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A Brunn-Minkowski-type inequality involving γ-mean variance and its applications

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Abstract

By means of the algebra, functional analysis, and inequality theories, we establish a Brunn-Minkowski-type inequality involving γ-mean variance:

$$\overline{\mathrm{Var}}^{[\gamma]}(f+g) \leqslant \overline{\mathrm{Var}}^{[\gamma]}f + \overline{\mathrm{Var}}^{[\gamma]}g, \quad \forall \gamma \in [1,2],$$

where $\overline{\mathrm{Var}}^{[\gamma]}\varphi$ is the γ-mean variance of the function $\varphi : \Omega \to (0,\infty)$. We also demonstrate the applications of this inequality to the performance appraisal of education and business.

Keywords: Brunn-Minkowski-type inequality, γ-mean variance, performance appraisal, profit function, allowance function.
2010 MSC: 26D15, 26E60, 62J10.

1. Introduction

We begin by recalling some basic concepts and previous results which are related to the investigation of the present paper.

Let Ω be an m-dimensional, closed and bounded domain in $\mathbb{R}^m$, $\mathbb{R} \triangleq (-\infty,\infty)$, and let

$$X \triangleq (X_1, X_2, \ldots, X_m) \in \Omega$$

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be an m-dimensional and continuous random variable with the probability density function $p : \Omega \to (0, \infty)$, where $\int_\Omega p = 1$. Then functionals

$$\mathrm{E}\varphi \triangleq \int_\Omega p\varphi,\ \ \mathrm{Var}\varphi \triangleq \mathrm{E}\varphi^2 - (\mathrm{E}\varphi)^2 \text{ and } \overline{\mathrm{Var}}\varphi \triangleq \sqrt{\mathrm{Var}\varphi}$$

are the *mathematical expectation*, *variance* and the *mean variance* of the random variable $\varphi(X)$, respectively, where the function $\varphi : \Omega \to \mathbb{R}$ is continuous (see [14–16]).

In [15], the authors studied the convergence of the following generalized integral:

$$\mathrm{E}\phi\,(\psi + \delta) \triangleq \int_1^\infty p\phi\,(\psi + \delta),$$

which is a *generalized mathematical expectation* of the random variable $\phi\,[\psi\,(X) + \delta\,(X)]$, where $X \in [1, \infty)$.

In [14, 16], the authors extended the classic variance $\mathrm{Var}\varphi$ of the random variable $\varphi : \Omega \to (0, \infty)$ and defined the γ-*order variance* as follows:

$$\mathrm{Var}^{[\gamma]}\varphi \triangleq \begin{cases} \frac{2}{\gamma(\gamma-1)}\left[\mathrm{E}\varphi^\gamma - (\mathrm{E}\varphi)^\gamma\right], & \gamma \neq 0, 1, \\ \lim_{\gamma\to 0} \mathrm{Var}^{[\gamma]}\varphi = 2\left[\log(\mathrm{E}\varphi) - \mathrm{E}(\log\varphi)\right], & \gamma = 0, \\ \lim_{\gamma\to 1} \mathrm{Var}^{[\gamma]}\varphi = 2\left[\mathrm{E}(\varphi\log\varphi) - (\mathrm{E}\varphi)\log(\mathrm{E}\varphi)\right], & \gamma = 1. \end{cases}$$

Since

$$\mathrm{Var}^{[\gamma]}\varphi \geqslant 0, \quad \forall \gamma \in \mathbb{R},$$

we say that the functional

$$\overline{\mathrm{Var}}^{[\gamma]}\varphi \triangleq \left(\mathrm{Var}^{[\gamma]}\varphi\right)^{1/\gamma}$$

is a γ-*mean variance* of the random variable $\varphi\,(X)$, where $\gamma \neq 0$,

$$\mathrm{Var}^{[2]}\varphi = \mathrm{Var}\varphi \ \text{ and } \ \overline{\mathrm{Var}}^{[2]}\varphi = \overline{\mathrm{Var}}\varphi.$$

In [14], the authors defined the *Dresher variance mean* of the random variable $\varphi(X)$, and obtained the *Dresher variance mean inequality* and the *Dresher-type inequality.* Also, they demonstrated the applications of these results in space science.

In [16], the authors generalized the traditional covariance and the variance of random variables, and defined ϕ-covariance, ϕ-variance, ϕ-Jensen variance, ϕ-Jensen covariance, integral variance, and γ-order variance, as well as they studied the relationships among these variances. Moreover, they dealt with a quasi-log concavity conjecture and the monotonicity of the interval function $\mathrm{JVar}_\phi\varphi\left(X_{[a,b]}\right)$. They also demonstrated the applications of these results in higher education and showed that the hierarchical teaching model is normally better than the traditional teaching model under the hypotheses that

$$X_I \subset X \sim N_k\,(\mu, \sigma)\,,\ k > 1.$$

The well-known Brunn-Minkowski's inequality can be described as ([3, 4]): if the real number $\gamma > 1$, then we have

$$(\mathrm{E}|f + g|^\gamma)^{1/\gamma} \leqslant (\mathrm{E}|f|^\gamma)^{1/\gamma} + (\mathrm{E}|g|^\gamma)^{1/\gamma}. \tag{1.1}$$

Inequality (1.1) is reversed if $f > 0,\ g > 0,\ \gamma < 1$, and $\gamma \neq 0$. Furthermore, the equality in (1.1) holds if and only if f/g is a constant function, where $(\mathrm{E}|\varphi|^\gamma)^{1/\gamma}$ is the γ-*mean* of the function $|\varphi|$ (see [7, 19]).

By means of moment space techniques, in [6], Dresher proved the following Brunn-Minkowski-type inequality:

If $\rho \geqslant 1 \geqslant \sigma \geqslant 0,\ f, g \geqslant 0$, then we have

$$\left[\frac{\mathrm{E}(f+g)^\rho}{\mathrm{E}(f+g)^\sigma}\right]^{1/(\rho-\sigma)} \leqslant \left(\frac{\mathrm{E}f^\rho}{\mathrm{E}f^\sigma}\right)^{1/(\rho-\sigma)} + \left(\frac{\mathrm{E}g^\rho}{Eg^\sigma}\right)^{1/(\rho-\sigma)}.$$

The above result is referred as the Dresher's inequality (for example, see Ref. [1, 5, 9]). Here

$$D_{\rho,\sigma}(f,p) \triangleq \left(\frac{\mathrm{E}f^{\rho}}{\mathrm{E}f^{\sigma}}\right)^{1/(\rho-\sigma)}$$

is the well-known *Dresher mean* of the function f (see [2, 12]).

