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Some new coupled fixed point theorems in ordered partial b-metric spaces

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Abstract

In this paper, we establish some new coupled fixed point theorems in ordered partial b-metric spaces. Also, an example is provided to support our new results. The results presented in this paper extend and improve several well-known comparable results.

Keywords: Common coupled fixed point, coupled coincidence point, partially ordered set, mixed g-monotone property, partial b-metric space.
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1. Introduction

Fixed point theory of nonlinear operators in metric spaces finds a lot of applications in convex optimization problems, see [13, 21, 27] and the references therein. In 1993, Czerwik [7] introduced the concept of the b-metric space. In 1994, Matthews [18] introduced the notion of partial metric spaces. After that, many researches have dealt with fixed point theories for various contraction mappings in b-metric spaces [5, 8, 15, 16, 22, 25, 26] and partial metric spaces [2, 3]. By combining these, Shukla [26] introduced a new generalization of metric space called partial b-metric space which was paid widespread attention immediately. Also, in [19] a modified version of partial b-metric space was introduced and many useful lemmas could be proved right away. Since then, several authors obtained more helpful results in this space [10, 19].

On the other hand, since the ordered set was introduced, many authors got many fixed point theorems in ordered metric space. In 2006, Bhaskar and Lakshmikantham [9] introduced the notion of a coupled fixed point and used the mixed monotone property to prove some coupled fixed point theorems. Three

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years later, Lakshmikantham and Ćirić [14] introduced the new concepts of coupled coincidence and coupled common fixed and used a mixed g-monotone property to prove some coupled common fixed point theorems which extended Bhaskar and Lakshmikantham's result from one mapping $F : X \times X \to X$ to two mapping $F : X \times X \to X$ and $g : X \to X$. Subsequently, many authors got a variety of coupled coincidence and coupled fixed point theorems in ordered metric spaces [11, 17, 20].

Recently, Aghajani and Arab [4] introduced a generalized contractive mapping with the altering distance functions and proved a new coupled common fixed point theorems in ordered b-metric space. Also, a number of articles on the topic of coupled fixed point theorems were obtained in ordered b-metric space and ordered partial metric space [1, 6, 23, 24]. But in ordered partial b-metric spaces, there are almost no research of them. In this paper, we use a more generalized contractive mapping to prove some coupled coincidence and coupled common fixed point theorems in ordered partial b-metric spaces. An example is provided to support our new results. The results presented in this paper extend and improve several well-known comparable results.

2. Preliminaries and definitions

First, we introduce some basic definitions and concepts as the following.

Definition 2.1 ([7])**.** A *b-metric* on nonempty set X is a mapping $d : X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and for all $x, y, z \in X$,

(1) $x = y \Leftrightarrow d(x, y) = 0$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

A *b-metric space* is a pair (X, d) such that X is a nonempty set and d is a b-metric on X. The number s is called the coefficient of (X, d).
It is obvious that a b-metric space with coefficient $s = 1$ is a metric space. There are examples of b-metric spaces which are not metric spaces (see, e.g., Akkouchi [5]).

Definition 2.2 ([18])**.** A *partial metric* on a nonempty set X is a function $p : X \times X \longrightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

(p2) $p(x, x) \leq p(x, y)$;

(p3) $p(x, y) = p(y, x)$;

(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. If p is a partial metric on X, then function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x, y) := 2p(x, y) - p(x, x) - p(y, y),$$

is ordinary equivalent metric on X.

Definition 2.3 ([19])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to \mathbb{R}^+$ is a partial b-metric, if for all $x, y, z \in X$, the following conditions are satisfied:

($\mathrm{p_{b1}}$) $x = y \Longleftrightarrow p_b(x, x) = p_b(x, y) = p_b(y, y)$;

($\mathrm{p_{b2}}$) $p_b(x,x) \leq p_b(x,y)$;

($\mathrm{p_{b3}}$) $p_b(x,y) = p_b(y,x)$;

($\mathrm{p_{b4}}$) $p_b(x,y) \leq s[p_b(x,z) + p_b(z,y) - p_b(z,z)] + \left(\frac{1-s}{2}\right)(p_b(x,x) + p_b(y,y))$.

The pair (X, p_b) is called a partial b-metric space.

Example 2.4 ([26])**.** Let $X = \mathbb{R}^+$, $q > 1$ be a constant, and $p_b : X \times X \to \mathbb{R}^+$ be defined by

$$p_b(x,y) = [\max\{x,y\}]^q + |x-y|^q, \quad \text{for all } x, y \in X.$$

Obviously, (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1}$, but it is neither a partial metric space nor a b-metric space.

Other examples of partial b-metric can be constructed thank to the following propositions.

Proposition 2.5 ([26])**.** *Let X be a nonempty set and let p be a partial metric and d be a b-metric with the coefficient $s \geq 1$ on X. Then the function $p_b : X \times X \to \mathbb{R}^+$ defined by $p_b(x,y) = p(x,y) + d(x,y)$, for all $x, y \in X$ is a partial b-metric on X with the coefficient s.*

Proposition 2.6 ([26])**.** *Let (X, p) be a partial metric space and $q \geq 1$. Then (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1}$, where p_b is defined by $p_b(x,y) = [p(x,y)]^q$.*

Proposition 2.7 ([19])**.** *Every partial b-metric p_b defines a b-metric d_{p_b}, where*

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y), \quad \textit{for all } x, y \in X.$$

Definition 2.8 ([19])**.** Let (X, p_b) be a partial b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be any sequence in X and $x \in X$. Then

(i) the sequence $\{x_n\}$ is said to be p_b-converges to x, if $\lim_{n\to\infty} p_b(x_n, x) = p_b(x,x)$,

(ii) the sequence $\{x_n\}$ is said to be p_b-Cauchy sequence in (X, p_b), if $\lim_{n,m\to\infty} p_b(x_n, x_m)$ exists and is finite.

(iii) (X, p_b) is said to be a p_b-complete partial b-metric space, if for every Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that

$$\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x_n, x) = p_b(x,x).$$

Thank to [19], we have the following important lemmas.

Lemma 2.9 ([19])**.**

(1) *A sequence $\{x_n\}$ is a p_b-Cauchy sequence in a partial b-metric space (X, p_b), if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{p_b}).*

(2) *A partial b-metric space (X, p_b) is p_b-complete, if and only if the b-metric space (X, d_{p_b}) is b-complete. Moreover, $\lim_{n\to\infty} d_{p_b}(x, x_n) = 0$, if and only if*

$$\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x_n, x) = p_b(x,x).$$

Lemma 2.10 ([19])**.** *Let (X, p_b) be a partial b-metric space with the coefficient $s \geq 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, respectively. Then we have*

$$\begin{aligned}\frac{1}{s^2}p_b(x,y) - \frac{1}{s}p_b(x,x) - p_b(y,y) \leq \liminf_{n\to\infty} p_b(x_n, y_n) &\leq \limsup_{n\to\infty} p_b(x_n, y_n) \\ &\leq sp_b(x,x) + s^2 p_b(y,y) + s^2 p_b(x,y).\end{aligned}$$

In particular, if $p_b(x,y)=0$, *then we have* $\lim_{n\to\infty} d_{p_b}(x_n,y_n)=0$. *Moreover, for each* $z\in X$, *we have*

$$\begin{aligned}\frac{1}{s}p_b(x,z)-p_b(x,x) &\le \liminf_{n\to\infty} p_b(x_n,z)\le \limsup_{n\to\infty} p_b(x_n,z)\\ &\le sp_b(x,z)+sp_b(x,x).\end{aligned}$$

In particular, if $p_b(x,x)=0$, *then we have*

$$\frac{1}{s}p_b(x,z)\le \liminf_{n\to\infty} p_b(x_n,z)\le \limsup_{n\to\infty} p_b(x_n,z)\le sp_b(x,z).$$

Definition 2.11 ([9])**.** An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F: X\times X\to X$, if $F(x,y)=x$ and $F(y,x)=y$.

Definition 2.12 ([14])**.** An element $(x,y)\in X\times X$ is called a coupled coincidence point of the mapping $F: X\times X\to X$ and $g: X\to X$, if $F(x,y)=gx$ and $F(y,x)=gy$, and in this case, (gx,gy) is called a coupled point of coincidence.

Definition 2.13 ([1])**.** An element $(x,x)\in X\times X$ is called a common fixed point of the mapping $F: X\times X\to X$ and $g: X\to X$, if $F(x,x)=gx=x$.

Definition 2.14 ([14])**.** Let X be a nonempty set. Then we say that the mappings $F: X\times X\to X$ and $g: X\to X$ are commutative, if $gF(x,y)=F(gx,gy)$.

