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Oscillation of solutions for a class of nonlinear fractional difference equations

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Abstract

In this paper, we investigate the oscillation of the following nonlinear fractional difference equations,

$$\Delta\left(a\left(t\right)\left[\Delta\left(r\left(t\right)\left(\Delta^{\alpha}x\left(t\right)\right)^{\gamma_1}\right)\right]^{\gamma_2}\right)+q\left(t\right)f\left(\sum_{s=t_0}^{t-1+\alpha}\left(t-s-1\right)^{\left(-\alpha\right)}x\left(s\right)\right)=0,$$

where $t \in N_{t_0+1-\alpha}$, γ_1 and γ_2 are the quotient of two odd positive number, and Δ^{α} denotes the Riemann-Liouville fractional difference operator of order $0 < \alpha \leq 1$.

Keywords: Fractional order, fractional difference equation, oscillation, nonlinear, difference equations, oscillatory solutions.
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1. Introduction

Fractional calculus is one of the most novel types of calculus having a broad range of applications in many scientific and engineering disciplines. Order of the derivatives in the fractional calculus might be any real number which separates the fractional calculus from the ordinary calculus where the derivatives are allowed only positive integer numbers. Therefore, fractional calculus might be considered as an extension of ordinary calculus. Fractional calculus is a highly valuable tool in the modeling of many sorts of scientific phenomena in various fields of science, engineering and physics [5, 25, 26]. Recently, many articles have

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investigated some aspects of differential equations with fractional-order derivatives, especially the existence, the uniqueness the methods for explicit and numerical solutions and the stability of solutions, we refer to [6, 8, 9, 15, 16, 23, 24] and the references cited therein. In recent years, oscillatory behaviour of fractional differential equations has been investigated by authors, see papers [2–4, 7, 10, 12, 14, 17–19, 27]. But the discrete analog of fractional difference equations are studied by very few authors, see [13, 20–22].

In [20], Sagayaraj et al. researched oscillation of the following fractional difference equations

$$\Delta\left(p(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma}\right)+q(t)\,f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0$$

for $t\in\mathbb{N}_{t_0+1-\alpha}$, $\alpha\in(0,1]$, and $\gamma>0$ is a quotient of odd positive integers.

In [22], Selvam et al. investigated the oscillation of a class of fractional difference equations with damping term of the following form

$$\Delta\left(c(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma}\right)+p(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma}+q(t)\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)^{\gamma}=0$$

for $t\in\mathbb{N}_{t_0+1-\alpha}$, $\alpha\in(0,1]$, and $\gamma>0$ is a quotient of odd positive integers.

In [13], Li investigated the oscillation of forced fractional difference equations with damping term of the form

$$(1+p(t))\,\Delta\left(\Delta^{\alpha}x(t)\right)+p(t)\,\Delta^{\alpha}x(t)+f(t,x(t))=g(t)\,,\ t\in\mathbb{N}_0$$

with initial condition $\Delta^{\alpha-1}x(t)\,|_{t=0}=x_0$, where $\alpha\in(0,1)$.

In [21], Sagayaraj et al. studied the oscillatory behavior of the fractional difference equations of the following form

$$\Delta\left(p(t)\,\Delta\left(\left[r(t)\left(\Delta^{\alpha}x(t)\right)\right]^{\eta}\right)\right)+F\left(t,\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0$$

for $t\in\mathbb{N}_{t_0+1-\alpha}$, $\alpha\in(0,1]$, and $\eta>0$ is a quotient of odd positive integers.

In this study, we investigate the following fractional differential equations

$$\Delta\left(a(t)\left[\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)\right]^{\gamma_2}\right)+q(t)\,f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0, \tag{1.1}$$

where $t\in\mathbb{N}_{t_0+1-\alpha}$, γ_1 and γ_2 are the quotient of two odd positive numbers. In [13, 20–22], and equation (1.1), Δ^{α} denotes the Riemann-Liouville fractional difference operator of order $0<\alpha\leq 1$. We shall make use of the following conditions in our results:

(C1) $a(t)$, $r(t)$ and $q(t)$ are positive sequences, $\sum_{s=t_0}^{\infty}\left(1/a^{1/\gamma_2}(s)\right)=\infty$, and the function of f belongs to $C(\mathbb{R},\mathbb{R})$, $f(x)/x\geq k$ for all $k\in\mathbb{R}_+$, $x\neq 0$.

As usual, a solution $x(t)$ of (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Equation (1.1) is called oscillatory if all of its solutions are oscillatory.