Let

$$\mathbf{C}(\Omega) \triangleq \{f | f : \Omega \to \mathbb{R} \ \text{ be continuous}\}.$$

Then for any $f, g \in \mathbf{C}(\Omega)$, we have the following Brunn-Minkowski-type inequality:

$$\overline{\mathrm{Var}}(f+g) \leqslant \overline{\mathrm{Var}}f + \overline{\mathrm{Var}}g. \tag{1.2}$$

Equality in (1.2) holds if and only if there exist two constants $C \in [0,\infty)$ and $C_0 \in \mathbb{R}$ such that

$$f \equiv Cg + C_0 \text{ or } g \equiv Cf + C_0. \tag{1.3}$$

The proof is based on the algebraic theory. We can define

$$\langle f, g\rangle \triangleq \mathrm{E}\left[(f-\mathrm{E}f)(g-\mathrm{E}g)\right] = \mathrm{Cov}(f,g), \quad \forall f, g \in \mathbf{C}(\Omega),$$

where $\mathbf{C}(\Omega)$ is a linear space in the real number field $\mathbb{R}$, $\mathrm{Cov}(f,g)$ is the covariance of the random variables $f(X)$ and $g(X)$ (see [16]). Then $\langle f, g\rangle$ is a *quasi-inner product* of the functions f and g, which satisfies the following conditions:

(H1.1) $\langle f, g\rangle = \langle g, f\rangle, \quad \forall f, g \in \mathbf{C}(\Omega)$;

(H1.2) $\langle \lambda f, g\rangle = \lambda\langle f, g\rangle, \quad \forall f, g \in \mathbf{C}(\Omega), \quad \forall \lambda \in \mathbb{R}$;

(H1.3) $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle, \quad \forall f, g, h \in \mathbf{C}(\Omega)$;

(H1.4) $\|f\| \triangleq \sqrt{\langle f, f\rangle} = \overline{\mathrm{Var}}f \geqslant 0, \quad \forall f \in \mathbf{C}(\Omega)$;

(H1.5) $\|f\| = 0 \Leftrightarrow f = \text{Constant}$.

Therefore, $\mathbf{C}(\Omega)$ is a *quasi-Euclidean space* (see [8, 14, 17]), and we have

$$\overline{\mathrm{Var}}(f+g) = \|f+g\| \leqslant \|f\| + \|g\| = \overline{\mathrm{Var}}f + \overline{\mathrm{Var}}g.$$

The inequality (1.2) is proved. Equality in (1.2) holds if and only if there exist two constants $C_1 \geqslant 0,\ C_2 \geqslant 0,\ C_1 \neq 0$ or $C_2 \neq 0$, such that

$$C_1(f - \mathrm{E}f) = C_2(g - \mathrm{E}g),$$

i.e., there exist two constants $C \in [0,\infty)$ and $C_0 \in \mathbb{R}$ such that (1.3) holds.

We say that $\|f\| \triangleq \overline{\mathrm{Var}}f$ is a *semi-norm* of the function f.

Brunn-Minkowski-type inequality has a wide range of applications, especially in probability and statistics, algebraic geometry, and space science (see [3, 4, 7, 8, 14, 17, 19]).

In this paper, we will introduce a new Brunn-Minkowski-type inequality, that is, we will extend inequality (1.2) to the case where $\lambda \in [1,2]$. Our main result is as follows:

Theorem 1.1 (Brunn-Minkowski-type inequality)**.** *Let $X \in \Omega$ be a continuous random variable and its probability density function $p : \Omega \to (0,\infty)$ be continuous, and let the functions $f : \Omega \to (0,\infty)$ and $g : \Omega \to (0,\infty)$ be continuous. If $\gamma \in [1,2]$, then we have the following Brunn-Minkowski-type inequality:*

$$\overline{\mathrm{Var}}^{[\gamma]}(f+g) \leqslant \overline{\mathrm{Var}}^{[\gamma]}f + \overline{\mathrm{Var}}^{[\gamma]}g. \tag{1.4}$$

Equality in (1.4) *holds if f/g is a constant function.*

We remark here that if

$$\gamma = 2,\ f \equiv Cg + C_0,$$

where C and C_0 are constants, then equality in (1.4) also holds. But if $\gamma > 2$, then inequality (1.4) does not hold in general. For instance, if $g > 0$ is a constant function, then the reverse inequality in (1.4) holds true, i.e., the following inequality

$$\overline{\mathrm{Var}}^{[\gamma]}(f + C) \geqslant \overline{\mathrm{Var}}^{[\gamma]} f + \overline{\mathrm{Var}}^{[\gamma]} C = \overline{\mathrm{Var}}^{[\gamma]} f \tag{1.5}$$

holds for $\gamma > 2$, $C > 0$.

Under the hypotheses of Theorem 1.1, we can think of the function g as a *perturbation* of the function f, and the function $f + g$ as an *expansion* of the function f. In order to study the stability of the *expansion function* $f + g$, we need to estimate the upper bounds of $\overline{\mathrm{Var}}^{[\gamma]}(f + g)$. According to Theorem 1.1, a sharp upper bound of $\overline{\mathrm{Var}}^{[\gamma]}(f + g)$ is

$$\overline{\mathrm{Var}}^{[\gamma]} f + \overline{\mathrm{Var}}^{[\gamma]} g.$$

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need several notations as follows:

$$\mathbf{x} \triangleq (x_1, \ldots, x_n),\ \phi(\mathbf{x}) \triangleq (\phi(x_1), \ldots, \phi(x_n)),\ \mathbf{p} \triangleq (p_1, \ldots, p_n),$$

$$\Omega^n \triangleq \left\{ \mathbf{p} \in (0, \infty)^n \left| \sum_{i=1}^n p_i = 1 \right. \right\},\ S \triangleq \left\{ (t_1, t_2) \in [0, \infty)^2 \left| t_1 + t_2 \leqslant 1 \right. \right\},$$

$$A(\mathbf{x}, \mathbf{p}) \triangleq \sum_{i=1}^n p_i x_i,\ w_{i,j}(\mathbf{x}, \mathbf{p}, t_1, t_2) \triangleq t_1 x_i + t_2 x_j + (1 - t_1 - t_2) A(\mathbf{x}, \mathbf{p}).$$

If $\mathbf{x} \in (0, \infty)^n$, $\mathbf{p} \in \Omega^n$, $\gamma \in \mathbb{R}$, then the nonnegative function

$$\mathrm{Var}^{[\gamma]}(\mathbf{x}, \mathbf{p}) \triangleq \begin{cases} \frac{2}{\gamma(\gamma-1)} \left[A(\mathbf{x}^\gamma, \mathbf{p}) - A^\gamma(\mathbf{x}, \mathbf{p}) \right], & \gamma \neq 0, 1, \\ 2 \left[\log A(\mathbf{x}, \mathbf{p}) - A(\log \mathbf{x}, \mathbf{p}) \right], & \gamma = 0, \\ 2 \left[A(\mathbf{x} \log \mathbf{x}, \mathbf{p}) - A(\mathbf{x}, \mathbf{p}) \log A(\mathbf{x}, \mathbf{p}) \right], & \gamma = 1 \end{cases}$$

is called a *discrete γ-order variance* of the vector $\mathbf{x}$.

In order to prove Theorem 1.1, we need the following four lemmas.

Lemma 2.1 ([14, Lemma 1]). *Let the function $\phi : J \to (-\infty, \infty)$, where J is an interval, be twice continuously differentiable. If $\mathbf{x} \in J^n$, $\mathbf{p} \in \Omega^n$, then we have the following identity:*

$$A(\phi(\mathbf{x}), \mathbf{p}) - \phi(A(\mathbf{x}, \mathbf{p})) = \sum_{1 \leqslant i < j \leqslant n} p_i p_j \left\{ \iint_S \phi''\left[w_{i,j}(\mathbf{x}, \mathbf{p}, t_1, t_2) \right] \mathrm{d}t_1 \mathrm{d}t_2 \right\} (x_i - x_j)^2. \tag{2.1}$$

We remark here that the proof of Lemma 2.1 is difficult (see the proof of Lemma 1 in [14]), which is based on the results in linear algebra.

