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A coupling method involving the Sumudu transform and the variational iteration method for a class of local fractional diffusion equations

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Abstract

In this article, a coupling of the variational iteration method with the Sumudu transform via the local fractional calculus operator is proposed for the first time. As a testing example, the exact solution for the local fractional diffusion equation in fractal one-dimensional space is obtained. The method provided an accurate and efficient technique for solving the local fractional PDEs.

Keywords: Diffusion equation, exact solution, variational iteration method, Sumudu transform, local fractional calculus.
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1. Introduction

Fractional heat transfer (see [1–5, 8, 9, 11, 13–15, 18, 19, 22]) is an abnormal phenomenon of dynamical systems to capture the relations in space and time with different kernels of non-differentiable and differentiable types. This interest spans the works of many scientists and engineers from the field of mathematical

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physics. In order to find the solutions of the fractional heat transfer problems, several methods, such as the heat-balance integral method (HBIM) [6], the Green function method (GFM) [7], the similarity variable method (SVM) [21], the variational iteration method (VIM) [20] and the Laplace transform variational iteration method (LTVIM) [10], were developed in recent years.

In this paper, we consider the local fractional diffusion equation in fractal one-dimensional space for description of the fractal heat transfer (see [10, 20–22]):

$$\frac{\partial^{\kappa}\Theta(x,t)}{\partial\tau^{\kappa}} - \frac{\partial^{2\kappa}\Theta(x,t)}{\partial x^{2\kappa}} = 0, \quad \kappa \in (0,1)\,, \tag{1.1}$$

where the local fractional derivative (LFD) of $\Theta(\tau)$ of order κ is given as follows (see, for example, [10, 16, 17, 20–23]):

$$\Theta_{\tau}^{(\kappa)}(\tau_0) = \frac{d^{\kappa}\Theta(\tau)}{d\tau^{\kappa}}\,|_{\tau=\tau_0} = \lim_{\tau\to\tau_0}\frac{\Delta^{\kappa}(\Theta(\tau)-\Theta(\tau_0))}{(\tau-\tau_0)^{\kappa}},$$

with the difference term given by [22]

$$\Delta^{\kappa}(\Theta(\tau)-\Theta(\tau_0)) \cong \Gamma(1+\kappa)\,\Delta(\Theta(\tau)-\Theta(\tau_0))\,.$$

The local fractional variational iteration method (LFVIM) was used to solve the local fractional PDEs in fractal-dimension space (see [20, 22]). Meanwhile, the local fractional Sumudu transform (LFST) was expressed by (see [17]):

$$S_{\kappa}\{\varphi(\tau)\} = \frac{1}{\Gamma(1+\kappa)}\int_0^{\infty} E_{\kappa}\left(-h^{-\kappa}\tau^{\kappa}\right)\frac{\varphi(\tau)}{h^{\kappa}}(d\tau)^{\kappa}, \quad 0<\beta\le 1,$$

and its inverse transform was defined by (see [17]):

$$S_{\kappa}^{-1}\{S_{\kappa}(\Theta(\tau))\} = \Theta(\tau)\,, \quad 0<\kappa\le 1,$$

where the local fractional integral of $\varphi(\tau)$ of order κ is defined by (see [17])

$${}_{\tau_0}I_{\tau}^{(\kappa)}\varphi(\tau) = \frac{1}{\Gamma(1+\kappa)}\int_{\tau_0}^{\tau}\varphi(\tau)(d\tau)^{\kappa} = \frac{1}{\Gamma(1+\kappa)}\lim_{\Delta\tau\to 0}\sum_{j=0}^{j=N-1}\varphi(\tau_j)(\Delta\tau)^{\kappa}\,,$$

with the partitions of the interval $[\tau,\tau_0]$ are given by (τ_j,τ_{j+1}) $(j=0,\cdots,N-1,\ \Delta\tau=\tau_{j+1}-\tau_j)$.

Motivated essentially by the above results, our aim in the present article is to propose a coupling method of LFVIM and LFST for solving the local fractional diffusion equation in fractal one-dimensional space at the first time (see also [12]).

This article is structured as follows: In Section 2, a coupling method of LFVIM and LFST is analyzed. The non-differentiable solution of local fractional diffusion equation is given in Section 3. Finally, Section 4 outlines our conclusions.

2. Analysis of the method

In this section, a coupling method of VIM with ST via the local fractional calculus operator is introduced.

