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On properties of solutions to the improved modified Boussinesq equation

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Abstract

In this paper, we investigate the Cauchy problem for the generalized IBq equation with damping in one dimensional space. When $\sigma = 1$, the nonlinear approximation of the global solutions is established under small condition on the initial value. Moreover, we show that as time tends to infinity, the solution is asymptotic to the superposition of nonlinear diffusion waves which are given explicitly in terms of the self-similar solution of the viscous Burgers equation. When $\sigma \geq 2$, we prove that our global solution converges to the superposition of diffusion waves which are given explicitly in terms of the solution of linear parabolic equation.

Keywords: IMBq equation with damping, large time behavior, diffusion waves.
2010 MSC: 35L30, 35L75.

1. Introduction

We investigate the Cauchy problem for the following generalized improved modified Boussinesq (IBq) equation with damping in one space dimension

$$u_{tt} - u_{xxtt} - u_{xx} - \nu u_{xxt} = \phi(u)_{xx} \tag{1.1}$$

with the initial value

$$t = 0 : u = f(x), \quad u_t = \partial_x g(x). \tag{1.2}$$

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Here $u = u(x,t)$ is the unknown function of $x \in \mathbb{R}$ and $t > 0$, $\nu > 0$ is a constant. The nonlinear term is as form of $\phi(u) = O(|u|^{1+\sigma})$ $(\sigma \geq 1)$.

Boussinesq [2, 3] deduced an important model

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta(u^2), \tag{1.3}$$

which approximately describes the propagation of long waves on shallow water. Equation (1.3) is called improved Boussinesq (IBq) equation by [11]. Equation (1.3) and its generalized form in n space dimensions

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta\phi(u) \tag{1.4}$$

can also describe the dynamical and thermodynamical properties of an harmonic monatomic and diatomic chains (see [14, 15]). Existence and nonexistence of global solutions, the global existence of small amplitude solutions for the Cauchy problem for (1.4) were obtained by Wang et al. [13, 19, 20]. Cho and Ozawa [4] studied the existence and scattering of global small amplitude solutions to (1.4).

To take into account internal friction (it is called this type of friction hydrodynamical), which is due to irreversible processes taking place within the system, the dissipation function depends on the time derivatives of the relative displacements, in [1] the authors obtained the following IBq equation with damping

$$u_{tt} - \Delta u - \Delta u_{tt} - \nu\Delta u_t = \Delta(u^2).$$

Equation (1.4) has the following generalized form

$$u_{tt} - \Delta u - \Delta u_{tt} - \nu\Delta u_t = \Delta\phi(u). \tag{1.5}$$

Polat [13] established the global existence and blow-up of solutions to (1.5) with the initial data.

$$(u, u_t)(x, 0) = (u_0, u_1)(x). \tag{1.6}$$

Under smallness condition on the initial data, Wang and Xu [23] obtained asymptotic behavior of global solutions to (1.5) and (1.6) by the contraction mapping principle. Later, global existence and asymptotic behavior of solutions were refined in [24]. More precisely, the decay estimate

$$\|\partial_x^k u(t)\|_{L^2} \leq C\mathscr{E}_1(1+t)^{-\frac{n}{4}-\frac{k}{2}} \tag{1.7}$$

is obtained, where $n\sigma \geq 1$ and $s \geq [n/2] + 1$, $\mathscr{E}_1 = \|u_0\|_{H^s\cap L^1} + \|u_1\|_{H^s\cap \dot{W}^{-1,1}}$ and $0 \leq k \leq s$. Moreover, when $n\sigma \geq 2$, the proof in [24] also implies

$$\|\partial_x^k (u - u_L)(t)\|_{L^2} \leq C\mathscr{E}_1^{1+\sigma}(1+t)^{-\frac{n}{4}-\frac{k}{2}}\eta(t) \tag{1.8}$$

for $0 \leq k \leq s$, where $u_L(t)$ is the solution to (1.5) and (1.6) with $\phi(u) \equiv 0$ and $\eta(t)$ is defined by

$$\eta(t) = \begin{cases} 1, \quad n = 1, \\ (1+t)^{-\frac{1}{2}}\log(2+t), \quad n\sigma = 2, \\ (1+t)^{-\frac{1}{2}}, \quad n\sigma \geq 3. \end{cases}$$

However, such a linear approximation does not hold for (1.1) with $\sigma = 1$. This comes from the slower decay of the solution for $n = 1$ and $\sigma = 1$. We note that, when $n = 1$, the decay estimate (1.7) for the problem (1.1), (1.2) is given by

$$\|\partial_x^k u(t)\|_{L^2} \leq CE_1(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \tag{1.9}$$

where $s \geq 0$, $0 \leq k \leq s$, and $\mathscr{E}_1 = \|f\|_{H^s\cap L^1} + \|g\|_{H^{s+1}\cap L^1}$.

