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A viscosity approximation method for nonself operators and equilibrium problems in Hilbert spaces

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Abstract

Viscosity approximate methods have recently received much attention due to the applications in convex optimization problems. In this paper, we study a viscosity iterative algorithm with computational errors. Strong convergence theorems of solutions are established in the framework of Hilbert spaces. The main results presented in this paper improve the corresponding results announced recently.

Keywords: Gradient projection method, monotone operator, normal cone, optimization, projection.
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1. Introduction

Variational inclusion problems have emerged as an effective and powerful tool for studying a wide class of unrelated problems arising in various branches of social, physical, engineering, pure and applied sciences in a unified and general framework, see, for example, [6, 7, 12]. Variational inclusion problems, which include equilibrium problems, fixed point problems, saddle point problems, and complementarity problems as special cases have been extended and generalized in different directions by using novel and innovative techniques and ideas, both for their own sake and for their applications, see, for example, [1, 8, 9, 11, 13], and the references therein. In the case that the given operator can be decomposed into the sum of two (or more) maximal monotone operators, whose resolvents are easier to evaluate than the resolvent of the original operator. Such a method is known as the operator splitting method. This can lead to the development of very efficient methods, since one can treat each part of the original operator independently; see, for example, [3, 10, 16], and the references therein. A useful feature of the forward-backward splitting method is that the

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resolvent step involves the subdifferential of the proper, convex and lower semicontinuous part only and the other part facilitates the problem decomposition. The simplest of these is the resolvent method. Viscosity approximation method, which was introduced by Moudafi [15], recently has received much attention due to the applications in convex optimization problems; see, for example, [19–21], and the references therein.

In this article, we propose a viscosity splitting method for solving variational inclusion problems, fixed point problems and equilibrium problems. Indeed, there are many problems needing more than one constraint. For these problems, we have to obtain some solution of a nonlinear problem which is also the solution of other nonlinear problems. We establish a strong convergence theorem of solutions in the framework of Hilbert spaces. The organization of this paper is as follows. In Section 2, we provide some necessary mathematical preliminaries. In Section 3, a viscosity splitting algorithm with computational errors is investigated. Some applications of the main results are also discussed in this section.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let P_C be the metric projection from H onto C.

Let $S : C \to H$ be a mapping. $F(S)$ stands for the fixed point set of S. Recall that S is said to be contractive iff there exists a constant $\kappa \in (0, 1)$ such that

$$\|Sx - Sy\| \le \kappa\|x - y\|, \quad \forall x, y \in C.$$

S is said to be firmly nonexpansive iff

$$\|Sx - Sy\|^2 \le \langle x - y, Sx - Sy\rangle, \quad \forall x, y \in C.$$

It is known that P_C is firmly nonexpansive. S is said to be nonexpansive iff

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C.$$

S is said to be strict pseudocontraction iff there exists a constant $\tau \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \tau\|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in C.$$

The class of strict pseudocontractions was introduced by Browder and Petryshyn [5] in 1967. It is clear that the class of strict pseudocontractions strictly include the class of nonexpansive mappings as a special cases. It is also known that strict pseudocontraction is Lipschitz continuous; see [5] and the references therein.

Recall that S is said to be monotone iff

$$\langle Sx - Sy, x - y\rangle \ge 0, \quad \forall x, y \in C.$$

S is said to be strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Sx - Sy, x - y\rangle \ge \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, we say that S is an α-strongly monotone mapping. S is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Sx - Sy, x - y\rangle \ge \alpha\|Sx - Sy\|^2, \quad \forall x, y \in C.$$

For such a case, we say that S is an α-inverse-strongly monotone mapping. It is clear that S is inverse-strongly monotone if and only if S^{-1} is strongly monotone.

A set-valued mapping $T : H \to 2^H$ is said to be monotone iff for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is maximal iff the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal iff, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone mapping of C into H and $N_C v$ the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \forall u \in C\}$$

and define a mapping T on C by

$$Tv = \begin{cases} \emptyset, & v \notin C, \\ Av + N_C v, & v \in C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ iff $\langle Av, u - v\rangle \geq 0$ for all $u \in C$; see [17] and the references therein. Let I denote the identity operator on H and $B : H \to 2^H$ be a maximal monotone operator. Then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r : H \to H$ by $J_r = (I + rB)^{-1}$. It is called the resolvent of B. We know that $B^{-1}0 = F(J_r)$ for all $r > 0$ and J_r is firmly nonexpansive.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following equilibrium problem in the terminology of Blum and Oettli [4].

