach algebras. Our main theorem extends the main results in [9].

The following definitions and results will be needed in this paper.

Let $\mathcal{A}$ always be a real Banach algebra, that is, $\mathcal{A}$ is a real Banach space in which an operator of multiplication is defined, subject to the following properties (for all $x, y, z \in \mathcal{A}$, $\alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;
2. $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leq \|x\|\|y\|$.

In the following, we assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be invertible, if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

A non-empty closed subset P of a Banach algebra $\mathcal{A}$ is called a cone, if

1. $\{\theta, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$.

For a given cone $P \subset \mathcal{A}$, a partial ordering '$\preceq$' with respect to P can be defined by $x \preceq y$, if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in int(P)$, where $int(P)$ denotes the interior of P. P is called a solid cone if $int(P) \neq \emptyset$.

Definition 1.1. Let X be a non-empty set and $\mathcal{A}$ be a real Banach algebra. Assume that the mapping $d : X \times X \to \mathcal{A}$ satisfies:

(1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$, if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,y) \preceq s[d(x,z) + d(z,y)]$ for all $x, y, z \in X$,

where $s \in P$ with $\rho(s) \geq 1$ (spectral radius $\rho(s)$ of s). Then d is called a cone metric on X, and (X,d) is called a cone b-metric space over the Banach algebra $\mathcal{A}$.

Remark 1.2. If $s = e$, then (X,d) in Definition 1.1 is reduced to the definition of a cone metric space over the Banach algebra $\mathcal{A}$ in [4]. If the Banach algebras replaced by Banach spaces in Definition 1.1, then (X,d) reduces to the cone b-metric space, in this case, $d : X \times X \to [0,\infty)$ and $s \geq 1$.

Definition 1.3 ([3, 4])**.** Let (X,d) be a cone b-metric space over the Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ a sequence in X. Then

1. $\{x_n\}$ converges to x whenever for each $c \in \mathcal{A}$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x) \ll c$, for all $n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$;
2. $\{x_n\}$ is a Cauchy sequence, if for each $c \in \mathcal{A}$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$, for all $n, m > N$;
3. (X,d) is a complete cone b-metric space, if every Cauchy sequence is convergent.

Definition 1.4. Let (X,d) be a b-cone metric space over the Banach algebra $\mathcal{A}$ and $T : X \to X$ be a mapping. We say that T is continuous, if for any $\{x_n\} \subset X$, $x_n \to x$ implies $Tx_n \to Tx$ $(n \to \infty)$.

Definition 1.5 ([9])**.** Let (X, d) be a b-cone metric space over the Banach algebra $\mathcal{A}$ and $\varphi : X \to \mathcal{A}$ be a mapping. A relation '$\leq$' (for the sake of differing from the partial ordering '$\preceq$' in $\mathcal{A}$, we denote it by '$\leq$') in X is defined as follows:

$$x, y \in X, \quad x \leq y \iff d(x, y) \preceq \varphi(x) - \varphi(y).$$

Obviously, '$\leq$' is a partial ordering in X and $x \leq y$ implies $\varphi(x) \succeq \varphi(y)$. Then (X, d) is called a partial ordering cone b-metric space over the Banach algebra $\mathcal{A}$.

Definition 1.6 ([9])**.** Let (X, d) be a partial ordering cone b-metric space over the Banach algebra $\mathcal{A}$. We say that $x, y \in X$ are comparable, if $x \leq y$ or $y \leq x$ holds.

Remark 1.7. Assume that $\mathcal{A}$ is a real Banach algebra, $u, v, w \in \mathcal{A}$. The following results are clear.

(i) If u and v are comparable, then $u - v$ and $v - u$ are comparable and $\theta \preceq (u - v) \vee (v - u)$.

(ii) If u and v, u and w together with v and w are comparable, then

$$(u - v) \vee (v - u) \preceq ((u - w) \vee (w - u)) + ((w - v) \vee (v - w)).$$

2. Main result

By [6, 8], we have the following:

Lemma 2.1. *Let $\mathcal{A}$ be a Banach algebra with a unit e, and $x \in \mathcal{A}$. If the spectral radius $\rho(x)$ of x is less than* 1*, i.e.,*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible, $(e - x)^{-1} = \sum_{i=0}^{\infty} x^i$, and $\rho((e - x)^{-1}) \leq (1 - \rho(x))^{-1}$.

Lemma 2.2 ([8])**.** *Let $x, y \in \mathcal{A}$. If x and y commute, then the spectral radius ρ satisfies the following properties:*

(i) $\rho(xy) \leq \rho(x)\rho(y)$;

(ii) $\rho(x + y) \leq \rho(x) + \rho(y)$;

(iii) $|\rho(x) - \rho(y)| \leq \rho(x - y)$.