The first main purpose of this paper is to establish nonlinear approximation to global solutions to the problem (1.1), (1.2) with $\sigma = 1$. We state the result as follows.

Theorem 1.1. *Let $\sigma = 1$ and $s \geq 1$. Assume that $f \in H^s \bigcap L^1$ and $g \in H^{s+1} \bigcap L^1$, and put $\mathscr{E}_1 = \|f\|_{H^s \bigcap L^1} + \|g\|_{H^{s+1} \bigcap L^1}$. Let $u(x,t)$ be the global solution to the problem (1.1), (1.2), and let ϖ be the approximation function defined by (3.10). Then for any $\varepsilon > 0$ and $0 \leq k \leq s$, there is a small positive constant δ_2 such that if $E_1 \leq \delta_2$, we have*

$$\|\partial_x^k (u - \varpi)(t)\|_{L^2} \leq C\mathscr{E}_1 (1+t)^{-\frac{3}{4}-\frac{k}{2}+\varepsilon}.$$

Theorem 1.1 implies that the global solution u to the problem (1.1), (1.2) is well approximated by the solution ϖ to the simpler problem (3.10). In the following result, we give the further approximation, i.e., we show that as time tends to infinity, the solution is asymptotic to the superposition of nonlinear diffusion waves which are given explicitly in terms of the self-similar solution of the viscous Burgers equation. The result is as follows:

Theorem 1.2. *Let $\sigma = 1$ and $s \geq 1$. Assume that $f, g \in H^{s+1} \cap L^1_{1/2}$ and put $\tilde{\mathscr{E}}_1 = \|(f,g)\|_{H^{s+1} \cap L^1}$ and $\mathscr{E}_2 = \|(f,g)\|_{H^{s+1} \cap L^1_{1/2}}$. Let u be the global solution to the problem (1.1), (1.2), and let $v_\pm$ be the nonlinear diffusion waves defined by (4.9) with the parameters in (4.11). Then there is a small positive constant δ_3 such that if $\tilde{E}_1 \leq \delta_3$, then we have*

$$\|\partial_x^k (u - v_+ - v_-)(t)\|_{L^2} \leq C\mathscr{E}_2 (1+t)^{-\frac{1}{2}-\frac{k}{2}}, \tag{1.10}$$

where $0 \leq k \leq s$.

When $\sigma \geq 2$, our global solution is approximated by the superposition of diffusion waves which are given explicitly in terms of the solution of linear parabolic equation. We state the results as follows:

Theorem 1.3. *Let $s \geq 1$ and $\sigma = 2$. Assume that $f, g \in H^{s+1} \bigcap L^1_1$. Put $\mathscr{E}_1 = \|(f,g)\|_{H^{s+1} \bigcap L^1}$ and $\mathscr{E}_3 = \|(f,g)\|_{H^{s+1} \bigcap L^1_1}$. Let u be the global solution to the problem (1.1), (1.2), and let $v_\pm$ be the diffusion waves defined by (5.2). There exists a small positive constant δ_3 such that if $\mathscr{E}_1 \leq \delta_3$, we have*

$$\|\partial_x^k (u - v_+ - v_-)(t)\|_{L^2} \leq CE_3 (1+t)^{-\frac{3}{4}-\frac{k}{2}} \log(2+t)$$

for $0 \leq k \leq s$.

Theorem 1.4. *Let $s \geq 1$ and $\sigma \geq 3$. Assume that $f, g \in H^{s+1} \bigcap L^1_1$. Put $\mathscr{E}_1 = \|(f,g)\|_{H^{s+1} \bigcap L^1}$ and $\mathscr{E}_3 = \|(f,g)\|_{H^{s+1} \bigcap L^1_1}$. Let u be the global solution to the problem (1.1), (1.2), and let $v_\pm$ be the diffusion waves defined by (5.2). There exists a small positive constant δ_3 such that if $E_1 \leq \delta_3$, we have*

$$\|\partial_x^k (u - v_+ - v_-)(t)\|_{L^2} \leq C\mathscr{E}_3 (1+t)^{-\frac{3}{4}-\frac{k}{2}}$$

for $0 \leq k \leq s$.

The global existence and asymptotic behavior of solutions to high order wave equation have been investigated by many authors. We may refer to [6, 8, 16–18, 21–24]. For quantum stochastic evolution inclusions and variational inclusions, some related results have been established in [12].

The paper is organized as follows. In Section 2 we review the previous results on the problem (1.1), (1.2). A nonlinear approximation of global solutions to (1.1), (1.2) with $\sigma = 1$ is established in Section 3. In Section 4, when $\sigma = 1$, we prove that global solution is asymptotic to the superposition of nonlinear diffusion waves which are given explicitly in terms of the self-similar solution of the viscous Burgers equation. Finally, large time behavior of global solutions is obtained for $\sigma \geq 2$ in Section 5.