In the local fractional differential operator, (1.1) can be written as follows:

$$\mathrm{E}_{\kappa}\Theta - \Pi_{\kappa}\Theta = 0, \tag{2.1}$$

where

$$\mathrm{E}_{\kappa} = \partial^{\kappa}/\partial\tau^{\kappa},$$

and

$$\Pi_\kappa = \partial^{2\kappa}/\partial x^{2\kappa}.$$

By owing to the idea of the LFVIM (see [20, 22]), the local fractional functional is determined by

$$\Theta_{n+1}(x,\tau) = \Theta_n(x,\tau) + {}_0I_\tau^{(\kappa)}\left\{\frac{\lambda(\eta-\tau)}{\Gamma(1+\kappa)}\left[\mathrm{E}_\kappa\Theta_n(x,\tau) - \Pi_\kappa\Theta_n(x,\tau)\right]\right\}. \tag{2.2}$$

By taking LFST of (2.2), we obtain

$$S_\kappa\{\Theta_{n+1}(x,\tau)\} = S_\kappa\{\Theta_n(x,\tau)\} + S_\kappa\left\{\frac{\lambda(\eta-\tau)}{\Gamma(1+\kappa)}\right\}S_\kappa\{[\mathrm{E}_\kappa\Theta_n(x,\tau) - \Pi_\kappa\Theta_n(x,\tau)]\}, \tag{2.3}$$

where

$$S_\kappa\left\{\frac{d^\kappa\Theta(\tau)}{d\tau^\kappa}\right\} = \frac{1}{h^\kappa}\left[S_\kappa\{\Theta(\tau)\} - \Theta(0)\right]. \tag{2.4}$$

By adopting local fractional variation (see [1, 3, 22]), we can rewrite (2.3) as

$$\delta^\kappa\{\Theta_{n+1}(x,\tau)\} = \delta^\kappa\{S_\kappa\{\Theta_n(x,\tau)\}\} + \delta^\kappa\left\{S_\kappa\left\{\frac{\lambda(\eta-\tau)}{\Gamma(1+k)}\right\}S_\kappa\left[\mathrm{E}_\kappa\Theta_n(x,\tau) - \Pi_\kappa\Theta_n(x,\tau)\right]\right\}. \tag{2.5}$$

Thus we find from (2.5) that

$$\delta^\kappa\{S_\kappa\{\Theta_{n+1}(x,\tau)\}\} = \delta^\kappa\{S_\kappa\{\Theta_n(x,\tau)\}\} + \delta^\kappa\left\{S_\kappa\left\{\frac{\lambda(\eta-\tau)}{\Gamma(1+k)}\right\}S_\kappa\left[\mathrm{E}_\kappa\Theta_n(x,\tau)\right]\right\} = 0. \tag{2.6}$$

In view of (2.6), we have

$$\delta^\kappa\{S_\kappa[\mathrm{E}_\kappa\Theta_n(x,\tau)]\} = \delta^\kappa\left\{\frac{1}{h^\kappa}S_\kappa\{\Theta_n(x,\tau) - \Theta_n(x,0)\}\right\} = \frac{1}{h^\kappa}\delta^\kappa S_\kappa\{\Theta_n(x,\tau)\}, \tag{2.7}$$

such that

$$1 + S_\kappa\left\{\frac{\lambda(\tau)}{\Gamma(1+\kappa)}\right\}\frac{1}{h^\kappa} = 0.$$

By using (2.7), we have

$$S_\kappa\left\{\frac{\lambda(\tau)}{\Gamma(1+\kappa)}\right\} = -h^\kappa,$$

such that the local fractional iteration algorithm is expressed by

$$S_\kappa\{\Theta_{n+1}(x,\tau)\} = S_\kappa\{\Theta_n(x,\tau)\} - h^\kappa S_\kappa\{[\mathrm{E}_\kappa\Theta_n(x,\tau) - \Pi_\kappa\Theta_n(x,\tau)]\}. \tag{2.8}$$

Therefore, the solution of (2.1) takes the following form

$$S_\kappa\{\Theta(x,\tau)\} = \lim_{n\to\infty} S_\kappa\{\Theta_n(x,\tau)\}. \tag{2.9}$$

By taking the inverse LFST, we find from (2.9) that

$$\Theta(x,\tau) = S_\kappa^{-1}\left\{\lim_{n\to\infty} S_\kappa\{\Theta_n(x,\tau)\}\right\}.$$

3. Solving local fractional diffusion equation in fractal one-dimensional space

In this section, an illustrative example is presented.