Definition 2.15 ([9])**.** Let $(X,\preceq)$ be a partially ordered set and $F: X\times X\to X$. The mapping F is said to have the mixed monotone property, if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x,y\in X$, we have

$$x_1,x_2\in X,\ x_1\preceq x_2\ \Rightarrow\ F(x_1,y)\preceq F(x_2,y),$$

and

$$y_1,y_2\in X,\ y_1\preceq y_2\ \Rightarrow\ F(x,y_1)\succeq F(x,y_2).$$

Definition 2.16 ([14])**.** Let $(X,\preceq)$ be a partially ordered set and $F: X\times X\to X$ and $g: X\to X$. The mapping F is said to have the mixed g-monotone property, if $F(x,y)$ is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, that is, for any $x,y\in X$, we have

$$x_1,x_2\in X,\ g(x_1)\preceq g(x_2)\ \Rightarrow\ F(x_1,y)\preceq F(x_2,y),$$

and

$$y_1,y_2\in X,\ g(y_1)\preceq g(y_2)\ \Rightarrow\ F(x,y_1)\succeq F(x,y_2).$$

Definition 2.17 ([12])**.** A function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function, if the following properties are satisfied:

1. ψ is continuous and nondecreasing;
2. $\psi(t)=0$, if and only if $t=0$.

3. Main results

Theorem 3.1. *Let* $(X,\preceq,p_b)$ *be a ordered partial b-metric space. Let* $F: X\times X\to X$ *and* $g: X\to X$ *be two mappings and* F *has the mixed* g*-monotone property with* g. *Suppose that there exists an altering distance function* ψ *and* $\theta:[0,\infty)^{10}\to[0,\infty)$ *is continuous with* $\theta(t_1,t_2,\cdots,t_{10})=0$ *implies* $t_1=t_2=t_5=t_6=0$

such that

$$\psi\left(sp_b(F(x,y),F(u,v))\right) \le \psi\left(M(x,y,u,v)\right) - \Theta(x,y,u,v) \tag{3.1}$$

for all $(x,y),(u,v) \in X \times X$ *with* $g(x) \preceq g(u)$ *and* $g(y) \succeq g(v)$*, where*

$$M(x,y,u,v) = \max\left\{ \begin{array}{c} p_b(gx,gu), p_b(gy,gv), p_b(gx,F(x,y)), \\ p_b(gy,F(y,x)), \frac{p_b(gu,F(u,v))}{2s}, \frac{p_b(gv,F(v,u))}{2s}, \\ \frac{p_b(gx,F(u,v))+p_b(gu,F(x,y))}{2s}, \frac{p_b(gy,F(v,u))+p_b(gv,F(y,x))}{2s} \end{array} \right\},$$

and

$$\Theta(x,y,u,v) = \theta\left(\begin{array}{c} p_b(gx,gu), p_b(gy,gv), p_b(gx,F(x,y)), p_b(gy,F(y,x)), \\ p_b(gu,F(u,v)), p_b(gv,F(v,u)), p_b(gx,F(u,v)), \\ p_b(gy,F(v,u)), p_b(gu,F(x,y)), p_b(gv,F(y,x)) \end{array} \right).$$

Further, suppose $F(X \times X) \subset g(X)$ *and* $g(X)$ *is a* p_b*-complete subspace of* (X,p_b)*. Also suppose that* X *satisfies the following properties:*

(i) *if a non-decreasing sequence* $\{x_n\}$ *in* X *converges to* $x \in X$*, then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$*;*

(ii) *if a non-increasing sequence* $\{y_n\}$ *in* X *converges to* $y \in X$*, then* $y_n \succeq y$ *for all* $n \in \mathbb{N}$*.*

If there exists $(x_0,y_0) \in X \times X$ *such that* $gx_0 \preceq F(x_0,y_0)$ *and* $gy_0 \succeq F(y_0,x_0)$*, then* F *and* g *have a coupled coincidence point.*

Proof. By the given condition, there exists $(x_0,y_0) \in X \times X$ such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$. Since $F(X\times X) \subset g(X)$, we can define $(x_1,y_1) \in X\times X$ such that $gx_1 = F(x_0,y_0)$ and $gy_1 = F(y_0,x_0)$, then $gx_0 \preceq F(x_0,y_0) = gx_1$ and $gy_0 \succeq F(y_0,x_0) = gy_1$. Going on in this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n,y_n) \text{ and } gy_{n+1} = F(y_n,x_n), \quad \forall\, n \ge 0. \tag{3.2}$$

Now we prove that

$$gx_n \preceq gx_{n+1} \text{ and } gy_n \succeq gy_{n+1}, \quad \forall\, n \ge 0.$$

We will use the mathematical induction. The conclusion holds for $n = 0$, suppose it holds for some $n > 0$. Since F has the mixed g-monotone property, $g(x_n) \preceq g(x_{n+1})$ and $g(y_n) \succeq g(y_{n+1})$, from (3.2) we have

$$\begin{cases} gx_{n+1} = F(x_n,y_n) \preceq F(x_{n+1},y_n) \preceq F(x_{n+1},y_{n+1}) = gx_{n+2}, & \forall\, n \ge 0, \\ gy_{n+1} = F(y_n,x_n) \succeq F(y_{n+1},x_n) \succeq F(y_{n+1},x_{n+1}) = gy_{n+2}, & \forall\, n \ge 0. \end{cases}$$

Thus, by the mathematical induction, we conclude that

$$\begin{cases} gx_0 \preceq gx_1 \preceq gx_2 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots, \\ gy_0 \succeq gy_1 \succeq gy_2 \succ \cdots \succeq gy_n \succeq gy_{n+1} \succeq \cdots. \end{cases} \tag{3.3}$$

From (3.2), (3.3), (3.1), and the property of ψ we have

$$\begin{aligned} \psi\left(p_b(gx_n,gx_{n+1})\right) &\le \psi\left(sp_b(gx_n,gx_{n+1})\right) = \psi\left(sp_b(F(x_{n-1},y_{n-1}),F(x_n,y_n))\right) \\ &\le \psi\left(M(x_{n-1},y_{n-1},x_n,y_n)\right) - \Theta(x_{n-1},y_{n-1},x_n,y_n), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} &M(x_{n-1},y_{n-1},x_n,y_n) \\ &= \max\left\{ \begin{array}{c} p_b(gx_{n-1},gx_n), p_b(gy_{n-1},gy_n), p_b(gx_{n-1},F(x_{n-1},y_{n-1})), \\ p_b(gy_{n-1},F(y_{n-1},x_{n-1})), \frac{p_b(gx_n,F(x_n,y_n))}{2s}, \frac{p_b(gy_n,F(y_n,x_n))}{2s}, \\ \frac{p_b(gx_{n-1},F(x_n,y_n))+p_b(gx_n,F(x_{n-1},y_{n-1}))}{2s}, \\ \frac{p_b(gy_{n-1},F(y_n,x_n))+p_b(gy_n,F(y_{n-1},x_{n-1}))}{2s} \end{array} \right\}\end{aligned}$$

$$= \max \left\{ \begin{array}{c} p_b(gx_{n-1}, gx_n), p_b(gy_{n-1}, gy_n), p_b(gx_{n-1}, gx_n), \\ p_b(gy_{n-1}, gy_n), p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1}), \\ \frac{p_b(gx_{n-1}, gx_{n+1}) + p_b(gx_n, gx_n)}{2s}, \frac{p_b(gy_{n-1}, gy_{n+1})) + p_b(gy_n, gy_n)}{2s} \end{array} \right\}$$
$$= \max \left\{ \begin{array}{c} p_b(gx_{n-1}, gx_n), p_b(gy_{n-1}, gy_n), p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1}), \\ \frac{p_b(gx_{n-1}, gx_{n+1}) + p_b(gx_n, gx_n)}{2s}, \frac{p_b(gy_{n-1}, gy_{n+1})) + p_b(gy_n, gy_n)}{2s} \end{array} \right\}. \tag{3.5}$$

and

$$\Theta(x_{n-1}, y_{n-1}, x_n, y_n) = \theta \left(\begin{array}{c} p_b(gx_{n-1}, gx_n), p_b(gy_{n-1}, gy_n), p_b(gx_{n-1}, gx_n), \\ p_b(gy_{n-1}, gy_n), p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1}), \\ p_b(gx_{n-1}, gx_{n+1}), p_b(gx_n, gx_n), p_b(gy_{n-1}, gy_{n+1}), p_b(gy_n, gy_n) \end{array} \right).$$

It follows from ($\mathrm{p_{b4}}$) that

$$\begin{aligned} \frac{p_b(gx_{n-1}, gx_{n+1}) + p_b(gx_n, gx_n)}{2s} &\leq \frac{sp_b(gx_{n-1}, gx_n) + sp_b(gx_n, gx_{n+1}) + (1-s)p_b(gx_n, gx_n)}{2s} \\ &\leq \frac{sp_b(gx_{n-1}, gx_n) + sp_b(gx_n, gx_{n+1})}{2s} \\ &\leq \max\{p_b(gx_{n-1}, gx_n), p_b(gx_n, gx_{n+1})\}. \end{aligned} \tag{3.6}$$

Similarly, we can show that

$$\begin{aligned} \frac{p_b(gy_{n-1}, gy_{n+1}) + p_b(gy_n, gy_n)}{2s} &\leq \frac{sp_b(gy_{n-1}, gy_n) + sp_b(gy_n, gy_{n+1}) + (1-s)p_b(gy_n, gy_n)}{2s} \\ &\leq \frac{sp_b(gy_{n-1}, gy_n) + sp_b(gy_n, gy_{n+1})}{2s} \\ &\leq \max\{p_b(gy_{n-1}, gy_n), p_b(gy_n, gy_{n+1})\}. \end{aligned} \tag{3.7}$$