2. Preliminaries

In this section, we introduce preliminary results of discrete fractional calculus.

Definition 2.1 ([1])**.** The v-th fractional sum f, for $v>0$, is defined by

$$\Delta^{-v}f(t)=\frac{1}{\Gamma(v)}\sum_{s=a}^{t-v}(t-s-1)^{v-1}f(s)\,,$$

where f is defined for $s\equiv a \bmod (1)$, $\Delta^{-v}f$ is defined for $t\equiv(a+v)\bmod(1)$ and $t^{(v)}=\frac{\Gamma(t+1)}{\Gamma(t-v+1)}$. The fractional sum $\Delta^{-v}f$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+v}$, where $\mathbb{N}_t=\{t,t+1,t+2,\cdots\}$.

Definition 2.2 ([1])**.** Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $v = m - \mu$. The μ-th fractional difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-v} f(t) = \Delta^{m} \Delta^{-v} f(t). \tag{2.1}$$

Lemma 2.3 ([11])**.** *Assume that A and B are nonnegative real numbers. Then,*

$$\lambda A B^{\lambda-1} - A^{\lambda} \le (\lambda - 1) B^{\lambda}$$

for all $\lambda > 1$.

3. Main results

In this section, we establish some oscillation criteria. Throughout this paper, we denote $i = 0,1,2,3$, $\delta_1(t,t_i) = \sum_{s=t_i}^{t-1} \left(1/a^{1/\gamma_2}(s)\right)$ and $\Delta\phi_+(s) = \max\{\Delta\phi(s), 0\}$.

Before we state and prove our main results, we give the following lemmas which will play an important role in the proof of our main results.

Lemma 3.1. *Let $x(t)$ be a solution of* (1.1) *and let*

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s),$$

then

$$\Delta(G(t)) = \Gamma(1-\alpha) \Delta^{\alpha} x(t).$$

Proof. Using Definition 2.1

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(1-\alpha)-1} x(s) = \Gamma(1-\alpha) \Delta^{-(1-\alpha)} x(t).$$

So, we obtain

$$\Delta(G(t)) = \Gamma(1-\alpha) \Delta\Delta^{-(1-\alpha)} x(t) = \Gamma(1-\alpha) \Delta^{\alpha} x(t).$$

The proof is complete. □

Lemma 3.2. *Assume $x(t)$ is an eventually positive solution of* (1.1) *and* (C1) *holds. If $\Delta^{\alpha} x(t) > 0$, then $\Delta\left(r(t)(\Delta^{\alpha} x(t))^{\gamma_1}\right) > 0$ for $t \ge t_0$.*

Proof. From the hypothesis, there exist a t_1 such that $x(t) > 0$ on $[t_1, \infty)$, so that $G(t) > 0$ on $[t_1, \infty)$, and from (1.1), we have

$$\Delta\left(a(t)\left[\Delta\left(r(t)(\Delta^{\alpha} x(t))^{\gamma_1}\right)\right]^{\gamma_2}\right) = -q(t) f\left(\sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)\right) < 0. \tag{3.1}$$

Then $a(t)\left[\Delta\left(r(t)(\Delta^{\alpha} x(t))^{\gamma_1}\right)\right]^{\gamma_2}$ is an eventually non increasing sequence on $[t_1, \infty)$. So, we know that $\Delta\left(r(t)(\Delta^{\alpha} x(t))^{\gamma_1}\right)$ is eventually of one sign. For $t_2 > t_1$ is sufficiently large, we claim $\Delta\left(r(t)(\Delta^{\alpha} x(t))^{\gamma_1}\right) > 0$ on $[t_2, \infty)$. Otherwise, assume that there exists a sufficiently large $t_3 > t_2$ such that $\Delta\left(r(t)(\Delta^{\alpha} x(t))^{\gamma_1}\right) < 0$ on $[t_3, \infty)$. For $[t_3, \infty)$, we have

$$a(t)\left[\Delta\left(r(t)(\Delta^{\alpha} x(t))^{\gamma_1}\right)\right]^{\gamma_2} \le a(t_3)\left[\Delta\left(r(t_3)(\Delta^{\alpha} x(t_3))^{\gamma_1}\right)\right]^{\gamma_2} = l < 0.$$

So,

$$\Delta\left(r(t)(\Delta^{\alpha} x(t))^{\gamma_1}\right) \le \frac{l^{1/\gamma_2}}{a^{1/\gamma_2}(t)}. \tag{3.2}$$