2. Decay property of solution operator

To prove our main results, we need to deduce the solution formula for the problem (1.1), (1.2), which will be used in the present paper (see also [24]). For this purpose, we first investigate the linear equation of (1.1):

$$u_{tt} - u_{xxtt} - u_{xx} - \nu u_{xxt} = 0. \tag{2.1}$$

Taking the Fourier transform, we have

$$(1+\xi^2)\hat{u}_{tt}+\nu\xi^2\hat{u}_t+\xi^2\hat{u}=0. \tag{2.2}$$

The corresponding initial values are given as

$$t=0:\hat{u}=\hat{f}(\xi),\hat{u}_t=i\xi\hat{g}(\xi). \tag{2.3}$$

The characteristic equation of (2.2) is

$$(1+\xi^2)\lambda^2+\nu\xi^2\lambda+\xi^2=0. \tag{2.4}$$

Let $\lambda=\lambda_\pm(\xi)$ be the corresponding eigenvalues of (2.4), we obtain

$$\lambda_\pm(\xi)=\frac{-\mu\xi^2\pm i\xi\sqrt{(4-\nu^2)\xi^2-4}}{2(1+\xi^2)}.$$

The solution to the problem (2.2)-(2.3) is given in the form

$$\hat{u}(\xi,t)=i\xi\hat{\mathcal{G}}(\xi,t)\hat{g}(\xi)+\hat{\mathcal{H}}(\xi,t)\hat{f}(\xi), \tag{2.5}$$

where

$$\hat{\mathcal{G}}(\xi,t)=\frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}(e^{\lambda_+(\xi)t}-e^{\lambda_-(\xi)t}) \tag{2.6}$$

and

$$\hat{\mathcal{H}}(\xi,t)=\frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}(\lambda_+(\xi)e^{\lambda_-(\xi)t}-\lambda_-(\xi)e^{\lambda_+(\xi)t}). \tag{2.7}$$

We define $\mathcal{G}(x,t)$ and $\mathcal{H}(x,t)$ by $\mathcal{G}(x,t)=\mathscr{F}^{-1}[\hat{\mathcal{G}}(\xi,t)](x)$ and $\mathcal{H}(x,t)=\mathscr{F}^{-1}[\hat{\mathcal{H}}(\xi,t)](x)$, respectively, where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. Then, applying $\mathscr{F}^{-1}$ to (2.5), we obtain

$$u(t)=\mathcal{G}(t)*\partial_x g+\mathcal{H}(t)*f. \tag{2.8}$$

By the Duhamel principle, we obtain the solution formula to (1.1), (1.2)

$$u(t)=\mathcal{G}(t)*\partial_x g+\mathcal{H}(t)*f+\int_0^t\mathcal{G}(t-\tau)*(I-\partial_x^2)^{-1}\partial_x^2\phi(u)(\tau)d\tau. \tag{2.9}$$

Next, we state the decay estimates of the solution operators $\mathcal{G}(t)$ and $\mathcal{H}(t)$ appearing in the solution formula (2.8), which was established in [23] and [24].

Lemma 2.1. *The solution of the problem* (2.2), (2.3) *satisfies*

$$(1+\xi^2)|\hat{u}_t(\xi,t)|^2+\xi^2|\hat{u}(\xi,t)|^2\le Ce^{-c\omega(\xi)t}((1+\xi^2)\xi^2|\hat{g}(\xi)|^2+\xi^2|\hat{f}(\xi)|^2)$$

for $\xi\in\mathbb{R}$ *and* $t\ge0$, *where* $\omega(\xi)=\frac{\xi^2}{1+\xi^2}$.

Lemma 2.2. *Let* $\hat{\mathcal{G}}(\xi,t)$ *and* $\hat{\mathcal{H}}(\xi,t)$ *be the fundamental solution of* (2.1) *in the Fourier space, which are given in* (2.6) *and* (2.7), *respectively. Then we have the estimates*

$$|\hat{\mathcal{G}}(\xi,t)|\le C|\xi|^{-1}(1+\xi^2)^{\frac{1}{2}}e^{-c\omega(\xi)t}$$

and

$$|\hat{\mathcal{H}}(\xi,t)|\le Ce^{-c\omega(\xi)t}$$

for $\xi\in\mathbb{R}$ *and* $t\ge0$, *where* $\omega(\xi)=\frac{\xi^2}{1+\xi^2}$.