Lemma 2.3 ([3])**.** *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x_0$, $y_n \to y_0$ as $n \to \infty$. Then $d(x_n, y_n) \to d(x_0, y_0)$ $(n \to \infty)$.*

Lemma 2.4 ([5])**.** *If E is a real Banach space with a solid cone P and if $\|x_n\| \to 0$ $(n \to \infty)$, then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.*

Definition 2.5. Two mappings $T, S : X \to X$ are said to be generalized φ-ordered contractive, if there exist $k_i \in P$ $(i = 1, ..., 5)$ with $0 \leq \sum_{i=1}^{5} \rho(k_i) < 1$ such that for any $x, y \in X$, if x and y are comparable, then Tx and Sy, Tx and x, Sy and y, Tx and y, and Sy and x are all comparable, and satisfies

$$\begin{aligned}
(\varphi(Tx) - \varphi(Sy)) &\vee (\varphi(Sy) - \varphi(Tx)) \\
&\preceq k_1[(\varphi(x) - \varphi(y)) \vee (\varphi(y) - \varphi(x))] + k_2[(\varphi(Tx) - \varphi(x)) \vee (\varphi(x) - \varphi(Tx))] \\
&\quad + k_3[(\varphi(Sy) - \varphi(y)) \vee (\varphi(y) - \varphi(Sy))] + k_4[(\varphi(y) - \varphi(Tx)) \vee (\varphi(Tx) - \varphi(y))] \\
&\quad + k_5[(\varphi(x) - \varphi(Sy)) \vee (\varphi(Sy) - \varphi(x))].
\end{aligned}$$

Remark 2.6. If $s = e$, $T = S$, and $k_2 = k_3 = k_4 = k_5 = \theta$, then Definition 2.5 reduces to the Definition 2.6 in [9].

Theorem 2.7. *Let (X, d) be a complete b-cone metric space over the Banach algebra $\mathcal{A}$ and let P be the underlying solid cone. If two continuous mappings $T, S : X \to X$ are generalized φ-ordered contractive with $\rho(s)\rho(k_1+k_3+k_4)+\rho(k_2+k_4) < 1$, and there exists $x_0 \in X$ such that x_0 and Sx_0 are comparable, then T and S have a common fixed point x^* in X. Moreover, the iterative sequence $x_{2n} = Tx_{2n-1}$ and $x_{2n-1} = Sx_{2n-2}$ $(n = 1, 2...)$ converges to x^* and*

$$d(x^*, x_0) \preceq s(e - s(e - k_2 - k_4)^{-1}(k_1 + k_3 + k_4))^{-1}[(\varphi(x_1) - \varphi(x_0)) \vee (\varphi(x_0) - \varphi(x_1))],$$

where s is as in Definition 1.1.

Proof. Let $x_0 \in X$ be an arbitrary point. Put $x_1 = Sx_0$ and $x_2 = Tx_1$, then let $x_3 = Sx_2$ and $x_4 = Tx_3$. By the induction, a sequence $\{x_n\}$ in X can be chose such that

$$x_{2n+2} = Tx_{2n+1} \quad \text{and} \quad x_{2n+1} = Sx_{2n}, \quad n = 0, 1, 2, ...$$

Since x_0 and $Sx_0 = x_1$ are comparable, we have by Definition 2.5 that $Sx_0 = x_1$ and $Tx_1 = x_2$ are comparable, and $Tx_1 = x_2$ and $Sx_2 = x_3$ are comparable. By the induction, it is easy to prove that for any natural number n, x_n and x_{n+1} are comparable. If n is an odd, then by Definition 2.5 and Remark 1.7 (ii) we have that

$$\begin{aligned}(\varphi(x_{n+1}) - \varphi(x_n)) \vee (\varphi(x_n) - \varphi(x_{n+1})) &= (\varphi(Tx_n) - \varphi(Sx_{n-1})) \vee (\varphi(Sx_{n-1}) - \varphi(Tx_n))\\
&\preceq k_1[(\varphi(x_n) - \varphi(x_{n-1})) \vee (\varphi(x_{n-1}) - \varphi(x_n))]\\
&\quad + k_2[(\varphi(Tx_n) - \varphi(x_n)) \vee (\varphi(x_n) - \varphi(Tx_n))]\\
&\quad + k_3[(\varphi(Sx_{n-1}) - \varphi(x_{n-1})) \vee (\varphi(x_{n-1}) - \varphi(Sx_{n-1}))]\\
&\quad + k_4[(\varphi(x_{n-1}) - \varphi(Tx_n)) \vee (\varphi(Tx_n) - \varphi(x_{n-1}))]\\
&\quad + k_5[(\varphi(x_n) - \varphi(Sx_{n-1})) \vee (\varphi(Sx_{n-1}) - \varphi(x_n))]\\
&\preceq k_1[(\varphi(x_n) - \varphi(x_{n-1})) \vee (\varphi(x_{n-1}) - \varphi(x_n))]\\
&\quad + k_2[(\varphi(x_{n+1}) - \varphi(x_n)) \vee (\varphi(x_n) - \varphi(x_{n+1}))]\\
&\quad + k_3[(\varphi(x_n) - \varphi(x_{n-1})) \vee (\varphi(x_{n-1}) - \varphi(x_n))]\\
&\quad + k_4[(\varphi(x_{n-1}) - \varphi(x_n)) \vee (\varphi(x_n) - \varphi(x_{n-1}))]\\
&\quad + k_4[(\varphi(x_n) - \varphi(x_{n+1})) \vee (\varphi(x_{n+1}) - \varphi(x_n))].\end{aligned}$$