We begin with the following local fractional diffusion equation in fractal one-dimensional space (see [3, 5, 18])

$$\frac{\partial^\kappa\Theta(x,t)}{\partial\tau^\kappa} - \frac{\partial^{2\kappa}\Theta(x,t)}{\partial x^{2\kappa}} = 0, \quad \tau \geq 0, \quad 0 < x < \pi, \tag{3.1}$$

subject to the initial-boundary value conditions given by

$$\begin{aligned}
&\Theta\left(x,0\right)=\sin_{\kappa}(x^{\kappa}) \quad 0<x<\pi,\\
&\Theta\left(\pi,\tau\right)=0, \quad \tau\geq 0,\\
&\Theta\left(0,\tau\right)=0, \quad \tau\geq 0,
\end{aligned}$$

where

$$\sin_{\kappa}\left(x^{\kappa}\right)=\sum_{i=0}^{\infty}\left(-1\right)^{\kappa}\frac{x^{(2i+1)\kappa}}{\Gamma\left(1+\left(2i+1\right)\kappa\right)},$$

represents the special function defined on Cantor sets (see [12, 22]) and the corresponding graph is demonstrated in Figure 1.

In view of (2.8), we have

$$S_{\kappa}\left\{\Theta_{n+1}\left(x,\tau\right)\right\}=S_{\kappa}\left\{\Theta_{n}\left(x,\tau\right)\right\}-h^{\kappa}S_{\kappa}\left\{\left[\mathrm{E}_{\kappa}\Theta_{n}\left(x,\tau\right)-\Pi_{\kappa}\Theta_{n}\left(x,\tau\right)\right]\right\},$$

and, alternatively,

$$\Theta_{n+1}\left(h,\tau\right)=\Theta_{n}\left(h,\tau\right)-h^{\kappa}S_{\kappa}\left\{\left[\mathrm{E}_{\kappa}\Theta_{n}\left(x,\tau\right)-\Pi_{\kappa}\Theta_{n}\left(x,\tau\right)\right]\right\}, \tag{3.2}$$

with the initial-value condition given by

$$\Theta_{0}\left(x,h\right)=S_{\kappa}\left\{\Theta_{0}\left(x,0\right)\right\}=\sin_{\kappa}(x^{\kappa}). \tag{3.3}$$

Thus, we have the following result from (3.2) and (3.3)

$$\begin{aligned}
\Theta_{1}\left(h,\tau\right)&=\Theta_{0}\left(h,\tau\right)-h^{\kappa}S_{\kappa}\left\{\left[\mathrm{E}_{\kappa}\Theta_{0}\left(x,\tau\right)-\Pi_{\kappa}\Theta_{0}\left(x,\tau\right)\right]\right\}\\
&=\Theta_{0}\left(h,\tau\right)-h^{\kappa}\left\{\frac{1}{h^{\kappa}}\left[\Theta_{0}\left(x,h\right)-\Theta_{0}\left(x,0\right)\right]-\Pi_{\kappa}\Theta_{0}\left(x,h\right)\right\}\\
&=\sin_{\kappa}(x^{\kappa})\sum_{i=0}^{1}\left(-1\right)^{i}h^{i\kappa},\\
\Theta_{2}\left(h,\tau\right)&=\Theta_{1}\left(h,\tau\right)-h^{\kappa}S_{\kappa}\left\{\left[\mathrm{E}_{\kappa}\Theta_{1}\left(x,\tau\right)-\Pi_{\kappa}\Theta_{1}\left(x,\tau\right)\right]\right\}\\
&=\Theta_{1}\left(h,\tau\right)-h^{\kappa}\left\{\frac{1}{h^{\kappa}}\left[\Theta_{1}\left(x,h\right)-\Theta_{1}\left(x,0\right)\right]-\Pi_{\kappa}\Theta_{1}\left(x,h\right)\right\}\\
&=\sin_{\kappa}(x^{\kappa})\sum_{i=0}^{2}\left(-1\right)^{i}h^{i\kappa},\\
\Theta_{3}\left(h,\tau\right)&=\Theta_{2}\left(h,\tau\right)-h^{\kappa}S_{\kappa}\left\{\left[\mathrm{E}_{\kappa}\Theta_{2}\left(x,\tau\right)-\Pi_{\kappa}\Theta_{2}\left(x,\tau\right)\right]\right\}\\
&=\Theta_{2}\left(h,\tau\right)-h^{\kappa}\left\{\frac{1}{h^{\kappa}}\left[\Theta_{2}\left(x,h\right)-\Theta_{2}\left(x,0\right)\right]-\Pi_{\kappa}\Theta_{2}\left(x,h\right)\right\}\\
&=\sin_{\kappa}(x^{\kappa})\sum_{i=0}^{3}\left(-1\right)^{i}h^{i\kappa},
\end{aligned}$$

and so on.