Summing both sides of (3.2) from t_3 to $t-1$, we obtain

$$r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}-r(t_3)\left(\Delta^{\alpha}x(t_3)\right)^{\gamma_1}\le\sum_{t_3}^{t-1}\frac{l^{1/\gamma_2}}{a^{1/\gamma_2}(s)}. \tag{3.3}$$

Letting $t\to\infty$ (3.3), we obtain a contradiction with $\Delta^{\alpha}x(t)>0$. This completes the proof. □

Lemma 3.3. *Assume $x(t)$ is an eventually positive solution of* (1.1) *such that $\Delta^{\alpha}x(t)>0$ and*

$$\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)>0$$

on $[t_1,\infty)$, where $t_1>t_0$ is sufficiently large. Then we have

$$\Delta G(t)\ge\frac{\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(t,t_1)\,a^{1/\gamma_1\gamma_2}(t)\left[\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)\right]^{1/\gamma_1}}{r^{1/\gamma_1}(t)} \tag{3.4}$$

for $t_1>t_0$.

Proof. Assume that x is an eventually positive solution of (1.1). Then we have that

$$a(t)\left[\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)\right]^{\gamma_2}$$

is non-increasing on $[t_1,\infty)$ by (3.1) and we have $\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)>0$. So,

$$\begin{aligned}r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}&\ge r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}-r(t_1)\left(\Delta^{\alpha}x(t_1)\right)^{\gamma_1}\\&=\sum_{t_1}^{t-1}\frac{\left(a(s)\left[\Delta\left(r(s)\left(\Delta^{\alpha}x(s)\right)^{\gamma_1}\right)\right]^{\gamma_2}\right)^{1/\gamma_2}}{a^{1/\gamma_2}(s)}\\&\ge a^{1/\gamma_2}(t)\,\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)\sum_{t_1}^{t-1}\frac{1}{a^{1/\gamma_2}(s)}\end{aligned}$$

and then,

$$\Delta^{\alpha}x(t)\ge\frac{a^{1/\gamma_1\gamma_2}(t)\left[\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)\right]^{1/\gamma_1}}{r^{1/\gamma_1}(t)}\left(\sum_{t_1}^{t-1}\frac{1}{a^{1/\gamma_2}(s)}\right)^{1/\gamma_1}.$$

That is,

$$\Delta G(t)\ge\frac{\Gamma(1-\alpha)\,a^{1/\gamma_1\gamma_2}(t)\left[\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)\right]^{1/\gamma_1}\delta_1^{1/\gamma_1}(t,t_1)}{r^{1/\gamma_1}(t)}.$$

So, the proof is complete. □

Theorem 3.4. *Assume* (C1) *and $\gamma_1\gamma_2=1$ hold. If there exists a positive sequence ϕ such that*

$$\lim_{t\to\infty}\sup\sum_{s=t_2}^{t-1}\left(k\phi(s)\,q(s)-\frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_1)}\right)=\infty, \tag{3.5}$$

then every solution of (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is non-oscillatory solution of (1.1). Then without loss of generality, we may assume that there is a solution $x(t)$ of (1.1) such that $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 3.2, we have $\Delta^{\alpha}x(t)>0$ and $\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)>0$ on $[t_2,\infty)$, where t_2 is sufficiently large. Define the following generalized Riccati function:

$$\omega(t)=\phi(t)\frac{a(t)\left[\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)\right]^{\gamma_2}}{G(t)}.$$

For $t\in[t_2,\infty)$, we have

$$\begin{aligned}\Delta\omega(t)&=\Delta\phi(t)\frac{a(t+1)\left[\Delta\left(r(t+1)\left(\Delta^{\alpha}x(t+1)\right)^{\gamma_1}\right)\right]^{\gamma_2}}{G(t+1)}\\&\quad+\phi(t)\frac{\Delta\left(a(t)\left[\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)\right]^{\gamma_2}\right)G(t+1)-a(t+1)\left[\Delta\left(r(t+1)\left(\Delta^{\alpha}x(t+1)\right)^{\gamma_1}\right)\right]^{\gamma_2}\Delta G(t)}{G(t)G(t+1)}.\end{aligned}$$

That is,

$$\Delta\omega(t)=\Delta\phi(t)\frac{\omega(t+1)}{\phi(t+1)}-\phi(t)\frac{q(t)f(G(t))}{G(t)}-\phi(t)\frac{a(t+1)\left[\Delta\left(r(t+1)\left(\Delta^{\alpha}x(t+1)\right)^{\gamma_1}\right)\right]^{\gamma_2}\Delta G(t)}{G^2(t+1)}.$$