By substituting (3.6) and (3.7) into (3.5), we obtain

$$\begin{aligned} M(x_{n-1}, y_{n-1}, x_n, y_n) &= \max\{p_b(gx_{n-1}, gx_n), p_b(gx_n, gx_{n+1}), p_b(gy_{n-1}, gy_n), p_b(gy_n, gy_{n+1})\} \\ &= \max\{\delta_{n-1}, \delta_n\}, \end{aligned} \tag{3.8}$$

where

$$\delta_n = \max\{p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1})\}.$$

By combining (3.4) and (3.8), we get

$$\psi(p_b(gx_n, gx_{n+1})) \leq \psi(\max\{\delta_{n-1}, \delta_n\}) - \Theta(x_{n-1}, y_{n-1}, x_n, y_n). \tag{3.9}$$

By the same way as above, we can show that

$$M(y_{n-1}, x_{n-1}, y_n, x_n) = \max\{\delta_{n-1}, \delta_n\},$$

and

$$\begin{aligned} \psi(p_b(gy_n, gy_{n+1})) &\leq \psi(M(y_{n-1}, x_{n-1}, y_n, x_n)) - \Theta(y_{n-1}, x_{n-1}, y_n, x_n) \\ &= \psi(\max\{\delta_{n-1}, \delta_n\}) - \Theta(y_{n-1}, x_{n-1}, y_n, x_n), \end{aligned} \tag{3.10}$$

where

$$\Theta(y_{n-1}, x_{n-1}, y_n, x_n) = \theta \left(\begin{array}{c} p_b(gy_{n-1}, gy_n), p_b(gx_{n-1}, gx_n), p_b(gy_{n-1}, gy_n), \\ p_b(gx_{n-1}, gx_n), p_b(gy_n, gy_{n+1}), p_b(gx_n, gx_{n+1}), \\ p_b(gy_{n-1}, gy_{n+1}), p_b(gy_n, gy_n), p_b(gx_{n-1}, gx_{n+1}), p_b(gx_n, gx_n) \end{array} \right).$$

Next we prove that $\delta_n \leq \delta_{n-1}$ for all $n \in \mathbb{N}$. In fact, suppose that $\delta_{n-1} < \delta_n$, then $\delta_n > 0$ (otherwise, $\delta_{n-1} < \delta_n = 0$, which is a contradiction). We consider the following two cases.

Case 1. $\max\{\delta_{n-1}, \delta_n\} = \delta_n = p_b(gx_n, gx_{n+1}) > 0$.

By (3.9) we have

$$\psi\left(p_b(gx_n, gx_{n+1})\right) \leq \psi\left(p_b(gx_n, gx_{n+1})\right) - \Theta(x_{n-1}, y_{n-1}, x_n, y_n),$$

which means $\Theta(x_{n-1}, y_{n-1}, x_n, y_n) = 0$. By the properties of θ, we can find $p_b(gx_n, gx_{n+1}) = 0$, which is a contradiction.

Case 2. $\max\{\delta_{n-1}, \delta_n\} = \delta_n = p_b(gy_n, gy_{n+1}) > 0$.

By (3.10) we have

$$\psi\left(p_b(gy_n, gy_{n+1})\right) \leq \psi\left(p_b(gy_n, gy_{n+1})\right) - \Theta(y_{n-1}, x_{n-1}, y_n, x_n),$$

which means $\Theta(y_{n-1}, x_{n-1}, y_n, x_n) = 0$. By the properties of θ we can find $p_b(gy_n, gy_{n+1}) = 0$, which is a contradiction.

Therefore, we have $\delta_n \leq \delta_{n-1}$ for all $n \in \mathbb{N}$ holds, thus the sequence $\{\delta_n\}$ is a non-increasing sequence of nonnegative real number, and so, there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta.$$

Since $\psi(\max\{x, y\}) = \max\{\psi(x), \psi(y)\}$, from (3.9) and (3.10) we have

$$\begin{aligned}\psi(\delta_n) &= \max\{\psi(p_b(gx_n, gx_{n+1})), \psi(p_b(gy_n, gy_{n+1}))\}\\ &\leq \psi(\delta_{n-1}) - \min\left\{\Theta(x_{n-1}, y_{n-1}, x_n, y_n), \Theta(y_{n-1}, x_{n-1}, y_n, x_n)\right\}.\end{aligned} \tag{3.11}$$

By taking the upper limit as $n \to \infty$ in (3.11), we have

$$\begin{aligned}\psi(\delta) &\leq \psi(\delta) - \liminf_{n\to\infty} \min\left\{\Theta(x_{n-1}, y_{n-1}, x_n, y_n), \Theta(y_{n-1}, x_{n-1}, y_n, x_n)\right\}\\ &\leq \psi(\delta) - \min\left\{\liminf_{n\to\infty} \Theta(x_{n-1}, y_{n-1}, x_n, y_n), \liminf_{n\to\infty} \Theta(y_{n-1}, x_{n-1}, y_n, x_n)\right\}.\end{aligned}$$

Therefore,

$$\liminf_{n\to\infty} \Theta(x_{n-1}, y_{n-1}, x_n, y_n) = 0 \ \text{ or } \ \liminf_{n\to\infty} \Theta(y_{n-1}, x_{n-1}, y_n, x_n) = 0.$$

Hence, by using the properties of θ, we get

$$\liminf_{n\to\infty} p_b(gx_n, gx_{n+1}) = 0 \ \text{ and } \ \liminf_{n\to\infty} p_b(gy_n, gy_{n+1}) = 0.$$

So,

$$\delta = \liminf_{n\to\infty} \delta_n = \liminf_{n\to\infty} \max\{p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1})\} = 0.$$

That is

$$\lim_{n\to\infty} p_b(gx_n, gx_{n+1}) = 0 \ \text{ and } \ \lim_{n\to\infty} p_b(gy_n, gy_{n+1}) = 0. \tag{3.12}$$

From ($\mathrm{p_{b2}}$) and (3.12), we have

$$\lim_{n\to\infty} p_b(gx_n, gx_n) = 0 \ \text{ and } \ \lim_{n\to\infty} p_b(gy_n, gy_n) = 0. \tag{3.13}$$

Next we prove that $\{gx_n\}, \{gy_n\}$ are p_b-Cauchy sequences in $g(X)$. For this, we have to show that $\{gx_n\}, \{gy_n\}$ are b-Cauchy sequences in $(g(X), d_{p_b})$. In other words, we need to show that for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $m, n \geq k$,

$$\max\{d_{p_b}(gx_m, gx_n), d_{p_b}(gy_m, gy_n)\} < \varepsilon.$$

Suppose to the contrary, there exists $\varepsilon > 0$ for which we can find subsequences $\{gx_{m_i}\}, \{gx_{n_i}\}$ of $\{gx_n\}$ and $\{gy_{m_i}\}, \{gy_{n_i}\}$ of $\{gy_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \quad \max\{d_{p_b}(gx_{m_i}, gx_{n_i}), d_{p_b}(gy_{m_i}, gy_{n_i})\} \geq \varepsilon. \tag{3.14}$$

That is,

$$\max\{d_{p_b}(gx_{m_i}, gx_{n_i-1}), d_{p_b}(gy_{m_i}, gy_{n_i-1})\} < \varepsilon. \tag{3.15}$$

From the definition of d_{p_b}, (3.12) and (3.13) we obtain

$$\begin{aligned}\lim_{n\to\infty} d_{p_b}(gx_n, gx_{n+1}) &= 2\lim_{n\to\infty} p_b(gx_n, gx_{n+1}) - \lim_{n\to\infty} p_b(gx_n, gx_n) - \lim_{n\to\infty} p_b(gx_{n+1}, gx_{n+1})\\ &= 0.\end{aligned}$$

Similarly, we have $\lim_{n\to\infty} d_{p_b}(gy_n, gy_{n+1}) = 0$. To sum up, we get

$$\lim_{n\to\infty} d_{p_b}(gx_n, gx_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d_{p_b}(gy_n, gy_{n+1}) = 0. \tag{3.16}$$

By using the triangle inequality, we get

$$d_{p_b}(gx_{m_i}, gx_{n_i}) \leq sd_{p_b}(gx_{m_i}, gx_{n_i-1}) + sd_{p_b}(gx_{n_i-1}, gx_{n_i}), \tag{3.17}$$

and

$$d_{p_b}(gy_{m_i}, gy_{n_i}) \leq sd_{p_b}(gy_{m_i}, gy_{n_i-1}) + sd_{p_b}(gy_{n_i-1}, gy_{n_i}). \tag{3.18}$$

Hence from (3.14), (3.17) and (3.18), we have

$$\begin{aligned}\varepsilon &\leq \max\{d_{p_b}(gx_{m_i}, gx_{n_i}), d_{p_b}(gy_{m_i}, gy_{n_i})\}\\ &\leq s\max\{d_{p_b}(gx_{m_i}, gx_{n_i-1}), d_{p_b}(gy_{m_i}, gy_{n_i-1})\}\\ &\quad + s\max\{d_{p_b}(gx_{n_i-1}, gx_{n_i}), d_{p_b}(gy_{n_i-1}, gy_{n_i})\}.\end{aligned} \tag{3.19}$$