Lemma 2.3. *Let* $k\ge0$ *and* $1\le p\le2$. *Then we have*

$$\|\partial_x^k\mathcal{G}(t)*\partial_x\varphi\|_{L^2}\le C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\varphi\|_{L^p}+Ce^{-ct}\|\partial_x^k\varphi\|_{L^2},$$
$$\|\partial_x^k\mathcal{H}(t)*\varphi\|_{L^2}\le C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\varphi\|_{L^p}+Ce^{-ct}\|\partial_x^k\varphi\|_{L^2},$$

and

$$\|\partial_x^k\mathcal{G}(t)*(I-\partial_x^2)^{-1}\partial_x^2\varphi\|_{L^2}\le C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}-\frac{1}{2}}\|\varphi\|_{L^p}+Ce^{-ct}\|\partial_x^k\varphi\|_{L^2}. \tag{2.10}$$

3. Approximate to solution to (1.1), (1.2) with $\sigma = 1$

In this section, our main aim is to obtain the nonlinear approximation to the global solutions. It follows from mean value theorem that

$$\begin{cases} e^{-\frac{\nu|\xi|^2 t}{2(1+|\xi|^2)}} = e^{-\frac{\nu}{2}|\xi|^2 t} + \bar{K}_1, \\ \sin \dfrac{|\xi|\sqrt{1-\frac{4-\nu^2}{|\xi|^2}|\xi|^2}t}{1+|\xi|^2} = \sin(|\xi|t) + \bar{K}_2, \\ \cos \dfrac{|\xi|\sqrt{1-\frac{4-\nu^2}{|\xi|^2}|\xi|^2}t}{1+|\xi|^2} = \cos(|\xi|t) + \bar{K}_3, \\ \dfrac{1}{\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}} = 1 + \bar{K}_4, \end{cases}$$

where

$$\begin{cases} \bar{K}_1 = \dfrac{\nu|\xi|^4 t}{2(1+|\xi|^2)} e^{-\frac{\nu}{2}|\xi|^2[\frac{\theta_1}{1+|\xi|^2}+(1-\theta_1)]t}, \\ \bar{K}_2 = -\dfrac{|\xi|^3(|\xi|^2+\frac{12-\nu^2}{4})t}{(1+|\xi|^2)(\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}+1+|\xi|^2)} \cos\Big[\dfrac{|\xi|\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}t}{1+|\xi|^2}\theta_2 + (1-\theta_2)|\xi|t\Big], \\ \bar{K}_3 = \dfrac{|\xi|^3(|\xi|^2+\frac{12-\nu^2}{4})t}{(1+|\xi|^2)(\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}+1+|\xi|^2)} \sin\Big[\dfrac{|\xi|\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}t}{1+|\xi|^2}\theta_3 + (1-\theta_3)|\xi|t\Big], \\ \bar{K}_4 = \dfrac{(4-\nu^2)|\xi|^2}{8(1-\frac{4-\nu^2}{4}|\xi|^2\theta_4)^{\frac{3}{2}}} \end{cases}$$

with $\theta_i(i = 1, 2, 3, 4) \in (0, 1)$.

When $|\xi| \le \delta$, where δ is a small positive constant, we obtain from the above four equalities

$$\begin{aligned} \hat{\mathcal{G}}(\xi,t) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} &= \frac{1+|\xi|^2}{|\xi|\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}} e^{-\frac{\nu|\xi|^2 t}{2(1+|\xi|^2)}} \sin \frac{|\xi|\sqrt{1-\frac{4-\nu^2}{|\xi|^2}|\xi|^2}t}{1+|\xi|^2} \\ &= \frac{1}{|\xi|} e^{-\frac{\nu}{2}\xi^2 t} \sin|\xi|t + \bar{J}_1 \end{aligned}$$

and

$$\begin{aligned} \hat{\mathcal{H}}(\xi,t) &= \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \\ &= \frac{v|\xi|}{2\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}} e^{-\frac{\nu|\xi|^2 t}{2(1+|\xi|^2)}} \sin \frac{|\xi|\sqrt{1-\frac{4-\nu^2}{|\xi|^2}|\xi|^2}t}{1+|\xi|^2} + e^{-\frac{\nu|\xi|^2 t}{2(1+|\xi|^2)}} \cos \frac{|\xi|\sqrt{1-\frac{4-\nu^2}{|\xi|^2}|\xi|^2}t}{1+|\xi|^2} \\ &= e^{-\frac{\nu}{2}\xi^2 t} \cos|\xi|t + \bar{J}_2. \end{aligned} \tag{3.1}$$

When $|\xi| \le \delta$, $\bar{J}_1$ and $\bar{J}_2$ satisfy

$$|\bar{J}_1| \le C(1+|\xi|^2 t)e^{-c|\xi|^2 t}$$

and

$$|\bar{J}_2| \le C(|\xi|+|\xi|^3 t)e^{-c|\xi|^2 t}.$$

Taking

$$\hat{\mathcal{G}}_0(\xi,t)=\frac{1}{|\xi|}e^{-\frac{\nu}{2}|\xi|^2t}\sin|\xi|t,\qquad \hat{\mathcal{H}}_0(\xi,t)=e^{-\frac{\nu}{2}|\xi|^2t}\cos|\xi|t.$$

Then

$$|(\hat{\mathcal{G}}-\hat{\mathcal{G}}_0)(\xi,t)|\le Ce^{-c|\xi|^2t},\qquad |(\hat{\mathcal{H}}-\hat{\mathcal{H}}_0)(\xi,t)|\le C|\xi|e^{-c|\xi|^2t} \tag{3.2}$$

for $|\xi|\le\delta$.