Using $f(x)/x>k$ and (3.4),

$$\begin{aligned}\Delta\omega(t)&\leq\Delta\phi_+(t)\frac{\omega(t+1)}{\phi(t+1)}-k\phi(t)q(t)\\&\quad-\phi(t)\frac{a(t+1)\left[\Delta\left(r(t+1)\left(\Delta^{\alpha}x(t+1)\right)^{\gamma_1}\right)\right]^{\gamma_2}\Gamma(1-\alpha)a(t)\left[\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)\right]^{\gamma_2}\delta_1^{1/\gamma_1}(t,t_2)}{G^2(t+1)r^{1/\gamma_1}(t)}.\end{aligned}$$

And, from (3.1), we have

$$a(t)\left[\Delta\left(r(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)\right]^{\gamma_2}\geq a(t+1)\left[\Delta\left(r(t+1)\left(\Delta^{\alpha}x(t+1)\right)^{\gamma_1}\right)\right]^{\gamma_2}.$$

So,

$$\Delta\omega(t)\leq\Delta\phi_+(t)\frac{\omega(t+1)}{\phi(t+1)}-k\phi(t)q(t)-\phi(t)\frac{\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\frac{\omega^2(t+1)}{\phi^2(t+1)}.$$

Setting $\lambda=2$, $A=\left(\frac{\phi(t)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}{r^{1/\gamma_1}(t)}\right)^{1/2}\frac{\omega(t+1)}{\phi(t+1)}$, and $B=\frac{1}{2}\left(\frac{r^{1/\gamma_1}(t)}{\phi(t)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}\right)^{1/2}\Delta\phi_+(t)$, using Lemma 2.3, we obtain

$$\Delta\omega(t)\leq-k\phi(t)q(t)+\frac{r^{1/\gamma_1}(t)\left(\Delta\phi_+(t)\right)^2}{4\phi(t)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(t,t_2)}. \tag{3.6}$$

Summing both sides of (3.6) from t_2 to $t-1$, we have

$$\sum_{s=t_2}^{t-1}\left(k\phi(s)q(s)-\frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}\right)\leq\omega(t_1)-\omega(t)\leq\omega(t_1)<\infty. \tag{3.7}$$

Letting $t\to\infty$ in (3.7),

$$\lim_{t\to\infty}\sup\sum_{s=t_2}^{t-1}\left(k\phi(s)q(s)-\frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\Gamma(1-\alpha)\delta_1^{1/\gamma_1}(s,t_2)}\right)\leq\omega(t_1)<\infty.$$

We obtain a contradiction with (3.5). So, the proof is complete. □

Theorem 3.5. *Assume* (C1) *and* $\gamma_1\gamma_2 = 1$ *hold. Let* ϕ *be a positive sequence. Furthermore, we assume that there exists a double sequence such that*

$$H(t,t) = 0 \text{ for } t \geq 0, \quad H(t,s) > 0 \text{ for } t > s \geq 0,$$
$$\Delta_2 H(t,s) = H(t,s+1) - H(t,s) \leq 0 \text{ for } t > s \geq 0.$$

If

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)} \sum_{t_0}^{t-1} H(t,s)\left(k\phi(s)\,q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)}\right) = \infty, \tag{3.8}$$

then every solution of (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is non-oscillatory solution of (1.1). Then without loss of generality, we may assume that there is a solution $x(t)$ of (1.1) such that $x(t) > 0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 3.2, we have $\Delta^\alpha x(t) > 0$ and $\Delta\left(r(t)\left(\Delta^\alpha x(t)\right)^{\gamma_1}\right) > 0$ on $[t_2,\infty)$, where t_2 is sufficiently large. Let $\omega(t)$, be defined as in Theorem 3.4. So, we have (3.6). Multiplying both sides by $H(t,s)$ and then summing from t_2 to $t-1$, we have

$$\sum_{t_2}^{t-1} H(t,s)\,k\phi(s)\,q(s) \leq -\sum_{t_2}^{t-1} H(t,s)\,\Delta\omega(s) + \sum_{t_2}^{t-1} H(t,s)\frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)},$$
$$\sum_{t_2}^{t-1} H(t,s)\left(k\phi(s)\,q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)}\right) \leq -\sum_{t_2}^{t-1} H(t,s)\,\Delta\omega(s).$$