By taking the lower limit as $i \to \infty$ in (3.19) and using (3.15), (3.16), we have

$$\begin{aligned}\varepsilon &\leq \liminf_{i\to\infty} \max\{d_{p_b}(gx_{m_i}, gx_{n_i}), d_{p_b}(gy_{m_i}, gy_{n_i})\}\\ &\leq s\liminf_{i\to\infty} \max\{d_{p_b}(gx_{m_i}, gx_{n_i-1}), d_{p_b}(gy_{m_i}, gy_{n_i-1})\}\\ &\leq s\limsup_{i\to\infty} \max\{d_{p_b}(gx_{m_i}, gx_{n_i-1}), d_{p_b}(gy_{m_i}, gy_{n_i-1})\} \leq s\varepsilon.\end{aligned} \tag{3.20}$$

Also, by using (3.15) and (3.16), taking the upper limit as $i \to \infty$ in (3.19), we obtain

$$\varepsilon \leq \limsup_{i\to\infty} \max\{d_{p_b}(gx_{m_i}, gx_{n_i}), d_{p_b}(gy_{m_i}, gy_{n_i})\} \leq s\varepsilon. \tag{3.21}$$

By the triangle inequality, we have

$$d_{p_b}(gx_{m_i}, gx_{n_i}) \leq sd_{p_b}(gx_{m_i}, gx_{m_i+1}) + sd_{p_b}(gx_{m_i+1}, gx_{n_i}), \tag{3.22}$$

and

$$d_{p_b}(gy_{m_i}, gy_{n_i}) \leq sd_{p_b}(gy_{m_i}, gy_{m_i+1}) + sd_{p_b}(gy_{m_i+1}, gy_{n_i}). \tag{3.23}$$

Therefore, from (3.14), (3.22) and (3.23), we have

$$\begin{aligned}\varepsilon &\leq \max\{d_{p_b}(gx_{m_i}, gx_{n_i}), d_{p_b}(gy_{m_i}, gy_{n_i})\}\\ &\leq s\max\{d_{p_b}(gx_{m_i}, gx_{m_i+1}), d_{p_b}(gy_{m_i}, gy_{m_i+1})\}\\ &\quad + s\max\{d_{p_b}(gx_{m_i+1}, gx_{n_i}), d_{p_b}(gy_{m_i+1}, gy_{n_i})\}.\end{aligned}$$

By taking the upper limit as $i \to \infty$ in the above inequality, and using (3.16), we have

$$\frac{\varepsilon}{s} \le \limsup_{i\to\infty} \max\{d_{p_b}(gx_{m_i+1}, gx_{n_i}), d_{p_b}(gy_{m_i+1}, gy_{n_i})\}. \tag{3.24}$$

Again, by the triangle inequality we have

$$d_{p_b}(gx_{m_i+1}, gx_{n_i-1}) \le sd_{p_b}(gx_{m_{i+1}}, gx_{m_i}) + sd_{p_b}(gx_{m_i}, gx_{n_i-1}), \tag{3.25}$$

and

$$d_{p_b}(gy_{m_i+1}, gy_{n_i-1}) \le sd_{p_b}(gy_{m_{i+1}}, gy_{m_i}) + sd_{p_b}(gy_{m_i}, gy_{n_i-1}). \tag{3.26}$$

From the inequality (3.25), (3.26) and (3.15), we have

$$\begin{aligned} \max\{d_{p_b}(gx_{m_i+1}, gx_{n_i-1}), d_{p_b}(gy_{m_i+1}, gy_{n_i-1})\} &\le s\max\{d_{p_b}(gx_{m_i+1}, gx_{m_i}), d_{p_b}(gy_{m_i+1}, gy_{m_i})\} \\ &\quad + s\max\{d_{p_b}(gx_{m_i}, gx_{n_i-1}), d_{p_b}(gy_{m_i}, gy_{n_i-1})\} \\ &< s\max\{d_{p_b}(gx_{m_i+1}, gx_{m_i}), d_{p_b}(gy_{m_i+1}, gy_{m_i})\} + s\varepsilon. \end{aligned}$$

By taking the upper limit as $i \to \infty$ in the above inequality, and using (3.16), we get

$$\limsup_{i\to\infty} \max\{d_{p_b}(gx_{m_i+1}, gx_{n_i-1}), d_{p_b}(gy_{m_i+1}, gy_{n_i-1})\} \le s\varepsilon. \tag{3.27}$$

On the other hand, because of the definition of d_{p_b} and (3.16), we have

$$\liminf_{i\to\infty} d_{p_b}(gx_{m_i}, gx_{n_i-1}) = 2\liminf_{i\to\infty} p_b(gx_{m_i}, gx_{n_i-1}), \tag{3.28}$$

and

$$\liminf_{i\to\infty} d_{p_b}(gy_{m_i}, gy_{n_i-1}) = 2\liminf_{i\to\infty} p_b(gy_{m_i}, gy_{n_i-1}). \tag{3.29}$$

Hence, from (3.28), (3.29) and (3.20), we obtain

$$\begin{aligned} \frac{\varepsilon}{s} &\le \liminf_{i\to\infty} \max\{d_{p_b}(gx_{m_i}, gx_{n_i-1}), d_{p_b}(gy_{m_i}, gy_{n_i-1})\} \\ &= 2\liminf_{i\to\infty} \max\{p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{n_i-1})\} \le \varepsilon. \end{aligned}$$

Thus, we get

$$\frac{\varepsilon}{2s} \le \liminf_{i\to\infty} \max\{p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{n_i-1})\} \le \frac{\varepsilon}{2}. \tag{3.30}$$

Similarly, from (3.15), (3.21), (3.24), (3.27) and definition of d_{p_b}, we can show that

$$\limsup_{i\to\infty} \max\{p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{n_i-1})\} \le \frac{\varepsilon}{2}, \tag{3.31}$$

$$\limsup_{i\to\infty} \max\{p_b(gx_{m_i}, gx_{n_i}), p_b(gy_{m_i}, gy_{n_i})\} \le \frac{s\varepsilon}{2}, \tag{3.32}$$

$$\frac{\varepsilon}{2s} \le \limsup_{i\to\infty} \max\{p_b(gx_{m_i+1}, gx_{n_i}), p_b(gy_{m_i+1}, gy_{n_i})\}, \tag{3.33}$$

$$\limsup_{i\to\infty} \max\{p_b(gx_{m_i+1}, gx_{n_i-1}), p_b(gy_{m_i+1}, gy_{n_i-1})\} \le \frac{s\varepsilon}{2}. \tag{3.34}$$

By using (3.1) with $(x, y) = (x_{m_i}, y_{m_i})$ and $(u, v) = (x_{n_i-1}, y_{n_i-1})$, we get

$$\begin{aligned} \psi\left(sp_b(gx_{m_i+1}, gx_{n_i})\right) &= \psi\left(sp_b(F(x_{m_i}, y_{m_i}), F(x_{n_i-1}, y_{n_i-1}))\right) \\ &\le \psi\left(M(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1})\right) - \Theta(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1}) \end{aligned} \tag{3.35}$$

where

$$\begin{aligned}
&M(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1})\\
&= \max\left\{\begin{array}{c} p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{n_i-1}), p_b(gx_{m_i}, F(x_{m_i}, y_{m_i})),\\ p_b(gy_{m_i}, F(y_{m_i}, x_{m_i})), \frac{p_b(gx_{n_i-1}, F(x_{n_i-1}, y_{n_i-1}))}{2s}, \frac{p_b(gy_{n_i-1}, F(y_{n_i-1}, x_{n_i-1}))}{2s},\\ \frac{p_b(gx_{m_i}, F(x_{n_i-1}, y_{n_i-1}))+p_b(gx_{n_i-1}, F(x_{m_i}, y_{m_i}))}{2s}, \frac{p_b(gy_{m_i}, F(y_{n_i-1}, x_{n_i-1}))+p_b(gy_{n_i-1}, F(y_{m_i}, x_{m_i}))}{2s} \end{array}\right\}\\
&= \max\left\{\begin{array}{c} p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{n_i-1}), p_b(gx_{m_i}, gx_{m_i+1}),\\ p_b(gy_{m_i}, gy_{m_i+1}), p_b(gx_{n_i-1}, gx_{n_i}), p_b(gy_{n_i-1}, gy_{n_i}),\\ \frac{p_b(gx_{m_i}, gx_{n_i})+p_b(gx_{n_i-1}, gx_{m_i+1})}{2s}, \frac{p_b(gy_{m_i}, gy_{n_i})+p_b(gy_{n_i-1}, gy_{m_i+1})}{2s} \end{array}\right\},
\end{aligned}$$

and

$$\Theta(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1}) = \theta\left(\begin{array}{c} p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{n_i-1}), p_b(gx_{m_i}, gx_{m_i+1}), p_b(gy_{m_i}, gy_{m_i+1}),\\ p_b(gx_{n_i-1}, gx_{n_i}), p_b(gy_{n_i-1}, gy_{n_i}), p_b(gx_{m_i}, gx_{n_i}),\\ p_b(gy_{m_i}, gy_{n_i}), p_b(gx_{n_i-1}, gx_{m_i+1}), p_b(gy_{n_i-1}, gy_{m_i+1}) \end{array}\right).$$