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \ \forall y \in C. \tag{2.1}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$, i.e., $EP(F) = \{x \in C : F(x, y) \geq 0, \quad \forall y \in C\}$.

To study equilibrium problem (2.1), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 ([14])**.** *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n + e_n,$$

where $\{\gamma_n\}$ is a real number sequence in $(0, 1)$ and $\{\delta_n\}$ and $\{e_n\}$ are nonnegative real number sequences such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} \delta_n < \infty$;

(iii) $\sum_{n=1}^{\infty} e_n < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.2 ([4])**.** *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4)*. Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$rF(z, y) + \langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : rF(z, y) + \langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the followings hold:

(a) *$F(T_r) = EP(F)$ is closed and convex.*

(b) T_r *is single-valued;*

(c) T_r *is firmly nonexpansive,*

Lemma 2.3 ([2])**.** *Let C be a nonempty closed convex subset of H, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4)*, and T_t and T_s are defined as in Lemma 2.2. Then*

$$\|T_s x - T_t x\| \leq \frac{|s-t|}{t}\|x - T_t x\|.$$

Lemma 2.4 ([18])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in H and let $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.5 ([22])**.** *Let C be a nonempty closed convex subset of H and let P_C be the metric projection from H onto C. Let $T : C \to H$ be a τ-strict pseudo-contraction with $F(T) \neq \emptyset$. Then, $F(P_C T) = F(T)$. Define $S : C \to H$ by $Sx = kx + (1-k)Tx$ for each $x \in C$. Then, as $k \in [\tau, 1)$, $S : C \to H$ is nonexpansive such that $F(S) = F(T)$. Moreover, $I - P_C S$ is demiclosed at origin.*

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to H$ be a strict pseudocontraction with constant $\tau \in [0,1)$ and let $f : C \to C$ be a contraction with constant $\kappa \in (0,1)$. Let F be a bifunction from $C\times C$ to R which satisfies* (A1)-(A4)*. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : H \to 2^H$ be a maximal monotone mapping. Assume $F(T) \cap (A+B)^{-1}(0) \cap EP(F) \neq \emptyset$. Let $\{r_n\}$ and $\{s_n\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$,*

$$\begin{cases} z_n = P_C(\delta_n x_n + (1-\delta_n)Tx_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, n \geq 1, \end{cases}$$

where $\{y_n\}$ is a sequence in C such that $s_n F(y_n, y) + \langle y - y_n, y_n - J_{r_n}(z_n - r_n A z_n + e_n)\rangle \geq 0$ for all $y \in C$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{r_n\}$, and $\{s_n\}$ satisfy the conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} s_n$, $0 < r \leq r_n \leq r' < 2\alpha$, $0 \leq \tau \leq \delta_n < 1$, and $\lim_{n\to\infty} |\delta_n - \delta_{n+1}| = \lim_{n\to\infty} |r_n - r_{n+1}| = \lim_{n\to\infty} |s_n - s_{n+1}| = 0$, where r and r' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{F(T)\cap(A+B)^{-1}(0)\cap EP(F)} f(q)$.

Proof. Fixing $p \in F(T) \cap (A+B)^{-1}(0) \cap EP(F)$, in view of Lemma 2.2, we find that $p = Tp = T_{s_n} p = J_{r_n}(p - r_n Ap)$ and $y_n = T_{s_n} J_{r_n}(z_n - r_n A z_n + e_n)$. For any $x, y \in C$, we see that

$$\begin{aligned} \|(I - r_n A)x - (I - r_n A)y\|^2 &= \|x-y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2. \end{aligned}$$