Lemma 3.1. *Let $k\ge 0$ and $1\le p\le 2$. Then we have*

$$\|\partial_x^k\mathcal{G}_0(t)*\partial_x\varphi\|_{L^2}\le C(1+t)^{-\frac12(\frac1p-\frac12)-\frac k2}\|\varphi\|_{L^p}+Ce^{-ct}\|\partial_x^k\varphi\|_{L^2}, \tag{3.3}$$

$$\|\partial_x^k\mathcal{H}_0(t)*\varphi\|_{L^2}\le C(1+t)^{-\frac12(\frac1p-\frac12)-\frac k2}\|\varphi\|_{L^p}+Ce^{-ct}\|\partial_x^k\varphi\|_{L^2}, \tag{3.4}$$

$$\|\partial_x^k\mathcal{G}_0(t)*\partial_x\varphi\|_{L^2}\le C(1+t)^{-\frac12(\frac1p-\frac12)-\frac k2}\|\varphi\|_{L^p}+Ce^{-ct}t^{-\frac{k-l}{2}}\|\partial_x^l\varphi\|_{L^2}, \tag{3.5}$$

and

$$\|\partial_x^k\mathcal{G}_0(t)*\partial_x\varphi\|_{L^2}\le Ct^{-\frac12(\frac1p-\frac12)-\frac k2}\|\varphi\|_{L^p}. \tag{3.6}$$

Proof. We only give the proof of (3.5). The Plancherel theorem entails that

$$\|\partial_x^k\mathcal{G}_0(t)*\partial_x\phi\|_{L^2}^2=\int_{|\xi|\le1}|\xi|^{2k+2}|\hat{\mathcal{G}}_0(\xi,t)|^2|\hat{\varphi}(\xi)|^2d\xi+\int_{|\xi|\ge1}|\xi|^{2k+2}|\hat{\mathcal{G}}_0(\xi,t)|^2|\hat{\varphi}(\xi)|^2d\xi=:I_1+I_2. \tag{3.7}$$

By (3.1), Hölder inequality and Hausdorff inequality, we have

$$I_1\le C(1+t)^{-\frac12(\frac2p-1)-k}\|\varphi\|_{L^p}^2. \tag{3.8}$$

Owing to (3.1), I_2 can be estimated as

$$I_2\le Ce^{-ct}\sup_{|\xi|\ge1}(|\xi|^{2(k-l)}e^{-c|\xi|^2t})\int_{|\xi|\ge1}|\xi|^{2l}|\hat{\varphi}|^2d\xi\le Ce^{-ct}t^{-(k-l)}\|\partial_x^l\varphi\|_{L^2}^2. \tag{3.9}$$

Inserting (3.8) and (3.9) into (3.7) yields (3.5). Thus we have completed the proof. □

Let

$$\varpi(t)=\mathcal{G}_0(t)*\partial_xg+\mathcal{H}_0(t)*f+\frac{\phi''(0)}{2}\int_0^t\mathcal{G}_0(t-\tau)*\partial_x^2\varpi^2(\tau)d\tau. \tag{3.10}$$

In order to obtain nonlinear approximation of global solutions to the Cauchy problem (1.1), (1.2), we need the following lemma, which comes from [9] (see also [25]).

Lemma 3.2. *Assume that $\phi=\phi(v)$ is a smooth function. Suppose that $\phi(v)=O(|v|^{1+\theta})$ ($\theta\ge1$ is an integer) when $|v|\le\nu_0$. Then for integer $m\ge0$, if $v\in W^{m,q}(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$ and $\|v\|_{L^\infty}\le\nu_0$, then $\phi(v)\in W^{m,r}(\mathbb{R}^n)$. Furthermore, the following inequality holds:*

$$\|\partial_x^m\phi(v)\|_{L^r}\le C\|v\|_{L^p}\|\partial_x^mv\|_{L^q}\|v\|_{L^\infty}^{\theta-1},$$

where $1\le p,q,r\le+\infty$ and $\frac1r=\frac1p+\frac1q$.

Lemma 3.3. *Let $s\ge1$. Assume that $f,g\in H^s(\mathbb{R})\bigcap L^1(\mathbb{R})$. Put*

$$\mathscr{E}_0:=\|f\|_{H^s\bigcap L^1}+\|g\|_{H^s\bigcap L^1}.$$

$\varpi(t)$ is defined by (3.10). If $\mathscr{E}_0$ is suitably small, then

$$\|\partial_x^k\varpi(t)\|_{L^2}\le C\mathscr{E}_0(1+t)^{-\frac14-\frac k2} \tag{3.11}$$

for $0\le k\le s$.