Similarly, we have

$$\begin{aligned}
\psi\left(sp_b(gy_{m_i+1}, gy_{n_i})\right) &= \psi\left(sp_b(F(y_{m_i}, x_{m_i}), F(y_{n_i-1}, x_{n_i-1}))\right)\\
&\le \psi\left(M(y_{m_i}, x_{m_i}, y_{n_i-1}, x_{n_i-1})\right) - \Theta(y_{m_i}, x_{m_i}, y_{n_i-1}, x_{n_i-1}),
\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}
M(y_{m_i}, x_{m_i}, y_{n_i-1}, x_{n_i-1}) &= \max\left\{\begin{array}{c} p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{n_i-1}), p_b(gx_{m_i}, gx_{m_i+1}),\\ p_b(gy_{m_i}, gy_{m_i+1}), p_b(gx_{n_i-1}, gx_{n_i}), p_b(gy_{n_i-1}, gy_{n_i}),\\ \frac{p_b(gx_{m_i}, gx_{n_i})+p_b(gx_{n_i-1}, gx_{m_i+1})}{2s}, \frac{p_b(gy_{m_i}, gy_{n_i})+p_b(gy_{n_i-1}, gy_{m_i+1})}{2s} \end{array}\right\}\\
&= M(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1}),
\end{aligned}$$

and

$$\Theta(y_{m_i}, x_{m_i}, y_{n_i-1}, x_{n_i-1}) = \theta\left(\begin{array}{c} p_b(gy_{m_i}, gy_{n_i-1}), p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{m_i+1}), p_b(gx_{m_i}, gx_{m_i+1}),\\ p_b(gy_{n_i-1}, gy_{n_i}), p_b(gx_{n_i-1}, gx_{n_i}), p_b(gy_{m_i}, gy_{n_i}),\\ p_b(gx_{m_i}, gx_{n_i}), p_b(gy_{n_i-1}, gy_{m_i+1}), p_b(gx_{n_i-1}, gx_{m_i+1}) \end{array}\right).$$

By combining (3.35) and (3.36), we have

$$\begin{aligned}
\psi(s\max\{p_b(gx_{m_i+1}, gx_{n_i}), p_b(gy_{m_i+1}, gy_{n_i})\}) &= \max\{\psi(sp_b(gx_{m_i+1}, gx_{n_i})), \psi(sp_b(gy_{m_i+1}, gy_{n_i}))\}\\
&\le \psi(M(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1}))\\
&\quad - \min\left\{\Theta(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1}), \Theta(y_{m_i}, x_{m_i}, y_{n_i-1}, x_{n_i-1})\right\}.
\end{aligned}$$

By taking the upper limit as $i \to \infty$ in the above inequality and using (3.12), (3.31), (3.32), (3.33) and (3.34), we have

$$\begin{aligned}
\psi\left(\frac{\varepsilon}{2}\right) &= \psi\left(s\cdot\frac{\varepsilon}{2s}\right) \le \psi(s\limsup_{i\to\infty}\max\{p_b(gx_{m_i+1}, gx_{n_i}), p_b(gy_{m_i+1}, gy_{n_i})\})\\
&\le \psi\left(\max\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 0, 0, 0, 0, \frac{\frac{s\varepsilon}{2}+\frac{s\varepsilon}{2}}{2s}, \frac{\frac{s\varepsilon}{2}+\frac{s\varepsilon}{2}}{2s}\right\}\right) - \liminf_{i\to\infty}\min\left\{\begin{array}{c}\Theta(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1}),\\ \Theta(y_{m_i}, x_{m_i}, y_{n_i-1}, x_{n_i-1})\end{array}\right\}\\
&= \psi\left(\frac{\varepsilon}{2}\right) - \min\left\{\liminf_{i\to\infty}\Theta(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1}), \liminf_{i\to\infty}\Theta(y_{m_i}, x_{m_i}, y_{n_i-1}, x_{n_i-1})\right\},
\end{aligned}$$

which implies that

$$\liminf_{i\to\infty}\Theta(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1}) = 0 \ \text{ or } \ \liminf_{i\to\infty}\Theta(y_{m_i}, x_{m_i}, y_{n_i-1}, x_{n_i-1}) = 0.$$

Hence, by using the properties of θ, we get

$$\liminf_{i\to\infty} p_b(gx_{m_i}, gx_{n_i-1}) = 0 \text{ and } \liminf_{i\to\infty} p_b(gy_{m_i}, gy_{n_i-1}) = 0,$$

which is a contradiction to (3.30). Thus, $\{gx_n\}, \{gy_n\}$ are b-Cauchy sequences in $(g(X), d_{p_b})$. By Lemma 2.9, $\{gx_n\}, \{gy_n\}$ are p_b-Cauchy sequences in $(g(X), p_b)$. Since $g(X)$ is p_b-complete subspace of (X, p_b), there exist $gx, gy \in g(X)$, such that $\{gx_n\}$ and $\{gy_n\}$ p_b-converges to gx and gy, respectively. By using Lemma 2.9 again, we have

$$\lim_{n\to\infty} p_b(gx_n, gx) = \lim_{n,m\to\infty} p_b(gx_n, gx_m) = p_b(gx, gx), \tag{3.37}$$

$$\lim_{n\to\infty} p_b(gy_n, gy) = \lim_{n,m\to\infty} p_b(gy_n, gy_m) = p_b(gy, gy).$$

Since $\{gx_n\}$ is b-Cauchy sequence in (X, d_{p_b}), so $\lim_{n,m\to\infty} d_{p_b}(gx_n, gx_m) = 0$. By using

$$d_{p_b}(gx_n, gx_m) = 2p_b(gx_n, gx_m) - p_b(gx_n, gx_n) - p_b(gx_m, gx_m),$$

and (3.13) we obtain that $\lim_{n,m\to\infty} p_b(gx_n, gx_m) = 0$. Thus, it follows from (3.37) that

$$\lim_{n\to\infty} p_b(gx_n, gx) = \lim_{n,m\to\infty} p_b(gx_n, gx_m) = p_b(gx, gx) = 0. \tag{3.38}$$

On using similar steps as above we can show that

$$\lim_{n\to\infty} p_b(gy_n, gy) = \lim_{n,m\to\infty} p_b(gy_n, gy_m) = p_b(gy, gy) = 0. \tag{3.39}$$

By (3.3) and the properties (i) and (ii), we have $gx_n \preceq gx,\ \ gy_n \succeq gy$ for all $n \in N$. From (3.1), we have

$$\begin{aligned}\psi\left(sp_b(gx_{n+1}, F(x,y))\right) &= \psi\left(sp_b(F(x_n,y_n), F(x,y))\right)\\ &\le \psi\left(M(x_n,y_n,x,y)\right) - \Theta(x_n,y_n,x,y),\end{aligned} \tag{3.40}$$

where

$$M(x_n,y_n,x,y) = \max\left\{\begin{array}{c} p_b(gx_n, gx), p_b(gy_n, gy), p_b(gx_n, gx_{n+1}),\\ p_b(gy_n, gy_{n+1}), \frac{p_b(gx,F(x,y))}{2s}, \frac{p_b(gy,F(y,x))}{2s},\\ \frac{p_b(gx_n,F(x,y))+p_b(gx,gx_{n+1})}{2s}, \frac{p_b(gy_n,F(y,x))+p_b(gy,gy_{n+1})}{2s}\end{array}\right\}, \tag{3.41}$$

and

$$\Theta(x_n,y_n,x,y) = \theta\left(\begin{array}{c} p_b(gx_n, gx), p_b(gy_n, gy), p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1}),\\ p_b(gx, F(x,y)), p_b(gy, F(y,x)), p_b(gx_n, F(x,y)),\\ p_b(gy_n, F(y,x)), p_b(gx, gx_{n+1}), p_b(gy, gy_{n+1})\end{array}\right).$$

By taking the upper limit as $n \to \infty$ in (3.41), and using (3.12), (3.38), (3.39) and Lemma 2.10, we obtain

$$\begin{aligned}\limsup_{n\to\infty} M(x_n,y_n,x,y) &\le \max\left\{0,0,0,0,\frac{p_b(gx,F(x,y))}{2s}, \frac{p_b(gy,F(y,x))}{2s},\right.\\ &\qquad\qquad\left.\frac{sp_b(gx,F(x,y))+0}{2s}, \frac{sp_b(gy,F(y,x))+0}{2s}\right\}\\ &\le \max\left\{p_b(gx,F(x,y)), p_b(gy,F(y,x))\right\}.\end{aligned} \tag{3.42}$$