By using the condition imposed on $\{r_n\}$, we see that $\|(I - r_n A)x - (I - r_n A)y\| \leq \|x - y\|$. This proves that $I - r_n A$ is nonexpansive. Put $T_n = P_C(\delta_n I + (1-\delta_n)T)$. In view of Lemma 2.5, we find that $T_n : C \to C$ is nonexpansive with $F(T) = F(T_n)$ for every $n \geq 1$. It follows that

$$\|x_{n+1} - p\| \leq \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\|$$

$$
\begin{aligned}
&\le \alpha_n\|x_n - f(p)\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|J_{r_n}\big(z_n - r_nAz_n + e_n\big) - J_{r_n}\big(p - r_nAp\big)\| \\
&\le \alpha_n\kappa\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|\big(z_n - r_nAz_n + e_n\big) - \big(p - r_nAp\big)\| \\
&\le \alpha_n\kappa\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|T_nx_n - p\| + \gamma_n\|e_n\| \\
&\le \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\kappa}\} + \|e_n\| \\
&\vdots \\
&\le \max\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1-\kappa}\} + \sum_{i=1}^{\infty}\|e_i\| < \infty,
\end{aligned}
$$

which implies that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Letting $\lambda_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, we have

$$
\|\lambda_{n+1} - \lambda_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - y_n\| + \|y_{n+1} - y_n\|. \tag{3.1}
$$

Put $u_n = z_n - r_nAz_n + e_n$. Since B is monotone, we see that

$$
\langle J_{r_{n+1}}u_n - J_{r_n}u_n, \frac{u_n - J_{r_{n+1}}u_n}{r_{n+1}} - \frac{u_n - J_{r_n}u_n}{r_n}\rangle \ge 0.
$$

It follows that

$$
\langle J_{r_{n+1}}u_n - J_{r_n}u_n, (1 - \frac{r_{n+1}}{r_n})(u_n - J_{r_n}u_n)\rangle \ge \|J_{r_{n+1}}u_n - J_{r_n}u_n\|^2.
$$

This in turn implies that

$$
\frac{|r_{n+1} - r_n|}{r_n}\|u_n - J_{r_n}u_n\| \ge \|J_{r_{n+1}}u_n - J_{r_n}u_n\|. \tag{3.2}
$$

Putting $\zeta_n = J_{r_n}(z_n - r_nAz_n + e_n)$, we have

$$
\begin{aligned}
\|\zeta_{n+1} - \zeta_n\| &\le \|J_{r_{n+1}}\big(z_{n+1} - r_{n+1}Az_{n+1} + e_{n+1}\big) - J_{r_{n+1}}\big(z_n - r_nAz_n + e_n\big)\| \\
&\quad + \|J_{r_{n+1}}\big(z_n - r_nAz_n + e_n\big) - J_{r_n}\big(z_n - r_nAz_n + e_n\big)\| \\
&\le \|\big(z_{n+1} - r_{n+1}Az_{n+1} + e_{n+1}\big) - \big(z_n - r_nAz_n + e_n\big)\| + \|J_{r_{n+1}}u_n - J_{r_n}u_n\|.
\end{aligned} \tag{3.3}
$$

Substituting (3.2) into (3.3), we find that

$$
\begin{aligned}
\|\zeta_{n+1} - \zeta_n\| &\le \|\big(z_{n+1} - r_{n+1}Az_{n+1} + e_{n+1}\big) - \big(z_n - r_nAz_n + e_n\big)\| + \frac{|r_{n+1} - r_n|}{r_n}\|u_n - J_{r_n}u_n\| \\
&\le \|z_{n+1} - z_n\| + \|(z_n - r_{n+1}Az_n) - (z_n - r_nAz_n)\| \\
&\quad + \frac{|r_{n+1} - r_n|}{r_n}\|u_n - J_{r_n}u_n\| + \|e_{n+1}\| + \|e_n\| \\
&\le \|z_{n+1} - z_n\| + |r_{n+1} - r_n|(\|Az_n\| + \frac{\|u_n - J_{r_n}u_n\|}{r_n}) + \|e_{n+1}\| + \|e_n\|.
\end{aligned} \tag{3.4}
$$