Lemma 2.2 (Minkowski's inequality [3, 4, 17, 18]). *Let $\mathbf{x}, \mathbf{y} \in (0, \infty)^n$. If $\gamma \in (1, \infty)$, then we have*

$$\left[\sum_{i=1}^n (x_i + y_i)^\gamma \right]^{1/\gamma} \leqslant \left(\sum_{i=1}^n x_i^\gamma \right)^{1/\gamma} + \left(\sum_{i=1}^n y_i^\gamma \right)^{1/\gamma}. \tag{2.2}$$

Inequality (2.2) is reversed if $\gamma \in (0, 1)$. Equality in (2.2) holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

We remark here that inequality (2.2) had been extended in [18] and the authors obtained the following Minkowski-type inequality:

$$[\mathrm{per}H_n(\mathbf{x}+\mathbf{y},\alpha)]^{1/|\alpha|} \geqslant [\mathrm{per}H_n(\mathbf{x},\alpha)]^{1/|\alpha|} + [\mathrm{per}H_n(\mathbf{y},\alpha)]^{1/|\alpha|},$$

where

$$\alpha \in [0,1]^n,\ 0 < |\alpha| \triangleq \sum_{j=1}^{n} \alpha_j \leqslant 1,\ \mathbf{x},\mathbf{y} \in (0,\infty)^n,$$

$H_n(\mathbf{x},\alpha)$ is the Hardy matrix and $\mathrm{per}H_n(\mathbf{x},\alpha)$ is the Hardy function. In [8, 17], the authors obtained the following Minkowski-type inequality: Let $A \in \mathbb{R}^{n\times n}$. If $A^T = A,\ A \geqslant 0$, then for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, we have

$$\sqrt{(\mathbf{x}+\mathbf{y})^T A(\mathbf{x}+\mathbf{y})} \leqslant \sqrt{\mathbf{x}^T A\mathbf{x}} + \sqrt{\mathbf{y}^T A\mathbf{y}}, \tag{2.3}$$

where $\mathbf{x},\mathbf{y}$ are column vectors, and (2.3) was used to study the circuit layout system.

Lemma 2.3 (Minkowski-type inequality)**.** *Let $(a,b) \in (0,\infty)^2$ and $(c,d) \in [0,\infty)^2$. If $\gamma \in [1,2]$, then we have*

$$\left[(a+b)^{\gamma-2}(c+d)^2\right]^{1/\gamma} \leqslant \left(a^{\gamma-2}c^2\right)^{1/\gamma} + \left(b^{\gamma-2}d^2\right)^{1/\gamma}. \tag{2.4}$$

Equality in (2.4) holds if (a,b) and (c,d) are linearly dependent.

Proof. By continuity considerations, we may assume that $\gamma \in (1,2)$. Set

$$\theta \triangleq \frac{2}{\gamma} \in (1,2),\ \ (u,v) \triangleq \left(\frac{c}{a},\frac{d}{b}\right) \in [0,\infty)^2,$$

and

$$\mathbb{H}(u,v,a,b,\theta) \triangleq (a+b)^{1-\theta}(ua+vb)^\theta - (au^\theta + bv^\theta).$$

Without loss of generality, we may assume that $0 \leqslant u \leqslant v$. By the Lagrange mean value theorem and the fact that

$$\mathbb{H}(v,v,a,b,\theta) = 0,$$

we know that there exists $\zeta \in [u,v]$ such that

$$\begin{aligned}
&\left[(a+b)^{\gamma-2}(c+d)^2\right]^{1/\gamma} - \left[(a^{\gamma-2}c^2)^{1/\gamma} + (b^{\gamma-2}d^2)^{1/\gamma}\right]\\
&= (a+b)^{1-\theta}(ua+vb)^\theta - (au^\theta + bv^\theta)\\
&= \mathbb{H}(u,v,a,b,\theta) - \mathbb{H}(v,v,a,b,\theta)\\
&= (u-v)\left.\frac{\partial \mathbb{H}}{\partial u}\right|_{u=\zeta}\\
&= -\theta a(v-u)\left[(a+b)^{1-\theta}(\zeta a+vb)^{\theta-1} - \zeta^{\theta-1}\right]\\
&\leqslant -\theta a(v-u)\left[(a+b)^{1-\theta}(\zeta a+\zeta b)^{\theta-1} - \zeta^{\theta-1}\right]\\
&= 0.
\end{aligned}$$

This shows that inequality (2.4) holds, and equality in (2.4) holds if (a,b) and (c,d) are linearly dependent. The proof is completed. □

Lemma 2.4 (Minkowski-type inequality)**.** *Let $\mathbf{x},\mathbf{y} \in [0,\infty)^n$ and $\mathbf{p} \in \Omega^n$. If $\gamma \in [1,2]$, then we have*

$$\left[\mathrm{Var}^{[\gamma]}(\mathbf{x}+\mathbf{y},\mathbf{p})\right]^{1/\gamma} \leqslant \left[\mathrm{Var}^{[\gamma]}(\mathbf{x},\mathbf{p})\right]^{1/\gamma} + \left[\mathrm{Var}^{[\gamma]}(\mathbf{y},\mathbf{p})\right]^{1/\gamma}. \tag{2.5}$$

Equality in (2.5) holds if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

Proof. By the continuity considerations, we may assume that

$$1<\gamma<2,\ p_i>0,\quad \forall\, i:1\leqslant i\leqslant n,\ \text{ and } x_i\neq x_j,\ y_i\neq y_j,\forall\, i,j:\ i\neq j, 1\leqslant i,j\leqslant n.$$

Let $G\triangleq\{\Delta S_1,\Delta S_2,\ldots,\Delta S_l\}$ be a partition of the triangle:

$$S\triangleq\left\{(t_1,t_2)\in[0,\infty)^2:\ t_1+t_2\leqslant 1\right\},$$

and let the area of each ΔS_i be denoted by

$$|\Delta S_i|\triangleq \mathrm{Area}\Delta S_i,$$

as well as let

$$\|G\|\triangleq\max_{1\leqslant k\leqslant l}\max_{\mathbf{X},\mathbf{Y}\in\Delta S_k}\{\|\,\mathbf{X}-\mathbf{Y}\,\|\},$$

be the *diameter* of the partition G, where $\|\mathbf{X}-\mathbf{Y}\|$ is the Euclidean norm of the vector $\mathbf{X}-\mathbf{Y}$. By the definition of the Riemann integral, for any $(\xi_{k,1},\xi_{k,2})\in\Delta S_k$, we have

$$\iint_S [w_{i,j}(\mathbf{x},\mathbf{p},t_1,t_2)]^{\gamma-2}\mathrm{d}t_1\mathrm{d}t_2=\lim_{\|G\|\to 0}\sum_{k=1}^{l}[w_{i,j}(\mathbf{x},\mathbf{p},\xi_{k,1},\xi_{k,2})]^{\gamma-2}|\Delta S_k|. \tag{2.6}$$