Remark 3.4. If "$f, g \in H^s(\mathbb{R})\bigcap L^1(\mathbb{R})$" is replaced by "$f \in H^s(\mathbb{R})\bigcap L^1(\mathbb{R})$ and $g \in H^{s+1}(\mathbb{R})\bigcap L^1(\mathbb{R})$", put

$$\mathscr{E}_1 := \|f\|_{H^s\bigcap L^1} + \|g\|_{H^{s+1}\bigcap L^1}.$$

If $\mathscr{E}_1$ is suitably small, then

$$\|\partial_x^k \varpi(t)\|_{L^2} \le CE_1(1+t)^{-\frac{1}{4}-\frac{k}{2}}. \tag{3.12}$$

Proof. We only prove (3.11). Set

$$M(t) = \sum_{k=0}^{s} \sup_{0\le\tau\le t} (1+\tau)^{\frac{1}{4}+\frac{k}{2}} \|\partial_x^k \varpi(t)\|_{L^2}.$$

By applying (3.10) and Minkowski inequality, we arrive at

$$\begin{aligned} \|\partial_x^k \varpi(t)\|_{L^2} &\le \|\partial_x^k \mathcal{G}_0(t) * \partial_x g\|_{L^2} + \|\partial_x^k \mathcal{H}_0(t) * f\|_{L^2} + \int_0^{\frac{t}{2}} \|\partial_x^k \mathcal{G}_0(t-\tau) * \partial_x^2(\frac{\phi''(0)}{2}\varpi^2)\|_{L^2}(\tau)d\tau \\ &\quad + \int_{\frac{t}{2}}^{t} \|\partial_x^k \mathcal{G}_0(t-\tau) * \partial_x^2(\frac{\phi''(0)}{2}\varpi^2)\|_{L^2}(\tau)d\tau \\ &\triangleq I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.13}$$

It follows from (3.3) with $p = 1$ that

$$I_1 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|g\|_{H^s\bigcap L^1}. \tag{3.14}$$

Due to (3.4) with $p = 1$ to I_2, it holds that

$$I_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|f\|_{H^s\bigcap L^1}. \tag{3.15}$$

Equation (3.5), Lemma 3.2, and Gagliardo-Nirenberg inequality entail that

$$\begin{aligned} I_3 &\le C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} \|\varpi^2\|_{L^1} d\tau + C\int_0^{\frac{t}{2}} e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}} \|\partial_x^k \varpi^2\|_{L^2} d\tau \\ &\le CM^2(t)\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{1}{4}-\frac{k}{2}-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}} d\tau \\ &\quad + CM^2(t)\int_0^{\frac{t}{2}} e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{4}-\frac{k}{2}} d\tau \\ &\le CM^2(t)(1+t)^{-\frac{1}{4}-\frac{k}{2}}. \end{aligned} \tag{3.16}$$

By exploiting (3.6), Lemma 3.2, and Gagliardo-Nirenberg inequality, we get

$$\begin{aligned} I_4 &\le C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|\partial_x^k \varpi^2\|_{L^2} d\tau \\ &\le CM^2(t)\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{4}-\frac{k}{2}} d\tau \\ &\le CM^2(t)(1+t)^{-\frac{1}{4}-\frac{k}{2}}. \end{aligned} \tag{3.17}$$

Combining (3.13)-(3.17) yields

$$M(t) \le C\mathscr{E}_0 + CM^2(t),$$

This inequality can be solved as $M(t) \le C\mathscr{E}_0$ if $\mathscr{E}_0$ is sufficiently small. Thus we have completed the proof of lemma. ∎

Lemma 3.5. *Let $k \geq 0$ and $1 \leq p \leq 2$. Then we have*

$$\|\partial_x^k(\mathcal{G}-\mathcal{G}_0)(t) * \partial_x\varphi\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}-\frac{1}{2}}\|\varphi\|_{L^p} + Ce^{-ct}\|\partial_x^{k+1}\varphi\|_{L^2}, \tag{3.18}$$

$$\|\partial_x^k(\mathcal{G}-\mathcal{H}_0)(t) * \varphi\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}-\frac{1}{2}}\|\varphi\|_{L^p} + Ce^{-ct}\|\partial_x^{k}\varphi\|_{L^2}, \tag{3.19}$$

$$\|\partial_x^k\mathcal{G}_0(t) * \{(1-\partial_x^2)^{-1}-I\}\partial_x^2\varphi\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}-\frac{3}{2}}\|\varphi\|_{L^p} + Ce^{-ct}t^{-\frac{k+1-l}{2}}\|\partial_x^{l}\varphi\|_{L^2}, \tag{3.20}$$

and

$$\|\partial_x^k(\mathcal{G}-\mathcal{G}_0)(t) * (1-\partial_x^2)^{-1}\partial_x^2\varphi\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}-1}\|\varphi\|_{L^p} + Ce^{-ct}\|\partial_x^{k+l}\varphi\|_{L^2}.$$