Hence, we have

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\le \|T_{s_{n+1}}J_{r_{n+1}}\big(z_{n+1} - r_{n+1}Az_{n+1} + e_{n+1}\big) - T_{s_{n+1}}J_{r_n}\big(z_n - r_nAz_n + e_n\big)\| \\
&\quad + \|T_{s_{n+1}}J_{r_n}\big(z_n - r_nAz_n + e_n\big) - T_{s_n}J_{r_n}\big(z_n - r_nAz_n + e_n\big)\| \\
&\le \|\zeta_{n+1} - \zeta_n\| + \frac{\|T_{s_n}\zeta_n - \zeta_n\|}{s_n}|s_{n+1} - s_n|.
\end{aligned} \tag{3.5}
$$

Combining (3.4) with (3.5), we find that

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\le \|z_{n+1} - z_n\| + |r_{n+1} - r_n|(\|Az_n\| + \frac{\|u_n - J_{r_n}u_n\|}{r_n}) + \|e_{n+1}\| \\
&\quad + \|e_n\| + \frac{\|T_{s_n}\zeta_n - \zeta_n\|}{s_n}|s_{n+1} - s_n|.
\end{aligned} \tag{3.6}
$$

On the other hand, we have

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|x_{n+1}-x_n\| + \|T_{n+1}x_n - T_nx_n\| \\ &\le \|x_{n+1}-x_n\| + \|(\delta_{n+1}x_n + (1-\delta_{n+1})Tx_n) - (\delta_nx_n + (1-\delta_n)Tx_n)\| \\ &\le \|x_{n+1}-x_n\| + |\delta_{n+1}-\delta_n|\|Tx_n - x_n\|.\end{aligned}$$

From (3.6), we find

$$\begin{aligned}\|y_{n+1}-y_n\| \le \|x_{n+1}-x_n\| + |\delta_{n+1}-\delta_n|\|Tx_n-x_n\| + |r_{n+1}-r_n|(\|Az_n\| + \frac{\|u_n - J_{r_n}u_n\|}{r_n}) \\ + \|e_{n+1}\| + \|e_n\| + \frac{\|T_{s_n}\zeta_n - \zeta_n\|}{s_n}|s_{n+1}-s_n|,\end{aligned}$$

which together with (3.1) yields that

$$\|\lambda_{n+1}-\lambda_n\| - \|x_{n+1}-x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n)-y_n\| + \xi_n,$$

where $\xi_n = |\delta_{n+1}-\delta_n|\|Tx_n-x_n\| + |r_{n+1}-r_n|(\|Az_n\| + \frac{\|u_n-J_{r_n}u_n\|}{r_n}) + \|e_{n+1}\| + \|e_n\| + \frac{\|T_{s_n}\zeta_n-\zeta_n\|}{s_n}|s_{n+1}-s_n|$. This implies that

$$\limsup_{n\to\infty}\big(\|\lambda_{n+1}-\lambda_n\| - \|x_{n+1}-x_n\|\big) = 0.$$

Using Lemma 2.4, we see that $\lim_{n\to\infty}\|\lambda_n - x_n\| = 0$, which in turn implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.7}$$

Since J_{r_n} is firmly nonexpansive, we find that

$$\begin{aligned}\|\zeta_n - p\|^2 &\le \langle (z_n - r_nAz_n + e_n) - (p - r_nAp), \zeta_n - p\rangle \\ &= \frac{1}{2}\big(\|(z_n - r_nAz_n + e_n) - (p - r_nAp)\|^2 + \|\zeta_n - p\|^2 \\ &\quad - \|\big((z_n - r_nAz_n + e_n) - (p - r_nAp)\big) - (\zeta_n - p)\|^2\big) \\ &\le \frac{1}{2}\big(\|z_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|z_n - p\|) + \|\zeta_n - p\|^2 - \|z_n - \zeta_n - r_n(Az_n - Ap) + e_n\|^2\big) \\ &\le \frac{1}{2}\big(\|z_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|z_n - p\|) + \|\zeta_n - p\|^2 - \|z_n - \zeta_n\|^2 \\ &\quad - \|r_n(Az_n - Ap) - e_n\|^2 + 2\|z_n - \zeta_n\|\|r_n(Az_n - Ap) - e_n\|\big) \\ &\le \frac{1}{2}\big(\|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|z_n - p\|) + \|\zeta_n - p\|^2 - \|z_n - \zeta_n\|^2 \\ &\quad - \|r_n(Az_n - Ap) - e_n\|^2 + 2\|z_n - \zeta_n\|\|r_n(Az_n - Ap) - e_n\|\big).\end{aligned}$$