By (2.1) and (2.6), we have

$$\begin{aligned}\mathrm{Var}^{[\gamma]}(\mathbf{x},\mathbf{p})&=2\sum_{1\leqslant i<j\leqslant n}p_ip_j(x_i-x_j)^2\iint_S[w_{i,j}(\mathbf{x},\mathbf{p},t_1,t_2)]^{\gamma-2}\mathrm{d}t_1\mathrm{d}t_2\\ &=2\sum_{1\leqslant i<j\leqslant n}p_ip_j(x_i-x_j)^2\lim_{\|G\|\to 0}\sum_{k=1}^{l}[w_{i,j}(\mathbf{x},\mathbf{p},\xi_{k,1},\xi_{k,2})]^{\gamma-2}|\Delta S_k|\\ &=\lim_{\|G\|\to 0}\left\{2\sum_{1\leqslant i<j\leqslant n}p_ip_j(x_i-x_j)^2\sum_{k=1}^{l}[w_{i,j}(\mathbf{x},\mathbf{p},\xi_{k,1},\xi_{k,2})]^{\gamma-2}|\Delta S_k|\right\}\\ &=\lim_{\|G\|\to 0}\left\{\sum_{1\leqslant i<j\leqslant n,1\leqslant k\leqslant l}2p_ip_j|\Delta S_k|(x_i-x_j)^2[w_{i,j}(\mathbf{x},\mathbf{p},\xi_{k,1},\xi_{k,2})]^{\gamma-2}\right\},\end{aligned}$$

i.e.,

$$\mathrm{Var}^{[\gamma]}(\mathbf{x},\mathbf{p})=\lim_{\|G\|\to 0}\left\{\sum_{1\leqslant i<j\leqslant n,1\leqslant k\leqslant l}2p_ip_j|\Delta S_k|(x_i-x_j)^2[w_{i,j}(\mathbf{x},\mathbf{p},\xi_{k,1},\xi_{k,2})]^{\gamma-2}\right\}. \tag{2.7}$$

From (2.7) we obtain that

$$\begin{aligned}\mathrm{Var}^{[\gamma]}(\mathbf{x}+\mathbf{y},\mathbf{p})&=\lim_{\|G\|\to 0}\sum_{1\leqslant i<j\leqslant n,1\leqslant k\leqslant l}2p_ip_j|\Delta S_k|(x_i+y_i-x_j-y_j)^2[w_{i,j}(\mathbf{x}+\mathbf{y},\mathbf{p},\xi_{k,1},\xi_{k,2})]^{\gamma-2}\\ &\leqslant\lim_{\|G\|\to 0}\sum_{1\leqslant i<j\leqslant n,1\leqslant k\leqslant l}2p_ip_j|\Delta S_k|(|x_i-x_j|+|y_i-y_j|)^2[w_{i,j}(\mathbf{x}+\mathbf{y},\mathbf{p},\xi_{k,1},\xi_{k,2})]^{\gamma-2},\end{aligned} \tag{2.8}$$

where

$$w_{i,j}(\mathbf{x},\mathbf{p},\xi_{k,1},\xi_{k,2})=\xi_{k,1}x_i+\xi_{k,2}x_j+(1-\xi_{k,1}-\xi_{k,2})A(\mathbf{x},\mathbf{p})\in[0,\infty).$$

Set

$$a_{i,j,k}=w_{i,j}(\mathbf{x},\mathbf{p},\xi_{k,1},\xi_{k,2}),\ b_{i,j,k}=w_{i,j}(\mathbf{y},\mathbf{p},\xi_{k,1},\xi_{k,2}),$$

and

$$c_{i,j,k} = \sqrt{2p_ip_j|\Delta S_k|}|x_i - x_j|,\ d_{i,j,k} = \sqrt{2p_ip_j|\Delta S_k|}|y_i - y_j|.$$

By (2.8) and

$$w_{i,j}(\mathbf{x}+\mathbf{y}, \mathbf{p}, \xi_{k,1}, \xi_{k,2}) = w_{i,j}(\mathbf{x}, \mathbf{p}, \xi_{k,1}, \xi_{k,2}) + w_{i,j}(\mathbf{y}, \mathbf{p}, \xi_{k,1}, \xi_{k,2}) = a_{i,j,k} + b_{i,j,k}, \tag{2.9}$$

we get

$$\begin{aligned} \mathrm{Var}^{[\gamma]}(\mathbf{x}+\mathbf{y}, \mathbf{p}) &\leqslant \lim_{\|G\|\to 0} \sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} 2p_ip_j|\Delta S_k|\left(|x_i - x_j| + |y_i - y_j|\right)^2 \left[w_{i,j}(\mathbf{x}+\mathbf{y}, \mathbf{p}, \xi_{k,1}, \xi_{k,2})\right]^{\gamma-2} \\ &= \lim_{\|G\|\to 0} \sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} (a_{i,j,k} + b_{i,j,k})^{\gamma-2}(c_{i,j,k} + d_{i,j,k})^2. \end{aligned} \tag{2.10}$$

Since

$$(a_{i,j,k}, b_{i,j,k}) \in (0,\infty)^2,\ (c_{i,j,k}, d_{i,j,k}) \in (0,\infty)^2,\ 1 < \gamma < 2,$$

according to Lemma 2.3, we have

$$(a_{i,j,k} + b_{i,j,k})^{\gamma-2}(c_{i,j,k} + d_{i,j,k})^2 \leqslant \left[\left(a_{i,j,k}^{\gamma-2}c_{i,j,k}^2\right)^{1/\gamma} + \left(b_{i,j,k}^{\gamma-2}d_{i,j,k}^2\right)^{1/\gamma}\right]^{\gamma}. \tag{2.11}$$

By (2.10), (2.11), $1 < \gamma < 2$ and Lemma 2.2, we see that

$$\begin{aligned} \left[\mathrm{Var}^{[\gamma]}(\mathbf{x}+\mathbf{y}, \mathbf{p})\right]^{1/\gamma} &\leqslant \lim_{\|G\|\to 0} \left[\sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} (a_{i,j,k} + b_{i,j,k})^{\gamma-2}(c_{i,j,k} + d_{i,j,k})^2\right]^{1/\gamma} \\ &\leqslant \lim_{\|G\|\to 0} \left\{\sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} \left[(a_{i,j,k}^{\gamma-2}c_{i,j,k}^2)^{1/\gamma} + (b_{i,j,k}^{\gamma-2}d_{i,j,k}^2)^{1/\gamma}\right]^{\gamma}\right\}^{1/\gamma} \\ &\leqslant \lim_{\|G\|\to 0} \left[\left(\sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} a_{i,j,k}^{\gamma-2}c_{i,j,k}^2\right)^{1/\gamma} + \left(\sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} b_{i,j,k}^{\gamma-2}d_{i,j,k}^2\right)^{1/\gamma}\right] \\ &= \lim_{\|G\|\to 0} \left(\sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} a_{i,j,k}^{\gamma-2}c_{i,j,k}^2\right)^{1/\gamma} + \lim_{\|G\|\to 0} \left(\sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} b_{i,j,k}^{\gamma-2}d_{i,j,k}^2\right)^{1/\gamma} \\ &= \left[\mathrm{Var}^{[\gamma]}(\mathbf{x}, \mathbf{p})\right]^{1/\gamma} + \left[\mathrm{Var}^{[\gamma]}(\mathbf{y}, \mathbf{p})\right]^{1/\gamma}. \end{aligned}$$

This shows that inequality (2.5) is correct, and equality in (2.5) holds if x and y are linearly dependent. This ends the proof of Lemma 2.4. □

Now let us start to prove Theorem 1.1.