Proof. We only give the proof of (3.18). By applying the Plancherel theorem, we deduce that

$$\begin{aligned}\|\partial_x^k(\mathcal{G}-\mathcal{G}_0)(t) * \partial_x\varphi\|_{L^2}^2 &= \int_{|\xi|\leq\delta}|\xi|^{2k+2}|(\hat{\mathcal{G}}-\hat{\mathcal{G}}_0)(\xi,t)|^2|\hat{\varphi}(\xi)|^2d\xi \\ &\quad + \int_{|\xi|\geq\delta}|\xi|^{2k+2}|(\hat{\mathcal{G}}-\hat{\mathcal{G}}_0)(\xi,t)|^2|\hat{\varphi}(\xi)|^2d\xi \\ &=: I_1 + I_2.\end{aligned} \tag{3.21}$$

For the low frequency part I_1, using (3.2), Hölder inequality and Hausdorff inequality, we estimate as

$$I_1 \leq C(\int_{|\xi|\leq\delta}|\xi|^{(2k+2)q}e^{-cq|\xi|^2t}d\xi)^{\frac{1}{q}}\|\hat{\varphi}\|_{L^{p'}}^2 \leq C(1+t)^{-\frac{1}{2}(\frac{2}{p}-1)-k-1}\|\varphi\|_{L^p}^2, \tag{3.22}$$

where $\frac{1}{q}+\frac{2}{p'}=1$ and $\frac{1}{p}+\frac{1}{q}=1$. Also, for the high frequency part I_2, we have

$$I_2 \leq C\int_{|\xi|\geq\delta}|\xi|^{2k+2}e^{-ct}|\hat{\varphi}|^2d\xi \leq Ce^{-ct}\|\partial_x^{k+1}\varphi\|_{L^2}^2. \tag{3.23}$$

Combining (3.21), (3.22) and (3.23) yields (3.18). Thus we have completed the proof of lemma. □

In what follows, we prove Theorem 1.1.

Proof. We introduce the quantity

$$X(t) = \sum_{k=0}^{s}\sup_{0\leq\tau\leq t}(1+\tau)^{\frac{3}{4}+\frac{k}{2}-\varepsilon}\|\partial_x^k(u-\varpi)\|_{L^2},$$

where $\varepsilon > 0$ is a fixed small constant. Due to (2.9) and (3.10), we arrive at

$$\begin{aligned}(u-\varpi)(t) &= (\mathcal{G}-\mathcal{G}_0)(t) * \partial_x g + (\mathcal{H}-\mathcal{H}_0)(t) * f \\ &\quad + \int_0^t \mathcal{G}(t-\tau) * (I-\partial_x^2)^{-1}\partial_x^2(\phi(u) - \frac{\phi''(0)}{2}u^2)(\tau)d\tau \\ &\quad + \frac{\phi''(0)}{2}\int_0^t \mathcal{G}(t-\tau) * (I-\partial_x^2)^{-1}\partial_x^2\{(u+\varpi)(u-\varpi)\}(\tau)d\tau \\ &\quad + \frac{\phi''(0)}{2}\int_0^t (\mathcal{G}-\mathcal{G}_0)(t-\tau) * (I-\partial_x^2)^{-1}\partial_x^2(\varpi^2)(\tau)d\tau \\ &\quad + \frac{\phi''(0)}{2}\int_0^t \mathcal{G}_0(t-\tau) * \{(I-\partial_x^2)^{-1}-I\}\partial_x^2(\varpi^2)(\tau)d\tau.\end{aligned} \tag{3.24}$$