This implies that

$$\begin{aligned}\|\zeta_n - p\|^2 \le \|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|z_n - p\|) - \|z_n - \zeta_n\|^2 \\ + 2r_n\|z_n - \zeta_n\|\|Az_n - Ap\| + 2\|z_n - \zeta_n\|\|e_n\|.\end{aligned} \tag{3.8}$$

Since A is inverse-strongly monotone, we find that

$$\begin{aligned}\|\zeta_n - p\|^2 &\le \|(z_n - r_nAz_n) - (p - r_nAp) + e_n\|^2 \\ &\le \|(z_n - p) - r_n(Az_n - Ap)\|^2 + \|e_n\|(\|e_n\| + 2\|z_n - p\|) \\ &\le \|z_n - p\|^2 - r_n(2\alpha_n - r_n)\|Az_n - Ap\|^2 + \|e_n\|(\|e_n\| + 2\|z_n - p\|) \\ &\le \|x_n - p\|^2 - r_n(2\alpha_n - r_n)\|Az_n - Ap\|^2 + \|e_n\|(\|e_n\| + 2\|z_n - p\|).\end{aligned}$$

Hence, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\ &\le \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2-r_n(2\alpha_n-r_n)\gamma_n\|Ax_n-Ap\|^2+\|e_n\|(\|e_n\|+2\|e_n\|\|x_n-p\|).\end{aligned}$$

Using the conditions imposed on the control sequences, we find from (3.7) that $\lim_{n\to\infty}\|Ax_n-Ap\|=0$. From (3.8), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\ &\le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|\zeta_n-p\|^2\\ &\le \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|)-\gamma_n\|z_n-\zeta_n\|^2\\ &\quad +2r_n\|z_n-\zeta_n\|\|Az_n-Ap\|+2\|z_n-\zeta_n\|\|e_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n\|z_n-\zeta_n\|^2 &\le \alpha_n\|f(x_n)-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|+2\|z_n-\zeta_n\|)\\ &\quad +2r_n\|z_n-x_n\|\|Az_n-Ap\|+\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|).\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|z_n-\zeta_n\|=0.$$

On the other hand, we have

$$\|y_n-p\|^2\le\frac{1}{2}\big(\|\zeta_n-p\|^2+\|y_n-p\|^2-\|y_n-\zeta_n\|^2\big).$$

That is,

$$\begin{aligned}\|y_n-p\|^2 &\le \|\zeta_n-p\|^2-\|y_n-\zeta_n\|^2\\ &\le \|z_n-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|)-\|y_n-\zeta_n\|^2\\ &\le \|x_n-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|)-\|y_n-\zeta_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\ &\le \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|)-\gamma_n\|y_n-\zeta_n\|^2.\end{aligned}$$

This implies that

$$\gamma_n\|y_n-\zeta_n\|^2\le\alpha_n\|f(x_n)-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\|e_n\|(\|e_n\|+2\|z_n-p\|).$$

From (3.7), we find

$$\lim_{n\to\infty}\|\zeta_n-y_n\|=0.$$

Note that $\|y_n-x_n\|\le\frac{\|x_{n+1}-x_n\|}{\gamma_n}+\frac{\alpha_n}{\gamma_n}\|f(x_n)-x_n\|$. It follows that $\lim_{n\to\infty}\|x_n-y_n\|=0$. Since $P_\Gamma f$, where $\Gamma=F(T)\cap(A+B)^{-1}(0)\cap EP(F)$, is a contractive operator, we see that there exists a unique fixed point.