Proof. By continuity considerations, we may assume that $1 < \gamma < 2$. Let $T \triangleq \{\Delta\Omega_1, \ldots, \Delta\Omega_n\}$ be a partition of Ω, and let

$$\|T\| \triangleq \max_{1\leqslant i\leqslant n} \max_{\mathbf{X},\mathbf{Y}\in\Delta\Omega_i} \{\| \mathbf{X} - \mathbf{Y} \|\}$$

be the *diameter* of the partition T. Pick any $\xi_i \in \Delta\Omega_i$ for each $i = 1, 2, \ldots, n$. Set

$$\xi \triangleq (\xi_1, \xi_2, \ldots, \xi_n),\ f(\xi) \triangleq (f(\xi_1), f(\xi_2), \ldots, f(\xi_n)),$$

and

$$\overline{\mathbf{p}}(\xi) \triangleq (\overline{p}_1(\xi), \overline{p}_2(\xi), \ldots, \overline{p}_n(\xi)) = \frac{(p(\xi_1)|\Delta\Omega_1|, p(\xi_2)|\Delta\Omega_2|, \ldots, p(\xi_n)|\Delta\Omega_n|)}{\sum_{i=1}^n p(\xi_i)|\Delta\Omega_i|}.$$

Then

$$\overline{\mathbf{p}}(\xi) \in \Omega^n \text{ and } \lim_{\|T\|\to 0} \sum_{i=1}^n p(\xi_i)|\Delta\Omega_i| = \int_\Omega p = 1, \tag{2.12}$$

where $|\Delta\Omega_i|$ is the m-dimensional measure of $\Delta\Omega_i$, $i = 1, 2, \ldots, n$.

From (2.12), $\gamma > 1$ and the definition of the Riemann integral, we have

$$\begin{aligned}
\mathrm{Var}^{[\gamma]} f &= \frac{2}{\gamma(\gamma-1)} \left[\mathrm{E} f^\gamma - (\mathrm{E} f)^\gamma\right] \\
&= \frac{2}{\gamma(\gamma-1)} \left[\lim_{\|T\|\to 0} \sum_{i=1}^n p(\xi_i) f^\gamma(\xi_i)|\Delta\Omega_i| - \left(\lim_{\|T\|\to 0} \sum_{i=1}^n p(\xi_i) f(\xi_i)|\Delta\Omega_i|\right)^\gamma\right] \\
&= \frac{2}{\gamma(\gamma-1)} \left[\left(\lim_{\|T\|\to 0} \sum_{i=1}^n p(\xi_i)|\Delta\Omega_i|\right)\left(\lim_{\|T\|\to 0} \sum_{i=1}^n \overline{p}_i(\xi) f^\gamma(\xi_i)\right)\right. \\
&\quad \left. - \left(\lim_{\|T\|\to 0} \sum_{i=1}^n p(\xi_i)|\Delta\Omega_i|\right)^\gamma \left(\lim_{\|T\|\to 0} \sum_{i=1}^n \overline{p}_i(\xi) f(\xi_i)\right)^\gamma\right] \\
&= \frac{2}{\gamma(\gamma-1)} \left[\lim_{\|T\|\to 0} \sum_{i=1}^n \overline{p}_i(\xi) f^\gamma(\xi_i) - \left(\lim_{\|T\|\to 0} \sum_{i=1}^n \overline{p}_i(\xi) f(\xi_i)\right)^\gamma\right] \\
&= \lim_{\|T\|\to 0} \frac{2}{\gamma(\gamma-1)} \left[\sum_{i=1}^n \overline{p}_i(\xi) f^\gamma(\xi_i) - \left(\sum_{i=1}^n \overline{p}_i(\xi) f(\xi_i)\right)^\gamma\right] \\
&= \lim_{\|T\|\to 0} \mathrm{Var}^{[\gamma]}(f(\xi), \overline{\mathbf{p}}(\xi)),
\end{aligned}$$

i.e.,

$$\mathrm{Var}^{[\gamma]} f = \lim_{\|T\|\to 0} \mathrm{Var}^{[\gamma]}(f(\xi), \overline{\mathbf{p}}(\xi)). \tag{2.13}$$

By (2.13) and Lemma 2.4, we obtain that

$$\begin{aligned}
\overline{\mathrm{Var}}^{[\gamma]}(f+g) &= \lim_{\|T\|\to 0} \left[\mathrm{Var}^{[\gamma]}(f(\xi)+g(\xi), \overline{\mathbf{p}}(\xi))\right]^{1/\gamma} \\
&\leqslant \lim_{\|T\|\to 0} \left\{\left[\mathrm{Var}^{[\gamma]}(f(\xi), \overline{\mathbf{p}}(\xi))\right]^{1/\gamma} + \left[\mathrm{Var}^{[\gamma]}(g(\xi), \overline{\mathbf{p}}(\xi))\right]^{1/\gamma}\right\} \\
&= \lim_{\|T\|\to 0} \left[\mathrm{Var}^{[\gamma]}(f(\xi), \overline{\mathbf{p}}(\xi))\right]^{1/\gamma} + \lim_{\|T\|\to 0} \left[\mathrm{Var}^{[\gamma]}(g(\xi), \overline{\mathbf{p}}(\xi))\right]^{1/\gamma} \\
&= \overline{\mathrm{Var}}^{[\gamma]} f + \overline{\mathrm{Var}}^{[\gamma]} g.
\end{aligned}$$

The inequality (1.4) is proved.

Based on the above analysis we know that equality in (1.4) holds if f/g is a constant function. This completes the proof of Theorem 1.1. □

A large number of algebra, functional analysis and inequality theories are used in the proof of Theorem 1.1. Based on these tools, we obtained Lemmas 2.1, 2.2, and 2.3. And then according to these lemmas and the definition of the Riemann integral, we prove Lemma 2.4 which is the discrete form of the result of Theorem 1.1. Finally, by applying Lemma 2.4 and the definition of the Riemann integral, we finish the proof of Theorem 1.1.

3. Proof of inequality (1.5)

Proof. Recalling the proof of Theorem 1.1, and setting

$$g(t) = C > 0, \quad \forall t \in \Omega$$

and

$$\mathbf{y} = (C, C, \ldots, C) \in (0, \infty)^n.$$

By (2.9), we get

$$w_{i,j}(\mathbf{x}+\mathbf{y}, \mathbf{p}, \xi_{k,1}, \xi_{k,2}) = a_{i,j,k} + b_{i,j,k} = a_{i,j,k} + C > a_{i,j,k}. \tag{3.1}$$

From (2.7), (3.1) and $\gamma > 2$, we obtain that

$$\begin{aligned}
\mathrm{Var}^{[\gamma]}(\mathbf{x}+\mathbf{y}, \mathbf{p}) &= \lim_{\|G\|\to 0} \sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} 2p_ip_j|\Delta S_k|(x_i+y_i-x_j-y_j)^2[w_{i,j}(\mathbf{x}+\mathbf{y}, \mathbf{p}, \xi_{k,1}, \xi_{k,2})]^{\gamma-2} \\
&= \lim_{\|G\|\to 0} \sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} 2p_ip_j|\Delta S_k|(x_i-x_j)^2[w_{i,j}(\mathbf{x}+\mathbf{y}, \mathbf{p}, \xi_{k,1}, \xi_{k,2})]^{\gamma-2} \\
&\geqslant \lim_{\|G\|\to 0} \sum_{1\leqslant i<j\leqslant n, 1\leqslant k\leqslant l} 2p_ip_j|\Delta S_k|(x_i-x_j)^2\left(a_{i,j,k}\right)^{\gamma-2} \\
&= \mathrm{Var}^{[\gamma]}(\mathbf{x}, \mathbf{p}),
\end{aligned}$$

that is,

$$\mathrm{Var}^{[\gamma]}(\mathbf{x}+\mathbf{y}, \mathbf{p}) \geqslant \mathrm{Var}^{[\gamma]}(\mathbf{x}, \mathbf{p}). \tag{3.2}$$