Consequently, we have

$$\Theta_{n}\left(h,\tau\right)=\lim_{n\to\infty}\sin_{\kappa}(x^{\kappa})\sum_{i=0}^{n}\left(-1\right)^{i}h^{i\kappa}. \tag{3.4}$$

Therefore, by taking the inverse LFST of (3.4), we obtain

$$\Theta\left(x,\tau\right)=S_{\kappa}^{-1}\left\{\lim_{n\to\infty}S_{\kappa}\left\{\Theta_{n}\left(x,\tau\right)\right\}\right\}$$

$$
\begin{aligned}
&= S_\kappa^{-1}\left\{\lim_{n\to\infty} S_\kappa\left\{\sin_\kappa(x^\kappa)\sum_{i=0}^{n}(-1)^i h^{i\kappa}\right\}\right\}\\
&= S_\kappa^{-1}\left\{\lim_{n\to\infty} S_\kappa\left\{\sin_\kappa(x^\kappa)\frac{1}{1+h^\kappa}\right\}\right\}\\
&= \sin_\kappa(x^\kappa)E_\kappa(-\tau^\kappa),
\end{aligned}
$$

and the exact solution of (3.1) is shown in Figure 2, where

$$S_\kappa\left\{E_\kappa\left(-\tau^\kappa\right)\right\} = \frac{1}{1-h^\kappa},$$

with the following Mittag-Leffler function on Cantor sets [22]:

$$E_\kappa\left(\tau^\kappa\right) = \sum_{i=0}^{\infty}\frac{\tau^{i\kappa}}{\Gamma\left(1+i\kappa\right)},$$

and the corresponding graph when $\kappa = \ln 2/\ln 3$ represented in Figure 3.

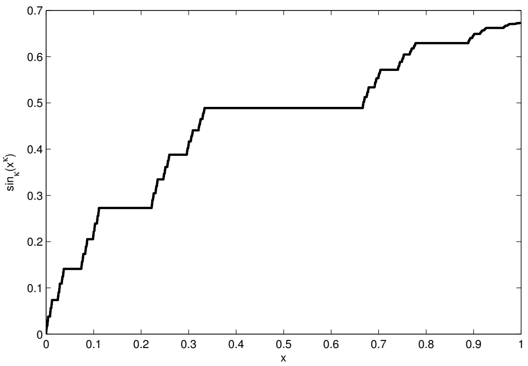

Figure 1: The special function defined on Cantor sets when $\kappa = \ln 2/\ln 3$.

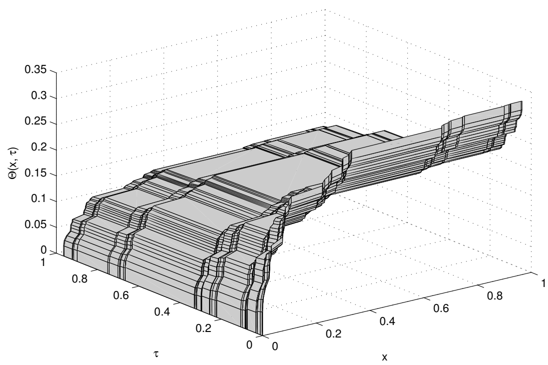

Figure 2: The non-differentiable solution for the local fractional diffusion equation when $\kappa = \ln 2/\ln 3$.

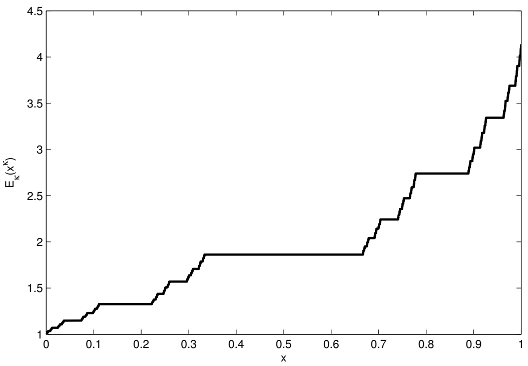

Figure 3: The Mittag-Leffler function on Cantor sets when $\kappa = \ln 2/\ln 3$.

4. Conclusions

The work has presented a coupling method involving the LFVIM with the LFST. The local fractional diffusion equation in fractal one-dimensional space was discussed. The non-differentiable exact solution for description of the fractal heat transfer was also obtained. The present methodology is proposed as an accurate and efficient tool for solving the local fractional PDEs in mathematical physics.

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