By using Lemma 2.10, (3.42) and the properties of ψ, and taking the upper limit as $n \to \infty$ in (3.40), we obtain

$$\begin{aligned}\psi\left(p_b(gx, F(x,y))\right) &= \psi\left(s \cdot \frac{p_b(gx,F(x,y))}{s}\right)\\ &\le \psi\left(s \limsup_{n\to\infty} p_b(gx_{n+1}, F(x,y))\right)\end{aligned}$$

$$\begin{aligned}
&= \limsup_{n\to\infty} \psi\left(sp_b(gx_{n+1}, F(x,y))\right) &(3.43)\\
&\le \limsup_{n\to\infty} \psi\left(M(x_n, y_n, x, y)\right) - \liminf_{n\to\infty} \Theta(x_n, y_n, x, y)\\
&\le \psi\left(\max\left\{p_b(gx, F(x,y)), p_b(gy, F(y,x))\right\}\right)\\
&\quad - \liminf_{n\to\infty} \Theta(x_n, y_n, x, y).
\end{aligned}$$

Similarly, we can show that

$$\psi\left(p_b(gy, F(y,x))\right) \le \psi\left(\max\left\{p_b(gx, F(x,y)), p_b(gy, F(y,x))\right\}\right) - \liminf_{n\to\infty} \Theta(y_n, x_n, y, x), \tag{3.44}$$

where

$$\Theta(y_n, x_n, y, x) = \theta\begin{pmatrix} p_b(gy_n, gy), p_b(gx_n, gx), p_b(gy_n, gy_{n+1}), p_b(gx_n, gx_{n+1}),\\ p_b(gy, F(y,x)), p_b(gx, F(x,y)), p_b(gy_n, F(y,x)),\\ p_b(gx_n, F(x,y)), p_b(gy, gy_{n+1}), p_b(gx, gx_{n+1}) \end{pmatrix}.$$

By combining (3.43) and (3.44) we obtain

$$\begin{aligned}
\psi(\max\{p_b(gx, F(x,y)), p_b(gy, F(y,x))\}) &= \max\{\psi(p_b(gx, F(x,y))), \psi(p_b(gy, F(y,x)))\}\\
&\le \psi\left(\max\left\{p_b(gx, F(x,y)), p_b(gy, F(y,x))\right\}\right)\\
&\quad - \min\{\liminf_{n\to\infty} \Theta(x_n, y_n, x, y), \liminf_{n\to\infty} \Theta(y_n, x_n, y, x)\}.
\end{aligned}$$

Accordingly, we get

$$\liminf_{n\to\infty} \Theta(x_n, y_n, x, y) = 0 \ \text{ or } \ \liminf_{n\to\infty} \Theta(y_n, x_n, y, x) = 0.$$

By using the properties of θ, we get $gx = F(x,y)$ and $gy = F(y,x)$. That is, (x,y) is a coupled coincidence point of the mappings F and g. □

Remark 3.2. The contractive conditions of Theorem 3.1 is new. As far as now, no author has investigated the problems. Theorem 3.1 improves and extends several well-known comparable results from b-metric spaces and partial metric spaces to ordered partial b-metric spaces.

Corollary 3.3. *Let $(X, \preceq, p_b)$ be a ordered partial b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and F has the mixed g-monotone property with g. Suppose that there exists an altering distance function ψ and $\phi : [0,\infty) \to [0,\infty)$ is continuous with $\phi(t) = 0$ implies $t = 0$ such that*

$$\psi\left(sp_b(F(x,y), F(u,v))\right) \le \psi\left(M(x,y,u,v)\right) - \phi\left(M(x,y,u,v)\right)$$

for all $(x,y), (u,v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, where

$$M(x,y,u,v) = \max\left\{\begin{array}{c} p_b(gx, gu), p_b(gy, gv), p_b(gx, F(x,y)),\\ p_b(gy, F(y,x)), \frac{p_b(gu, F(u,v))}{2s}, \frac{p_b(gv, F(v,u))}{2s},\\ \frac{p_b(gx, F(u,v)) + p_b(gu, F(x,y))}{2s}, \frac{p_b(gy, F(v,u)) + p_b(gv, F(y,x))}{2s} \end{array}\right\}.$$

Further, suppose $F(X \times X) \subset g(X)$ and $g(X)$ is a p_b-complete subspace of (X, p_b). Also, suppose that X satisfies the following properties:

(i) *if a non-decreasing sequence x_n in X converges to $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*

(ii) *if a non-increasing sequence y_n in X converges to $y \in X$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

If there exists $(x_0, y_0) \in X \times X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point.

Proof. Take

$$\theta(t_1, t_2, \cdots, t_{10}) = \phi\left(\max\left\{t_1, t_2, t_3, t_4, \frac{t_5}{2s}, \frac{t_6}{2s}, \frac{t_7 + t_8}{2s}, \frac{t_9 + t_{10}}{2s}\right\}\right),$$

in Theorem 3.1, then Corollary 3.3 holds. □

Remark 3.4. Corollary 3.3 improves and extends Theorem 2.2 in [4] from ordered b-metric space to ordered partial b-metric space.

Corollary 3.5. *Let $(X, \preceq, p_b)$ be a p_b-complete ordered partial b-metric space. Let $F : X \times X \to X$ be a mapping and F has the mixed monotone property on X. Suppose that there exists an altering distance function ψ and $\phi : [0, \infty) \to [0, \infty)$ is continuous with $\phi(t) = 0$ implies $t = 0$ such that*

$$\psi\left(sp_b(F(x, y), F(u, v))\right) \leq \psi\left(M(x, y, u, v)\right) - \phi\left(M(x, y, u, v)\right)$$

for all $(x, y), (u, v) \in X \times X$ with $x \preceq u$ and $y \succeq v$, where

$$M(x, y, u, v) = \max\left\{\begin{array}{c} p_b(x, u), p_b(y, v), p_b(x, F(x, y)), \\ p_b(y, F(y, x)), \frac{p_b(u,F(u,v))}{2s}, \frac{p_b(v,F(v,u))}{2s}, \\ \frac{p_b(x,F(u,v))+p_b(u,F(x,y))}{2s}, \frac{p_b(y,F(v,u))+p_b(v,F(y,x))}{2s} \end{array}\right\}.$$

Further, suppose that X satisfies the following properties:

(i) *if a non-decreasing sequence x_n in X converges to $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*

(ii) *if a non-increasing sequence y_n in X converges to $y \in X$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

If there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point.

Proof. It suffices to take $g = I_x$ in Corollary 3.3. □

Remark 3.6. Corollary 3.5 improves and extends Corollary 2.3 in [4] from ordered b-metric space to ordered partial b-metric space.

Corollary 3.7. *Let $(X, \preceq, p_b)$ be a ordered partial b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and F has the mixed g-monotone property with g. Suppose that there exists $k \in [0, 1)$ such that*

$$p_b(F(x, y), F(u, v)) \leq \frac{k}{s} \max\left\{\begin{array}{c} p_b(gx, gu), p_b(gy, gv), p_b(gx, F(x, y)), \\ p_b(gy, F(y, x)), \frac{p_b(gu,F(u,v))}{2s}, \frac{p_b(gv,F(v,u))}{2s}, \\ \frac{p_b(gx,F(u,v))+p_b(gu,F(x,y))}{2s}, \frac{p_b(gy,F(v,u))+p_b(gv,F(y,x))}{2s} \end{array}\right\}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further, suppose $F(X \times X) \subset g(X)$ and $g(X)$ is a p_b-complete subspace of (X, p_b). Also, suppose that X satisfies the following properties:

(i) *if a non-decreasing sequence x_n in X converges to $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*

(ii) *if a non-increasing sequence y_n in X converges to $y \in X$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

If there exists $(x_0, y_0) \in X \times X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point.

Proof. It suffices to take

$$\theta(t_1, t_2, \cdots, t_{10}) = (1 - k) \max\left\{t_1, t_2, t_3, t_4, \frac{t_5}{2s}, \frac{t_6}{2s}, \frac{t_7 + t_8}{2s}, \frac{t_9 + t_{10}}{2s}\right\},$$

and $\psi(t) = t$ in Theorem 3.1. □

Corollary 3.8. *Let $(X, \preceq, p_b)$ be a ordered partial b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and F has the mixed g-monotone property with g. Suppose that there exist non-negative real numbers $\alpha_1, \alpha_2, \cdots, \alpha_{10}$ with*

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2s(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}) < 1,$$

such that

$$\begin{aligned} sp_b(F(x,y), F(u,v)) \le{} & \alpha_1 p_b(gx, gu) + \alpha_2 p_b(gy, gv) + \alpha_3 p_b(gx, F(x,y)) + \alpha_4 p_b(gy, F(y,x)) \\ & + \alpha_5 p_b(gu, F(u,v)) + \alpha_6 p_b(gv, F(v,u)) + \alpha_7 p_b(gx, F(u,v)) \\ & + \alpha_8 p_b(gu, F(x,y)) + \alpha_9 p_b(gy, F(v,u)) + \alpha_{10} p_b(gv, F(y,x)) \end{aligned} \tag{3.45}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further, suppose $F(X \times X) \subset g(X)$ and $g(X)$ is a p_b-complete subspace of (X, p_b). Also, suppose that X satisfies the following properties:

(i) *if a non-decreasing sequence x_n in X converges to $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*

(ii) *if a non-increasing sequence y_n in X converges to $y \in X$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

If there exists $(x_0, y_0) \in X \times X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point.