By (3.2), we get

$$\mathrm{Var}^{[\gamma]}(f(\xi)+g(\xi), \overline{\mathbf{p}}(\xi)) \geqslant \mathrm{Var}^{[\gamma]}(f(\xi), \overline{\mathbf{p}}(\xi)). \tag{3.3}$$

By (2.13), (3.3) and $\overline{\mathrm{Var}}^{[\gamma]}C = 0$, we obtain that

$$\begin{aligned}
\overline{\mathrm{Var}}^{[\gamma]}(f+g) &= \lim_{\|T\|\to 0}\left[\mathrm{Var}^{[\gamma]}(f(\xi)+g(\xi), \overline{\mathbf{p}}(\xi))\right]^{1/\gamma} \\
&\geqslant \lim_{\|T\|\to 0}\left[\mathrm{Var}^{[\gamma]}(f(\xi), \overline{\mathbf{p}}(\xi))\right]^{1/\gamma} \\
&= \overline{\mathrm{Var}}^{[\gamma]}f \\
&= \overline{\mathrm{Var}}^{[\gamma]}f + \overline{\mathrm{Var}}^{[\gamma]}C.
\end{aligned}$$

This completes the proof of inequality (1.5). □

4. Application to the performance appraisal of education

In higher education, we are most concerned about the students' academic performance. Suppose that X is a random variable with probability density function given by

$$p(X) \triangleq \frac{2}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(X-\mu)^2}{2\sigma^2}\right], \ \mu, \sigma \in (0, \infty), \ X \in \Omega \triangleq (\mu, \infty). \tag{4.1}$$

Firstly, we shall prove the following equality:

$$\mathrm{Var}^{[\gamma]}\mathbb{A}\left(X; \mu, \alpha, c\right) = \frac{2c^\gamma\left(\sqrt{2}\sigma\right)^{\alpha\gamma}}{\gamma(\gamma-1)}\left\{\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{\alpha\gamma+1}{2}\right) - \left[\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)\right]^\gamma\right\}, \tag{4.2}$$

where

$$\mathbb{A}(X;\mu,\alpha,c) \triangleq c\,(X-\mu)^{\alpha}\,,\ \ X\in\Omega,\ \ \alpha,c\in(0,\infty) \tag{4.3}$$

is an *allowance function* [16], $\Gamma(s)$ is the gamma function, $\gamma > -1/\alpha$ and $\gamma \neq 0,1$.

Proof. Let $c=1$. Since $\int_\Omega p = \int_\mu^\infty p = 1$,

$$\int_\mu^\infty \frac{2\,(X-\mu)^{\alpha\gamma}}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(X-\mu)^2}{2\sigma^2}\right]\mathrm{d}X = \int_0^\infty \frac{2x^{\alpha\gamma}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x,$$

and

$$\int_\mu^\infty \frac{2\,(X-\mu)^{\alpha}}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(X-\mu)^2}{2\sigma^2}\right]\mathrm{d}X = \int_0^\infty \frac{2x^{\alpha}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x,$$

we have

$$\begin{aligned}\mathrm{Var}^{[\gamma]}\mathbb{A}(X;\mu,\alpha,c) &= \frac{2}{\gamma(\gamma-1)}\left[\int_\mu^\infty p(X)\mathbb{A}^\gamma(X;\mu,\alpha,c)\,\mathrm{d}X - \left(\int_\mu^\infty p(X)\mathbb{A}(X;\mu,\alpha,c)\,\mathrm{d}X\right)^\gamma\right]\\ &= \frac{2}{\gamma(\gamma-1)}\left\{\int_0^\infty \frac{2x^{\alpha\gamma}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x - \left[\int_0^\infty \frac{2x^{\alpha}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x\right]^\gamma\right\}.\end{aligned}$$

Since

$$\int_0^\infty \frac{2x^{\alpha}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x = \int_0^\infty \frac{2\left(\sqrt{2t}\sigma\right)^\alpha}{\sqrt{2\pi}\sigma}e^{-t}\mathrm{d}\left(\sqrt{2t}\sigma\right) = \frac{\left(\sqrt{2}\sigma\right)^\alpha}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right),$$

we see that

$$\begin{aligned}\mathrm{Var}^{[\gamma]}\mathbb{A}(X;\mu,\alpha,c) &= \frac{2}{\gamma(\gamma-1)}\left\{\frac{\left(\sqrt{2}\sigma\right)^{\alpha\gamma}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha\gamma+1}{2}\right) - \left[\frac{\left(\sqrt{2}\sigma\right)^{\alpha}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)\right]^\gamma\right\}\\ &= \frac{2\left(\sqrt{2}\sigma\right)^{\alpha\gamma}}{\gamma(\gamma-1)}\left\{\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{\alpha\gamma+1}{2}\right) - \left[\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)\right]^\gamma\right\}.\end{aligned}$$

That is, (4.2) holds for the case of $c=1$.

For $c>0$, from

$$\mathrm{Var}^{[\gamma]}\mathbb{A}(X;\mu,\alpha,c) = c^\gamma\mathrm{Var}^{[\gamma]}\mathbb{A}(X;\mu,\alpha,1)\,,$$

we see that (4.2) also holds. The formula (4.2) is proved. □

The usual teaching model assumes that the scores of each student in a class of a university is treated as a continuous random variable, written as X_I, which takes on some value in the real interval $I=[a_0,a_m]$, and its probability density function $p_I : I \to (0,\infty)$ is continuous. Suppose that we divide the students into m classes, written as

$$\mathrm{Class}\,[a_0,a_1]\,,\mathrm{Class}\,[a_1,a_2]\,,\ldots,\mathrm{Class}\,[a_{m-1},a_m]\,,$$

where $0\leqslant a_0\leqslant a_1\leqslant\cdots\leqslant a_m,\ m\geqslant 2$, and $a_i,\ a_{i+1},\ i=0,1,\cdots,m-1$, are the lowest and the highest allowable scores of the students of the Class $[a_i,a_{i+1}]$, respectively. We say that the set

$$\mathrm{HTM}\,\{a_0,\ldots,a_m,p_I\} \triangleq \{\mathrm{Class}[a_0,a_1],\mathrm{Class}[a_1,a_2],\ldots,\mathrm{Class}[a_{m-1},a_m],p_I\}$$

is a *hierarchical teaching model*. Particularly, the *traditional teaching model,* denoted by HTM $\{a_0,a_m,p_I\}$, is just a special HTM $\{a_0,\ldots,a_m,p_I\}$, where $m=1$ (see [16]).

If $a_0 = -\infty$ and $a_m = \infty$, then we say that HTM $\{-\infty, \ldots, \infty, p_{\mathbb{R}}\}$ and HTM $\{-\infty, \infty, p_{\mathbb{R}}\}$ are *generalized hierarchical teaching model* and *generalized traditional teaching model*, respectively (see [16]).