Next, we denote the unique fixed point of $P_\Gamma f$ by q, that is, $q=P_\Gamma f(q)$. Now, we are in a position to show $\limsup_{n\to\infty}\langle x_n-q,f(q)-q\rangle\le 0$. To show it, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n-q,f(q)-q\rangle=\lim_{i\to\infty}\langle x_{n_i}-q,f(q)-q\rangle.$$

Since $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some point $\bar{x}$. We may assume, without loss of generality, that $\{x_{n_i}\}$ converges weakly to $\bar{x}$. Now, we are in a position to show $\bar{x}\in F(T)\cap(A+B)^{-1}(0)\cap EP(F)$.

First, we prove $\bar{x} \in (A+B)^{-1}(0)$. Notice that

$$\frac{z_n - \zeta_n + e_n}{r_n} - Az_n \in B\zeta_n.$$

Let $\mu \in B\nu$. Since B is monotone, we find that

$$\left\langle \frac{z_n - \zeta_n + e_n}{r_n} - Az_n - \mu, \zeta_n - \nu \right\rangle \geq 0.$$

It follows that $\langle -A\bar{x} - \mu, \bar{x} - \nu\rangle \geq 0$. This implies that $-A\bar{x} \in B\bar{x}$, that is, $\bar{x} \in (A+B)^{-1}(0)$.

Next, we prove $\bar{x} \in F(T)$. From the restriction on $\{\delta_n\}$, we find $\delta_n \to \delta \in [\tau, 1)$. Define $S : C \to H$ by $Sx = \delta x + (1-\delta)Tx$. Then, S is a nonexpansive mapping with $F(S) = F(T) = F(P_C S)$. Note that

$$\begin{aligned} \|P_C Sx_n - x_n\| &\leq \|x_n - z_n\| + \|T_n x_n - P_C Sx_n\| \\ &\leq \|x_n - z_n\| + \|(\delta_n x_n + (1-\delta_n)Tx_n) - Sx_n\| \\ &\leq \|x_n - z_n\| + |\delta_n - \delta|\|Tx_n - x_n\|. \end{aligned}$$

This shows that $\lim_{n\to\infty} \|P_C Sx_n - x_n\| = 0$. From Lemma 2.5, we have $\bar{x} \in F(P_C S) = F(T)$.

Finally, we prove $\bar{x} \in EP(F)$. Notice that

$$s_n F(y_n, y) + \langle y - y_n, y_n - \zeta_n\rangle \geq 0, \quad \forall y \in C.$$

By using condition (A2), we see that $\langle y - y_n, y_n - \zeta_n\rangle \geq s_n F(y, y_n)$ for all $y \in C$. Replacing n by n_i, we arrive at

$$\langle y - y_{n_i}, \frac{y_{n_i} - \zeta_{n_i}}{s_{n_i}}\rangle \geq F(y, y_{n_i}), \quad \forall y \in C.$$

By using the condition $\liminf_{n\to\infty} s_n > 0$, we find that $0 \geq F(y, \bar{x})$. This proves that $\bar{x} \in EP(F)$. This shows that $\limsup_{n\to\infty}\langle x_n - q, f(q) - q\rangle \leq 0$. Note that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \alpha_n\langle f(x_n) - q, x_{n+1} - q\rangle + \beta_n\|x_n - q\|\|x_{n+1} - q\| + \gamma_n\|y_n - q\|\|x_{n+1} - q\| \\ &\leq \alpha_n\langle f(x_n) - f(q), x_{n+1} - q\rangle + \alpha_n\langle f(q) - q, x_{n+1} - q\rangle \\ &\quad + \beta_n\|x_n - q\|\|x_{n+1} - q\| + \gamma_n(\|x_n - q\| + e_n)\|x_{n+1} - q\| \\ &\leq \frac{\alpha_n\kappa + \beta_n + \gamma_n}{2}(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + \alpha_n\langle f(q) - q, x_{n+1} - q\rangle + e_n\|x_{n+1} - q\|. \end{aligned}$$

It follows that

$$\|x_{n+1} - q\|^2 \leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - q\|^2 + 2\alpha_n\langle f(q) - q, x_{n+1} - q\rangle + 2\|x_{n+1} - q\|e_n.$$

From Lemma 2.1, we find that $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof. □

From Theorem 3.1, we have the following results immediately.