Proof. By noting that α_i, $i = 1, 2, \cdots, 10$ are non-negative real numbers, from (3.45) we have

$$\begin{aligned} sp_b(F(x,y), F(u,v)) \le{} & \alpha_1 p_b(gx, gu) + \alpha_2 p_b(gy, gv) + \alpha_3 p_b(gx, F(x,y)) + \alpha_4 p_b(gy, F(y,x)) \\ & + \alpha_5 p_b(gu, F(u,v)) + \alpha_6 p_b(gv, F(v,u)) + \alpha_7 p_b(gx, F(u,v)) \\ & + \alpha_8 p_b(gu, F(x,y)) + \alpha_9 p_b(gy, F(v,u)) + \alpha_{10} p_b(gv, F(y,x)) \\ \le{} & k \max \left\{ \begin{array}{c} p_b(gx, gu), p_b(gy, gv), p_b(gx, F(x,y)), \\ p_b(gy, F(y,x)), \frac{p_b(gu, F(u,v))}{2s}, \frac{p_b(gv, F(v,u))}{2s}, \\ \frac{p_b(gx, F(u,v)) + p_b(gu, F(x,y))}{2s}, \frac{p_b(gy, F(v,u)) + p_b(gv, F(y,x))}{2s} \end{array} \right\}, \end{aligned}$$

where

$$k = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2s(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}) < 1.$$

From Corollary 3.7, we can find that F and g have a coupled coincidence point. □

Corollary 3.9. *Let $(X, \preceq, p_b)$ be a ordered partial b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and F has the mixed g-monotone property with g. Suppose that there exists $k \in [0, 1)$ such that*

$$p_b(F(x,y), F(u,v)) \le \frac{k}{s} \max \{p_b(gx, gu), p_b(gy, gv)\}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further, suppose $F(X \times X) \subset g(X)$ and $g(X)$ is a p_b-complete subspace of (X, p_b). Also, suppose that X satisfies the following properties:

(i) *if a non-decreasing sequence x_n in X converges to $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*

(ii) *if a non-increasing sequence y_n in X converges to $y \in X$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

If there exists $(x_0, y_0) \in X \times X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point.

Proof. Since

$$\begin{aligned} p_b(F(x,y), F(u,v)) \le{} & \frac{k}{s} \max \{p_b(gx, gu), p_b(gy, gv)\} \\ \le{} & \frac{k}{s} \max \left\{ \begin{array}{c} p_b(gx, gu), p_b(gy, gv), p_b(gx, F(x,y)), \\ p_b(gy, F(y,x)), \frac{p_b(gu, F(u,v))}{2s}, \frac{p_b(gv, F(v,u))}{2s}, \\ \frac{p_b(gx, F(u,v)) + p_b(gu, F(x,y))}{2s}, \frac{p_b(gy, F(v,u)) + p_b(gv, F(y,x))}{2s} \end{array} \right\}. \end{aligned}$$

From Corollary 3.7, we can find that F and g have a coupled coincidence point. □

Corollary 3.10. *Let $(X, \preceq, p_b)$ be a ordered partial b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and F has the mixed g-monotone property with g. Suppose that there exists $k \in [0,1)$ such that*

$$p_b(F(x,y), F(u,v)) \le \frac{k}{2s}(p_b(gx,gu) + p_b(gy,gv))$$

for all $(x,y),(u,v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further, suppose $F(X \times X) \subset g(X)$ and $g(X)$ is a p_b-complete subspace of (X, p_b). Also, suppose that X satisfies the following properties:

(i) *if a non-decreasing sequence x_n in X converges to $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*

(ii) *if a non-increasing sequence y_n in X converges to $y \in X$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

If there exists $(x_0, y_0) \in X \times X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point.

Proof. Since

$$p_b(F(x,y), F(u,v)) \le \frac{k}{2s}(p_b(gx,gu) + p_b(gy,gv)) \le \frac{k}{s}\max\left\{p_b(gx,gu), p_b(gy,gv)\right\}.$$

From Corollary 3.9, we can find that F and g have a coupled coincidence point. □

Remark 3.11. If we define $g = I_x, s = 1$, in Corollaries 3.9 and 3.10, then we can get some new results, which improve and extend Theorem 2.2 in [9], and Corollary 2 in [6] from ordered partial metric space to ordered partial b-metric space. These results also extend and generalize the corresponding results of [11, 17, 20].

Now we give an example to show the usability of Theorem 3.1.

Example 3.12. Let $X = [0,1]$ with usual ordering. Define $p_b(x,y) = (\max\{x,y\})^2$. Then $(X, \preceq, p_b)$ is a complete ordered partial b-metric space with coefficient $s = 2$.

Next we define

$$F(x,y) = \frac{1}{3}x(1-y), \quad \text{and} \quad g(x) = \frac{2}{3}x \quad \text{for all } x,y \in X,$$
$$\psi(t) = t, \quad \text{and} \quad \theta(t_1, t_2, \cdots, t_{10}) = \frac{1}{3}\max\left\{t_1, t_2, \cdots, t_6, \frac{t_7+t_8}{2s}, \frac{t_9+t_{10}}{2s}\right\}.$$

Clearly, F has the mixed g-monotone property with g and $F(X \times X) \subset g(X)$. Otherwise,

$$p_b(F(x,y), F(u,v)) = \left(\max\left\{\frac{1}{3}x(1-y), \frac{1}{3}u(1-v)\right\}\right)^2 = \max\left\{\frac{1}{9}x^2(1-y)^2, \frac{1}{9}u^2(1-v)^2\right\},$$

$$\begin{aligned}
M(x,y,u,v) &= \max\left\{\begin{array}{c} p_b(gx,gu), p_b(gy,gv), p_b(gx,F(x,y)), \\ p_b(gy,F(y,x)), \frac{p_b(gu,F(u,v))}{2s}, \frac{p_b(gv,F(v,u))}{2s}, \\ \frac{p_b(gx,F(u,v))+p_b(gu,F(x,y))}{2s}, \frac{p_b(gy,F(v,u))+p_b(gv,F(y,x))}{2s} \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \frac{4}{9}x^2, \frac{4}{9}u^2, \frac{4}{9}y^2, \frac{4}{9}v^2, \frac{1}{9}x^2(1-y)^2, \frac{1}{9}u^2(1-v)^2, \\ \frac{\frac{1}{9}y^2(1-x)^2}{4}, \frac{\frac{1}{9}v^2(1-u)^2}{4}, \frac{\frac{4}{9}x^2+\frac{4}{9}u^2}{4}, \frac{\frac{4}{9}y^2+\frac{4}{9}v^2}{4} \end{array}\right\} \\
&= \frac{4}{9}\max\left\{x^2, u^2, y^2, v^2\right\},
\end{aligned}$$

$$\Theta(x,y,u,v) = \frac{1}{3}M(x,y,u,v) = \frac{4}{27}\max\left\{x^2, u^2, y^2, v^2\right\}.$$

Therefore,

$$\psi\left(M(x,y,u,v)\right)-\Theta(x,y,u,v)=\frac{2}{3}\cdot\frac{4}{9}\max\left\{x^2,u^2,y^2,v^2\right\}=\frac{8}{27}\max\left\{x^2,u^2,y^2,v^2\right\}.$$

Then

$$\begin{aligned}\psi\left(sp_b(F(x,y),F(u,v))\right)&=\max\left\{\frac{2}{9}x^2(1-y)^2,\frac{2}{9}u^2(1-v)^2\right\}\\&\le\max\left\{\frac{2}{9}x^2,\frac{2}{9}u^2\right\}\\&\le\frac{2}{9}\max\left\{x^2,u^2,y^2,v^2\right\}\\&\le\frac{8}{27}\max\left\{x^2,u^2,y^2,v^2\right\}\\&=\psi\left(M(x,y,u,v)\right)-\Theta(x,y,u,v).\end{aligned}$$

At last, define $x_0=0,y_0=0$, then $gx_0\preceq F(x_0,y_0)$ and $gy_0\succeq F(y_0,x_0)$. So, the conditions of Theorem 3.1 are all satisfied. Since $F(0,0)=g(0)$ and $F(0,0)=g(0)$, $(0,0)$ is the coupled coincidence point of F and g.

4. Uniqueness of common fixed points

In this section we prove the existence and uniqueness of common fixed point. If $(X,\preceq)$ is a partially ordered set, first we define product space $X\times X$ with a partial order relation in the following way. For all $(x,y),(u,v)\in X\times X$,

$$(x,y)\preceq(u,v)\Longleftrightarrow x\preceq u,\ y\succeq v.$$

We say that (x,y) and (u,v) are comparable, if $(x,y)\preceq(u,v)$ or $(x,y)\succeq(u,v)$.