In the generalized traditional teaching model HTM $\{-\infty, \infty, p_{\mathbb{R}}\}$, the score of the student in a class is treated as a continuous random variable, written as $X_{\mathbb{R}}$, and its probability density function $p_{\mathbb{R}} : \mathbb{R} \to (0, \infty)$ is continuous. By the central limit theorem (see [10]), we may assume that the random variable $X_{\mathbb{R}}$ follows a normal distribution (see [11, 13]), i.e., the probability density function of $X_{\mathbb{R}}$ is

$$p_{\mathbb{R}}(X) \triangleq \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(X-\mu)^2}{2\sigma^2}\right], \ \mu, \sigma \in (0, \infty), \quad X \in \mathbb{R}, \tag{4.4}$$

where μ is the average score of the students and σ is the mean variance of the scores.

We remark here that if the score X_I of each student satisfies $X_I \in [0, 1]$, and $\mu \in [0, 1]$, then, by (4.4), we have

$$P(X_{\mathbb{R}} < 0) = \int_{-\infty}^{0} p_{\mathbb{R}}(t)\mathrm{d}t \approx 0 \text{ and } P(X_{\mathbb{R}} > 1) = \int_{1}^{\infty} p_{\mathbb{R}}(t)\mathrm{d}t \approx 0.$$

Hence we can use the generalized traditional teaching model instead of the traditional teaching model, approximately.

In the generalized traditional teaching model, in order to stimulate the learning enthusiasm of students, we may give the student a bonus payment $\mathbb{A}(X)$ if $X > \mu$, the function $\mathbb{A} : (\mu, \infty) \to (0, \infty)$ may be regarded as an *allowance function*, where $X = X_{(0,\infty)} \in (0, \infty)$ is a truncated random variable of the random variable $X_{\mathbb{R}}$, which is the score of the student. The probability density function $p(X)$ of X should be defined by (4.1). Indeed, by the definition of the truncated random variable (see [16]), we have

$$p(X) = \frac{p_{\mathbb{R}}(X)}{\int_{\mu}^{\infty} p_{\mathbb{R}}(t)\mathrm{d}t} = 2p_{\mathbb{R}}(X) = \frac{2}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(X-\mu)^2}{2\sigma^2}\right].$$

For example, we consider

$$\mathbb{A}(X) \equiv \mathbb{A}(X; \mu, \alpha_1, c_1) + \mathbb{A}(X; \mu, \alpha_2, c_2), \ \alpha_i > 0, \ c_i > 0, \ i = 1, 2.$$

Here we may interpret $\mathbb{A}(X; \mu, \alpha_1, c_1)$ as the allowance from the university, and $\mathbb{A}(X, \mu, \alpha_2, c_2)$ as the allowance from the parents of the student. Then, Theorem 1.1 implies the following Brunn-Minkowski-type inequality:

$$\overline{\mathrm{Var}}^{[\gamma]}\mathbb{A}(X) \leqslant \overline{\mathrm{Var}}^{[\gamma]}\mathbb{A}(X; \mu, \alpha_1, c_1) + \overline{\mathrm{Var}}^{[\gamma]}\mathbb{A}(X; \mu, \alpha_2, c_2), \quad \forall \gamma \in [1, 2], \tag{4.5}$$

where

$$\overline{\mathrm{Var}}^{[\gamma]}\mathbb{A}(X; \mu, \alpha_i, c_i) = \left[\mathrm{Var}^{[\gamma]}\mathbb{A}(X; \mu, \alpha_i, c_i)\right]^{1/\gamma}, \ i = 1, 2,$$

which can be calculated by (4.2). Equality in (4.5) holds if $\alpha_1 = \alpha_2$.

Inequality (4.5) provides a better upper bound of $\overline{\mathrm{Var}}^{[\gamma]}\mathbb{A}(X)$. Since the calculation of $\overline{\mathrm{Var}}^{[\gamma]}\mathbb{A}(X)$ is very difficult, the inequality (4.5) has important theoretical significance.

According to the theory of mathematical analysis, we know that an integral analogue of inequality (4.5) is the following: Let the allowance function $\mathbb{A}(X, \mu, \alpha, c)$ be defined by (4.3), where $\alpha \in [a, b]$, $0 < a < b$. If $\gamma \in [1, 2]$, then for any constants $\mu > 0$ and $c > 0$, we have the following Brunn-Minkowski-type inequality:

$$\overline{\mathrm{Var}}^{[\gamma]}\left[\int_a^b \mathbb{A}(X; \mu, \alpha, c)\,\mathrm{d}\alpha\right] \leqslant \int_a^b \overline{\mathrm{Var}}^{[\gamma]}\mathbb{A}(X; \mu, \alpha, c)\,\mathrm{d}\alpha,$$

i.e.,

$$\overline{\mathrm{Var}}^{[\gamma]}\left[\frac{1}{b-a}\int_a^b \mathbb{A}(X; \mu, \alpha, c)\,\mathrm{d}\alpha\right] \leqslant \frac{1}{b-a}\int_a^b \overline{\mathrm{Var}}^{[\gamma]}\mathbb{A}(X; \mu, \alpha, c)\,\mathrm{d}\alpha,$$

where

$$\frac{1}{b-a}\int_a^b \mathbb{A}\left(X;\mu,\alpha,c\right)\mathrm{d}\alpha$$

is the *mean allowance function* for the parameter α, and

$$\frac{1}{b-a}\int_a^b \overline{\mathrm{Var}}^{[\gamma]}\mathbb{A}\left(X;\mu,\alpha,c\right)\mathrm{d}\alpha$$

is the *mean of the γ-mean variance* of the allowance function for the parameter α.

5. Application to the performance appraisal of business

In the field of business, we are most concerned about the commercial profit.

Suppose that $\mathbf{C}$ is a product for sale. Then the sales X of $\mathbf{C}$ in a day is a random variable. Assume that $X \in (\mu,\infty),\ \mu>0$, where μ is the minimum quantity of sale, and the probability density function $p:\Omega\to(0,\infty)$ of X is a continuous function. Then

$$P\left(\mu\leqslant X<x\right)=\int_\mu^x p(t)\mathrm{d}t,\ x\in\Omega\triangleq(\mu,\infty),\ \int_\mu^\infty p(t)\mathrm{d}t=1,$$

where $P\left(\mu\leqslant X<x\right)$ is the probability of the random event “ $\mu\leqslant X<x$ ”.

Assume that the price of the product is p_0, where $p_0>0$ is a constant, then the *income function* $\mathbf{i}$, the income of $\mathbf{C}$ in a day, is

$$\mathbf{i}\triangleq\mathbf{i}(X)=p_0X,\ X\in(\mu,\infty),$$

where $\mathbf{i}$ is a random variable.

Assume that the *cost function* (or *expenditure function*) $\mathbf{e}$, the cost of $\mathbf{C}$ in a day, is

$$\mathbf{e}\triangleq\mathbf{e}(X),\ X\in(\mu,\infty),$$

then $\mathbf{e}$ is also a random variable. Here we assume that function $\mathbf{e}(X)$ is continuous, and

$$0\leqslant\mathbf{e}(X)<\mathbf{i}(X),\quad\forall X\in(\mu,\infty).$$

We say that $\mathbf{e}_0=p_0\mu$ is the *fixed cost*, and $\mathbf{e}_1=\mathbf{e}-p_0\mu$ is the *variable cost*.

Therefore, the *profit function* $\mathbf{p}$, the profit of $\mathbf{C}$ in a day, is

$$\mathbf{p}=\mathbf{p}(X)\triangleq\mathbf{i}(X)-\mathbf{e}(X)=p_0X-\mathbf{e}(X)>0,\quad\forall X\in(\mu,\infty),$$

where $\mathbf{p}$ is still a random variable.