Corollary 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $f : C \to C$ be a contraction with constant $\kappa \in (0,1)$. Let F be a bifunction from $C \times C$ to R which satisfies (A1)-(A4). Assume $F(T) \cap EP(F) \neq \emptyset$. Let $\{s_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty}\|e_n\| < \infty$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, n \geq 1$, where $\{y_n\}$ is a sequence in C such that $s_n F(y_n, y) + \langle y - y_n, y_n - z_n - e_n\rangle \geq 0$ for all $y \in C$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{r_n\}$, and $\{s_n\}$ satisfy the conditions: $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, $0 < \liminf_{n\to\infty} s_n$, $0 < r \leq r_n \leq r' < 2\alpha$, $0 \leq \tau \leq \delta_n < 1$, and $\lim_{n\to\infty}|\delta_n - \delta_{n+1}| = \lim_{n\to\infty}|s_n - s_{n+1}| = 0$, where r and r' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{F(T)\cap EP(F)}f(q)$.*

Corollary 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $f : C \to C$ be a contraction with constant $\kappa \in (0,1)$. Let F be a bifunction from $C \times C$ to R which satisfies* (A1)-(A4). *Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : H \to 2^H$ be a maximal monotone mapping. Assume $(A + B)^{-1}(0) \cap EP(F) \neq \emptyset$. Let $\{r_n\}$ and $\{s_n\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, n \geq 1$, where $\{y_n\}$ is a sequence in C such that $s_n F(y_n, y) + \langle y - y_n, y_n - J_{r_n}(z_n - r_n A z_n + e_n)\rangle \geq 0$ for all $y \in C$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, and $\{s_n\}$ satisfy the conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} s_n$, $0 < r \leq r_n \leq r' < 2\alpha$, and $\lim_{n\to\infty} |r_n - r_{n+1}| = \lim_{n\to\infty} |s_n - s_{n+1}| = 0$, where r and r' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{(A+B)^{-1}(0)\cap EP(F)} f(q)$.*

Corollary 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to H$ be a strict pseudocontraction with constant $\tau \in [0,1)$ and let $f : C \to C$ be a contraction with constant $\kappa \in (0,1)$. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : H \to 2^H$ be a maximal monotone mapping such that its domain is in C. Assume $F(T) \cap (A+B)^{-1}(0) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$,*

$$\begin{cases} z_n = P_C(\delta_n x_n + (1 - \delta_n) T x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{r_n}(z_n - r_n A z_n + e_n), n \geq 1. \end{cases}$$

Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{r_n\}$ satisfy the conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < r \leq r_n \leq r' < 2\alpha$, $0 \leq \tau \leq \delta_n < 1$, and $\lim_{n\to\infty} |\delta_n - \delta_{n+1}| = \lim_{n\to\infty} |r_n - r_{n+1}| = 0$, where r and r' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{F(T)\cap(A+B)^{-1}(0)} f(q)$.

Let G be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4), and let W be a multivalued mapping of H into itself defined by

$$Wx = \begin{cases} \{z \in H : G(x, y) \geq \langle y - x, z\rangle \ \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then W is a maximal monotone operator with the domain $D(W) \subset C$ and $EP(G) = W^{-1}(0)$. Hence, we have the following result on common solutions of a pair of equilibrium problems.

Corollary 3.5. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $f : C \to C$ be a contraction with constant $\kappa \in (0,1)$. Let F and G be two bifunctions from $C \times C$ to R which satisfies* (A1)-(A4). *Assume $EP(G) \cap EP(F) \neq \emptyset$. Let $\{r_n\}$ and $\{s_n\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$,*

$$\begin{cases} z_n = (I + r_n W)^{-1}(x_n + e_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, n \geq 1, \end{cases}$$

where $\{y_n\}$ is a sequence in C such that $s_n F(y_n, y) + \langle y - y_n, y_n - z_n\rangle \geq 0$ for all $y \in C$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, and $\{s_n\}$ satisfy the conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} s_n$, $0 < r \leq r_n \leq r' < 2\alpha$, and $\lim_{n\to\infty} |r_n - r_{n+1}| = \lim_{n\to\infty} |s_n - s_{n+1}| = 0$, where r and r' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{EP(G)\cap EP(F)} f(q)$.

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