By using summation by parts formula, we have

$$\begin{aligned}\sum_{t_2}^{t-1} H(t,s)\left(k\phi(s)\,q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)}\right) &\leq H(t,t_2)\,\omega(t_2) + \sum_{t_2}^{t-1}\omega(s+1)\,\Delta_2 H(t,s)\\ &\leq H(t,t_2)\,\omega(t_2)\\ &\leq H(t,t_0)\,\omega(t_2).\end{aligned}$$

Then,

$$\begin{aligned}&\sum_{t_0}^{t-1} H(t,s)\left(k\phi(s)\,q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)}\right)\\ &\quad= \sum_{t_0}^{t_2-1} H(t,s)\left(k\phi(s)\,q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)}\right)\\ &\qquad+ \sum_{t_2}^{t-1} H(t,s)\left(k\phi(s)\,q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)}\right)\\ &\quad\leq H(t,t_0)\,\omega(t_2) + H(t,t_0)\sum_{t_0}^{t_2-1}\left|k\phi(s)\,q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)}\right|.\end{aligned}$$

So,

$$\begin{aligned}&\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)} \sum_{t_0}^{t-1} H(t,s)\left(k\phi(s)\,q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)}\right)\\ &\quad\leq \omega(t_2) + \sum_{t_0}^{t_2-1}\left|k\phi(s)\,q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)}\right| < \infty,\end{aligned}$$

which contradicts (3.8). So the proof is complete. □

By choosing the sequences H and ϕ in appropriate manners, we can derive a lot of oscillation criteria for (1.1). For instance, we can choose the double sequence $H(t,s) = (t-s)^{\lambda}$ with $\lambda \geq 1$, $t \geq s \geq 0$. So, we have the following corollary.

Corollary 3.6. *Under the conditions of Theorem* 3.5 *and*

$$\lim_{t\to\infty} \sup \frac{1}{(t-t_0)^{\lambda}} \sum_{t_0}^{t-1} (t-s)^{\lambda} \left(k\phi(s)\, q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)} \right) = \infty,$$

every solution of (1.1) *is oscillatory.*

4. Applications

Example 4.1. Consider the following fractional difference equation

$$\begin{aligned} &\Delta\left(t^{1/5}\left[\Delta\left(\left(\Delta^{\alpha} x(t)\right)^5\right)\right]^{1/5}\right) \\ &\quad + t^{-2}\left[\left(\sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)\right) + \left(\sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)\right)^2\right] = 0, \quad t \geq 2. \end{aligned} \tag{4.1}$$

This corresponds to (1.1) with $t_0 = 2$, $\gamma_1 = 5$, $\gamma_2 = 1/5$, $\alpha \in (0,1]$, $a(t) = t^{1/5}$, $r(t) = 1$, $q(t) = t^{-2}$, and $f(x)/x \geq 1 = k$. On the other hand,

$$\sum_{s=t_2}^{\infty} \frac{1}{a^{1/\gamma_2}(s)} = \sum_{s=t_2}^{\infty} \frac{1}{s} = \infty.$$

Letting $\phi(t) = t$ in (3.5),

$$\begin{aligned} &\lim_{t\to\infty} \sup \sum_{s=t_2}^{t-1} \left(k\phi(s)\, q(s) - \frac{r^{1/\gamma_1}(s)\left(\Delta\phi_+(s)\right)^2}{4\phi(s)\,\Gamma(1-\alpha)\,\delta_1^{1/\gamma_1}(s,t_2)} \right) \\ &\qquad = \lim_{t\to\infty} \sup \sum_{s=t_2}^{t-1} \left(s^{-1} - \frac{1}{4s\Gamma(1-\alpha)\,\delta_1^{1/5}(s,t_2)} \right) \\ &\qquad = \lim_{t\to\infty} \sup \sum_{s=t_2}^{t-1} \frac{1}{s} \left(1 - \frac{1}{4\Gamma(1-\alpha)\left(\sum_{s=t_2}^{\infty} \frac{1}{s}\right)^{1/5}} \right) \\ &\qquad = \infty. \end{aligned}$$

So (3.5) holds, and then we deduce that (4.1) is oscillatory by Theorem 3.4.

5. Conclusion

In this paper, we have established some oscillation criteria for a class of nonlinear fractional difference equations by using some inequalities and Riccati transformation. Then, an example illustrating the results was presented. As a result, it can be seen that this approach can also be applied to research the oscillation of fractional difference equations with more complicated forms.

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