Theorem 1.1 implies the following Brunn-Minkowski-type inequality:

$$\overline{\mathrm{Var}}^{[\gamma]}\mathbf{p}\geqslant\overline{\mathrm{Var}}^{[\gamma]}\mathbf{i}-\overline{\mathrm{Var}}^{[\gamma]}\ \mathbf{e},\quad\forall\ \gamma\in[1,2]\,, \tag{5.1}$$

equality in (5.1) holds if $\mathbf{e}(X)/X$ is a constant function.

Roughly, inequality (5.1) says that the γ-mean variance of the profit function is greater than or equal to the γ-mean variance of the income function minus the γ-mean variance of the expenditure function. Equality in (5.1) holds if $\mathbf{e}(X)\equiv CX$, where C is a constant, and $0<C<p_0$. That is to say, if $\mathbf{e}(X)\equiv CX$, then

$$\overline{\mathrm{Var}}^{[\gamma]}\mathbf{p}\equiv\overline{\mathrm{Var}}^{[\gamma]}\mathbf{i}-\overline{\mathrm{Var}}^{[\gamma]}\mathbf{e}\equiv\left(p_0-C\right)\overline{\mathrm{Var}}^{[\gamma]}X.$$

If $\gamma=2$, then equality in (5.1) also holds if $\mathbf{e}(X)\equiv CX+C_0$, which is linearly increasing, where C,C_0 are constants, and

$$CX+C_0<pX\quad\forall X\in\Omega\Leftrightarrow 0\leqslant C_0\leqslant(p-C)\mu.$$

In the field of business, we need to have a very small γ-mean variance $\overline{\mathrm{Var}}^{[\gamma]}\mathbf{p}$.

If $\overline{\mathrm{Var}}^{[\gamma]}\mathbf{i}$ and $\overline{\mathrm{Var}}^{[\gamma]}\mathbf{e}$ are fixed, then by inequality (5.1), we have

$$\min\left\{\overline{\mathrm{Var}}^{[\gamma]}\mathbf{p}\right\} = \overline{\mathrm{Var}}^{[\gamma]}\mathbf{i} - \overline{\mathrm{Var}}^{[\gamma]}\mathbf{e}, \quad \forall\, \gamma \in [1,2]$$

and

$$\min\left\{\overline{\mathrm{Var}}\mathbf{p}\right\} = \overline{\mathrm{Var}}\mathbf{i} - \overline{\mathrm{Var}}\mathbf{e}.$$

If we choose the cost function $\mathbf{e}(X) \equiv CX + C_0$, which is linearly increasing, where the constants C, $C_0 > 0$, then $\overline{\mathrm{Var}}\mathbf{p}$ is the minimal. That is,

$$\overline{\mathrm{Var}}\mathbf{p}\Big|_{\mathbf{e}(X)\equiv CX+C_0} = \min\left\{\overline{\mathrm{Var}}\mathbf{p}\right\}.$$

If the company $\mathbf{C}$ sells m products $P_j,\ j = 1, 2, \ldots, m$, then the sales $\mathbf{X} \triangleq (X_1, X_2, \cdots, X_m)$ of $\mathbf{C}$ in a day is an m-dimensional and continuous random vector variable. The income function $\mathbf{i}$ associated with the income of $\mathbf{C}$ in a day is

$$\mathbf{i} \triangleq \mathbf{i}(\mathbf{X}) = \sum_{j=1}^{m} p_{0,j} X_j = \mathbf{p}_0 \mathbf{X},$$

where X_j and $p_{0,j}$ are the sales and the price of the product P_j $(j = 1, 2, \ldots, m)$ respectively. $\mathbf{p}_0\mathbf{X}$ is the inner product of the vectors $\mathbf{p}_0 \triangleq (p_{0,1}, p_{0,2}, \ldots, p_{0,m})$ and $\mathbf{X}$. It is worth noting that the inequality (5.1) also holds true for this case.

Based on the above analysis, we suggest that the cost function $\mathbf{e}$ should be linearly increasing, that is, $\mathbf{e}(X) \equiv CX + C_0$, where the constants $C, C_0 \in (0, \infty)$. In this way, it can maintain the stability of the profit function $\mathbf{p}$, this also enables us to control the cost function $\mathbf{e}$.

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References

[1] E. F. Beckenbach, R. Bellman, *Inequalities*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin-Göttingen-Heidelberg, (1961). 1
[2] M. Bjelica, *Asymptotic planarity of Dresher mean values*, Mat. Vesnik, **57** (2005), 61–63. 1
[3] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (N.S.), **39** (2002), 355–405. 1, 1, 2.2
[4] R. J. Gardner, S. Vassallo, *The Brunn-Minkowski inequality, Minkowski's first inequality, and their duals*, J. Math. Anal. Appl., **245** (2000), 502–512. 1, 1, 2.2
[5] J. M. Danskin, *Dresher's inequality*, Amer. Math. Monthly, **59** (1952), 687–688. 1
[6] M. Dresher, *Moment spaces and inequalities*, Duke Math. J., **20** (1953), 261–271. 1
[7] C. B. Gao, J. J. Wen, *Theory of surround system and associated inequalities*, Comput. Math. Appl., **63** (2012), 1621–1640. 1, 1
[8] T. Y. Han, S. H. Wu, J. J. Wen, *Sharp upper bound involving circuit layout system*, J. Nonlinear Sci. Appl., **9** (2016), 1922–1935. 1, 2
[9] K. Hu, *Some problems of analytic inequalities*, (Chinese) Wuhan University Press, Wuhan, (2003). 1
[10] O. Johnson, *Information theory and the central limit theorem*, Imperial College Press, London, (2004). 4
[11] S. Nadarajah, *A generalized normal distribution*, J. Appl. Stat., **32** (2005), 685–694. 4
[12] Z. Páles, *Inequalities for sums of powers*, J. Math. Anal. Appl., **131** (1988), 265–270. 1
[13] M. K. Varanasi, B. Aazhang, *Parametric generalized Gaussian density estimation*, J. Acoust. Soc. Am., **86** (1989), 1404–1415. 4

[14] J. J. Wen, T. Y. Han, S. S. Cheng, *Inequalities involving Dresher variance mean*, J. Inequal. Appl., **2013** (2013), 29 pages. 1, 1, 2.1, 2

[15] J. J. Wen, T. Y. Han, C. B. Gao, *Convergence tests on constant Dirichlet series*, Comput. Math. Appl., **62** (2011), 3472–3489. 1

[16] J. J. Wen, Y. Huang, S. S. Cheng, *Theory of φ-Jensen variance and its applications in higher education*, J. Inequal. Appl., **2015** (2015), 40 pages. 1, 1, 4, 4

[17] J. J. Wen, S. H. Wu, C. B. Gao, *Sharp lower bounds involving circuit layout system*, J. Inequal. Appl., **2013** (2013), 22 pages. 1, 2.2, 2

[18] J. J. Wen, S. H. Wu, T. Y. Han, *Minkowski-type inequalities involving Hardy function and symmetric functions*, J. Inequal. Appl., **2014** (2014), 17 pages. 2.2, 2

[19] J. J. Wen, J. Yan, S. H. Wu, *Isoperimetric inequalities in surround system and space science*, J. Inequal. Appl., **2016** (2016), 28 pages. 1, 1