Theorem 4.1. *In addition to hypotheses of Theorem* 3.1*, suppose that for every* (x,y) *and* (x^*,y^*) *in* $X\times X$*, there exists* $(u,v)\in X\times X$ *such that* $(F(u,v),F(v,u))$ *is comparable to* $(F(x,y),F(y,x))$ *and to* $(F(x^*,y^*),F(y^*,x^*))$*. Also we assume that* F *commutes with* g*. Then* F *and* g *have a unique common fixed point, that is, there exists* $x\in X$ *such that* $x=gx=F(x,x)$*.*

Proof. From Theorem 3.1, there exists at least a coupled coincidence point. Suppose (x,y) and (x^*,y^*) are coupled coincidence points of F and g, that is, $gx=F(x,y),gy=F(y,x),gx^*=F(x^*,y^*)$ and $gy^*=F(y^*,x^*)$. Next we prove $gx=gx^*,gy=gy^*$. By the assumptions, there exists $(u,v)\in X\times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and to $(F(x^*,y^*),F(y^*,x^*))$. Without loss of generality, we can assume that

$$(F(x,y),F(y,x))\preceq(F(u,v),F(v,u)),(F(x^*,y^*),F(y^*,x^*))\preceq(F(u,v),F(v,u)). \tag{4.1}$$

Put $u_0=u,v_0=v$ and choose $u_1,v_1\in X$ such that $g(u_1)=F(u_0,v_0)$ and $g(v_1)=F(v_0,u_0)$. By continuing this process, we can define sequences $\{gu_n\},\{gv_n\}$ such that

$$gu_{n+1}=F(u_n,v_n)\ \text{ and }\ gv_{n+1}=F(v_n,u_n),\ \ \forall\, n\ge 0.$$

Since

$$\begin{aligned}(F(x,y),F(y,x))&=(gx,gy),\\(F(u,v),F(v,u))&=(gu_1,gv_1).\end{aligned}$$

By using (4.1) we have $gx \preceq gu_1$ and $gy \succeq gv_1$. By using the mixed g-monotone property, we have

$$\begin{aligned} gx &= F(x,y) \preceq F(u_1,y) \preceq F(u_1,v_1) = gu_2, \\ gy &= F(y,x) \succeq F(v_1,x) \succeq F(v_1,u_1) \succeq gv_2. \end{aligned}$$

By going on this, we can show that $gx \preceq gu_n$ and $gy \succeq gv_n$, for all $n \geq 1$. Thus from (3.1) we have

$$\begin{aligned} \psi\left(sp_b(gx, gu_{n+1})\right) &= \psi\left(sp_b(F(x,y), F(u_n,v_n))\right) \\ &\leq \psi\left(M(x,y,u_n,v_n)\right) - \Theta(x,y,u_n,v_n), \end{aligned}$$

where

$$\begin{aligned} M(x,y,u_n,v_n) &= \max\left\{ \begin{array}{c} p_b(gx,gu_n), p_b(gy,gv_n), p_b(gx,F(x,y)), \\ p_b(gy,F(y,x)), \frac{p_b(gu_n,F(u_n,v_n))}{2s}, \frac{p_b(gv_n,F(v_n,u_n))}{2s}, \\ \frac{p_b(gx,F(u_n,v_n))+p_b(gu_n,F(x,y))}{2s}, \frac{p_b(gy,F(v_n,u_n))+p_b(gv_n,F(y,x))}{2s} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} p_b(gx,gu_n), p_b(gy,gv_n), p_b(gx,F(x,y)), \\ p_b(gy,F(y,x)), \frac{p_b(gu_n,gu_{n+1})}{2s}, \frac{p_b(gv_n,gv_{n+1})}{2s}, \\ \frac{p_b(gx,gu_{n+1})+p_b(gu_n,F(x,y))}{2s}, \frac{p_b(gy,gv_{n+1})+p_b(gv_n,F(y,x))}{2s} \end{array} \right\}. \end{aligned}$$

It follows from (p_{b4}) that

$$\begin{aligned} \frac{p_b(gu_n, gu_{n+1})}{2s} &\leq \frac{sp_b(gx,gu_n) + sp_b(gx,gu_{n+1})}{2s} \\ &\leq \max\{p_b(gx,gu_n), p_b(gx,gu_{n+1})\}. \end{aligned}$$

Similarly, we can show that

$$\frac{p_b(gv_n, gv_{n+1})}{2s} \leq \max\{p_b(gy,gv_n), p_b(gy,gv_{n+1})\}.$$

Therefore

$$\begin{aligned} M(x,y,u_n,v_n) &\leq \max\left\{p_b(gx,gu_n), p_b(gy,gv_n), p_b(gx,gu_{n+1}), p_b(gy,gv_{n+1})\right\} \\ &= \max\left\{\gamma_{n-1}, \gamma_n\right\}, \end{aligned}$$

where $\gamma_n = \max\{p_b(gx,gu_{n+1}), p_b(gy,gv_{n+1})\}$. Hence

$$\psi\left(sp_b(gx,gu_{n+1})\right) \leq \psi\left(\max\left\{\gamma_{n-1},\gamma_n\right\}\right) - \Theta(x,y,u_n,v_n), \tag{4.2}$$

where

$$\Theta(x,y,u_n,v_n) = \theta\left(\begin{array}{c} p_b(gx,gu_n), p_b(gy,gv_n), p_b(gx,F(x,y)), p_b(gy,F(y,x)), \\ p_b(gu_n,F(u_n,v_n)), p_b(gv_n,F(v_n,u_n)), p_b(gx,F(u_n,v_n)), \\ p_b(gu_n,F(x,y)), p_b(gy,F(v_n,u_n), p_b(gv_n,F(y,x)) \end{array} \right),$$

Similarly,

$$\psi\left(sp_b(gy,gv_{n+1})\right) \leq \psi\left(\max\left\{\gamma_{n-1},\gamma_n\right\}\right) - \Theta(y,x,v_n,u_n), \tag{4.3}$$

where

$$\Theta(y,x,v_n,u_n) = \theta\left(\begin{array}{c} p_b(gy,gv_n), p_b(gx,gu_n), p_b(gy,F(y,x)), p_b(gx,F(x,y)), \\ p_b(gv_n,F(v_n,u_n)), p_b(gu_n,F(u_n,v_n)), p_b(gy,F(v_n,u_n)), \\ p_b(p_b(gv_n,F(y,x)), p_b(gx,F(u_n,v_n), p_b(gu_n,F(x,y))) \end{array} \right).$$

In the same way of Theorem 3.1 (Case 1 and Case 2), we can prove that $\gamma_n \leq \gamma_{n-1}$ for all $n \in \mathbb{N}$ holds. Therefore, the sequence $\{\gamma_n\}$ is a non-increasing sequence of nonnegative real number, and so, there exists $\gamma \geq 0$ such that $\lim_{n\to\infty} \gamma_n = \gamma$. Next we prove $\gamma = 0$.

By combining (4.2) and (4.3), we get

$$\psi(\gamma_n) \le \psi(s\gamma_n) \le \psi(\gamma_{n-1}) - \min\left\{ \begin{array}{c} \Theta(x, y, u_n, v_n), \\ \Theta(y, x, v_n, u_n) \end{array} \right\}. \tag{4.4}$$

By taking the upper limit as $n \to \infty$ in (4.4), we have

$$\psi(\gamma) \le \psi(\gamma) - \min\left\{ \begin{array}{c} \liminf_{n\to\infty} \Theta(x, y, u_n, v_n), \\ \liminf_{n\to\infty} \Theta(y, x, v_n, u_n) \end{array} \right\}.$$

So,

$$\liminf_{n\to\infty} \Theta(x, y, u_n, v_n) = 0, \quad \text{or} \quad \liminf_{n\to\infty} \Theta(y, x, v_n, u_n) = 0.$$

Hence, by using the properties of θ, we get

$$\liminf_{n\to\infty} p_b(gx, gu_n) = 0, \ \text{ and } \ \liminf_{n\to\infty} p_b(gy, gv_n) = 0.$$

That is,

$$\gamma = \liminf_{n\to\infty} \gamma_{n-1} = \max\{\liminf_{n\to\infty} p_b(gx, gu_n), \liminf_{n\to\infty} p_b(gy, gv_n)\} = 0,$$

which concludes

$$\lim_{n\to\infty} p_b(gx, gu_{n+1}) = 0, \ \text{ and } \ \lim_{n\to\infty} p_b(gy, gv_{n+1}) = 0. \tag{4.5}$$

In the same way, we can get

$$\lim_{n\to\infty} p_b(gx^*, gu_{n+1}) = 0, \ \text{ and } \ \lim_{n\to\infty} p_b(gy^*, gv_{n+1}) = 0. \tag{4.6}$$

From (4.5) and (4.6), we have

$$p_b(gx, gx^*) \le \lim_{n\to\infty} sp_b(gx, gu_{n+1}) + \lim_{n\to\infty} sp_b(gu_{n+1}, gx^*) = 0.$$

That is $gx = gx^*$. Similarly, $gy = gy^*$. This implies the uniqueness of coupled coincidence point. On the other hand, (y, x) is also the coupled coincidence point of F and g. So, $gx = gy$.

Define $t = gx$. By the commutativity of F and g, we have

$$gt = g(gx) = gF(x, y) = F(gx, gy) = F(t, t).$$

Thus, (gt, gt) is a coupled coincidence point. It follows that $gt = gx = t$, that is, $t = gt = F(t, t)$. Therefore, (t, t) is a common fixed point of F and g. Finally, we prove the uniqueness, assume that (s, s) is another common fixed point, that is $s = gs = F(s, s)$. Since (gs, gs) is a coupled coincidence point of F and g, we have $gs = gt$, that is $s = t$, which is the desired result. □

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