From the definition of $\preceq$ and Lemma 2, we obtain that

$$\begin{aligned}d(x_{n+1}, x_n) &\preceq (\varphi(x_{n+1}) - \varphi(x_n)) \vee (\varphi(x_n) - \varphi(x_{n+1}))\\
&\preceq (e - k_2 - k_4)^{-1}(k_1 + k_3 + k_4)[(\varphi(x_n) - \varphi(x_{n-1})) \vee (\varphi(x_{n-1}) - \varphi(x_n))]\\
&\ \vdots\\
&\preceq [(e - k_2 - k_4)^{-1}(k_1 + k_3 + k_4)]^n[(\varphi(x_1) - \varphi(x_0)) \vee (\varphi(x_0) - \varphi(x_1))]\\
&= K^n D,\end{aligned} \tag{2.1}$$

where

$$K = (e - k_2 - k_4)^{-1}(k_1 + k_3 + k_4),$$

and

$$D = (\varphi(x_1) - \varphi(x_0)) \vee (\varphi(x_0) - \varphi(x_1)).$$

Similarly, if n is an even, then (2.1) is also holds. By Definition 2.5, Lemmas 2.1 and 2.2, and the condition $\rho(s)\rho(k_1 + k_3 + k_4) + \rho(k_2 + k_4) < 1$, we obtain

$$\begin{aligned}\rho(K) &= \rho((e - k_2 - k_4)^{-1}(k_1 + k_3 + k_4)) \le \rho((e - k_2 - k_4)^{-1})\rho(k_1 + k_3 + k_4)\\
&\le (1 - \rho(k_2 + k_4))^{-1}\rho(k_1 + k_3 + k_4) < \frac{1}{\rho(s)} \le 1.\end{aligned} \tag{2.2}$$

For any natural numbers n, m with $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\preceq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\preceq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &\vdots \\ &\preceq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} d(x_{m-1}, x_m) \\ &\preceq sK^n D + s^2 K^{n+1} D + \cdots + s^{m-n} K^{m-1} D \\ &\preceq s^n K^n D + s^{n+1} K^{n+1} D + \cdots + s^{m-1} K^{m-1} D \\ &= [(sK)^n + (sK)^{n+1} + \cdots + (sK)^{m-1}] D \\ &\preceq \left(\sum_{i=0}^{\infty} (sK)^i \right) (sK)^n D \\ &= (e - sK)^{-1} (sK)^n D. \end{aligned}$$

By Lemmas 2.1 and 2.2, and Eq. (2.2), we have $\rho(sK) \leq \rho(s)\rho(K) < 1$, and

$$\|(sK)^n\| \to 0, \quad n \to \infty, \quad \text{and} \quad \|(e - sK)^{-1}\| < \infty. \tag{2.3}$$

By Lemma 2.4 and the fact that $\|(e - sK)^{-1}(sK)^n D\| \to 0$ $(n \to \infty)$ (by (2.3)), it follows that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $m > n > N$, we have

$$d(x_n, x_m) \preceq (e - sK)^{-1}(sK)^n D \ll c,$$

which implies that the sequence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Moreover, $x_{2n} \to x^*$ and $x_{2n+1} \to x^*$. Now we prove that $x^* = Tx^* = Sx^*$. The continuity of T and S implies that

$$x_{2n+2} = Tx_{2n+1} \to Tx^*, \quad x_{2n+1} = Sx_{2n} \to Sx^*, \quad n \to \infty.$$

Hence $Tx^* = Sx^* = x^*$. That is, x^* is a common fixed point of T and S.

Moreover, from Lemma 2.3 and Eq. (2.1), we get

$$\begin{aligned} d(x^*, x_0) &= \lim_{n\to\infty} d(x_n, x_0) \\ &\preceq \lim_{n\to\infty} \sum_{i=1}^{n} s^i d(x_{i-1}, x_i) \preceq \lim_{n\to\infty} \sum_{i=1}^{n} s^i K^{i-1} D \preceq s \sum_{i=1}^{\infty} (sK)^{i-1} D \\ &= s(e - sK)^{-1}[(\varphi(x_1) - \varphi(x_0)) \vee (\varphi(x_0) - \varphi(x_1))] \\ &= s(e - s(e - k_2 - k_4)^{-1}(k_1 + k_3 + k_4))^{-1}[(\varphi(x_1) - \varphi(x_0)) \vee (\varphi(x_0) - \varphi(x_1))]. \end{aligned}$$

□

Remark 2.8. If $T = S$, $s = e$ and $k_2 = k_3 = k_4 = k_5 = \theta$, then Theorem 2.7 reduces to Theorem 3.1 in [9].

Remark 2.9. If $T = S$, $s = e$ and $k_4 = k_5 = \theta$, then Theorem 2.7 reduces to Theorem 3.3 in [9].

Remark 2.10. If $T = S$, $s = e$ and $k_2 = k_3 = \theta$, then Theorem 2.7 reduces to Theorem 3.5 in [9].

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