Owing to (3.24) and Minkowski inequality, we get

$$\begin{aligned}\|\partial_x^k(u-\varpi)(t)\|_{L^2} \le\ & \|\partial_x^k(\mathcal{G}-\mathcal{G}_0)(t)*\partial_x g\|_{L^2}+\|\partial_x^k(H-H_0)(t)*f\|_{L^2}\\ &+\int_0^{\frac{t}{2}}\|\partial_x^k\mathcal{G}(t-\tau)*(I-\partial_x^2)^{-1}\partial_x^2(\phi(u)-\frac{\phi''(0)}{2}u^2)(\tau)\|_{L^2}d\tau\\ &+\int_{\frac{t}{2}}^t\|\partial_x^k\mathcal{G}(t-\tau)*(I-\partial_x^2)^{-1}\partial_x^2(\phi(u)-\frac{\phi''(0)}{2}u^2))(\tau)\|_{L^2}d\tau\\ &+\frac{|\phi''(0)|}{2}\int_0^{\frac{t}{2}}\|\partial_x^k\mathcal{G}(t-\tau)*(I-\partial_x^2)^{-1}\partial_x^2\{(u+\varpi)(u-\varpi)\}(\tau)\|_{L^2}d\tau\\ &+\frac{|\phi''(0)|}{2}\int_{\frac{t}{2}}^t\|\partial_x^k\mathcal{G}(t-\tau)*(I-\partial_x^2)^{-1}\partial_x^2\{(u+\varpi)(u-\varpi)\}(\tau)\|_{L^2}d\tau\\ &+\frac{|\phi''(0)|}{2}\int_0^{\frac{t}{2}}\|\partial_x^k(\mathcal{G}-\mathcal{G}_0)(t-\tau)*(I-\partial_x^2)^{-1}\partial_x^2(\varpi^2)(\tau)\|_{L^2}d\tau\\ &+\frac{|\phi''(0)|}{2}\int_{\frac{t}{2}}^t\|\partial_x^k(\mathcal{G}-\mathcal{G}_0)(t-\tau)*(I-\partial_x^2)^{-1}\partial_x^2(\varpi^2)(\tau)\|_{L^2}d\tau\\ &+\frac{|\phi''(0)|}{2}\int_0^{\frac{t}{2}}\|\partial_x^k\mathcal{G}_0(t-\tau)*\{(I-\partial_x^2)^{-1}-I\}\partial_x^2(\varpi^2)(\tau)\|_{L^2}d\tau\\ &+\frac{|\phi''(0)|}{2}\int_{\frac{t}{2}}^t\|\partial_x^k\mathcal{G}_0(t-\tau)*\{(I-\partial_x^2)^{-1}-I\}\partial_x^2(\varpi^2)(\tau)\|_{L^2}d\tau\\ \triangleq\ & J_1+J_2+J_{31}+J_{32}+J_{41}+J_{42}+J_{51}+J_{52}+J_{61}+J_{62}.\end{aligned} \tag{3.25}$$

By (3.18), we have

$$J_1\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|g\|_{L^1}+\|g\|_{H^{s+1}}). \tag{3.26}$$

Making use of (3.19), we obtain

$$J_2\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|f\|_{L^1}+\|f\|_{H^s}). \tag{3.27}$$

Thanks to Lemma 3.2 and (1.9), we obtain

$$\|\phi(u)-\frac{\phi''(0)}{2}u^2(\tau)\|_{L^1}\le C\|u(\tau)\|_{L^\infty}\|u(\tau)\|_{L^2}^2\le C\mathscr{E}_1^3(1+\tau)^{-1} \tag{3.28}$$

and

$$\|\partial_x^k(\phi(u)-\frac{\phi''(0)}{2}u^2)(\tau)\|_{L^2}\le C\|u(\tau)\|_{L^\infty}^2\|\partial_x^k u(\tau)\|_{L^2}\le C\mathscr{E}_1^3(1+\tau)^{-\frac{5}{4}-\frac{k}{2}}. \tag{3.29}$$

It follows from (2.10) and (3.28)-(3.29) that

$$\begin{aligned}J_{31}\le\ & C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|(\phi(u)-\frac{\phi''(0)}{2}u^2)(\tau)\|_{L^1}d\tau\\ &+\int_0^{\frac{t}{2}}e^{-c(t-\tau)}\|\partial_x^k(\phi(u)-\frac{\phi''(0)}{2}u^2)(\tau)\|_{L^2}d\tau\\ \le\ & C\mathscr{E}_1^3\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-1}d\tau+C\mathscr{E}_1^3\int_0^{\frac{t}{2}}e^{-c(t-\tau)}(1+\tau)^{-\frac{5}{4}-\frac{k}{2}}d\tau\\ \le\ & C\mathscr{E}_1^3(1+t)^{-\frac{3}{4}-\frac{k}{2}+\varepsilon}.\end{aligned} \tag{3.30}$$

By using (2.10) with $p=2$ and (3.29), it holds that

$$\begin{aligned} J_{32} &\le C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^k(\phi(u)-\frac{\phi''(0)}{2}u^2)(\tau)\|_{L^2}d\tau \\ &\le C\mathscr{E}_1^3\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}-\frac{k}{2}}d\tau \\ &\le C\mathscr{E}_1^3(1+t)^{-\frac{3}{4}-\frac{k}{2}}. \end{aligned} \tag{3.31}$$

From Lemma 3.2, Gagliardo-Nirenberg inequality and (1.9), (3.12), we arrive at

$$\|(u^2-\varpi^2)(\tau)\|_{L^1}\le C\|u+\varpi\|_{L^2}\|u-\varpi\|_{L^2}\le C\mathscr{E}_1X(t)(1+\tau)^{-1+\varepsilon} \tag{3.32}$$

and

$$\|\partial_x^k(u^2-\varpi^2)(\tau)\|_{L^2}\le C\mathscr{E}_1X(t)(1+\tau)^{-\frac{5}{4}-\frac{k}{2}+\varepsilon}. \tag{3.33}$$

For the term J_{41}, applying (2.10) and (3.32), (3.33) yields

$$\begin{aligned} J_{41} &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|u^2-\varpi^2(\tau)\|_{L^1}d\tau + C\int_0^{\frac{t}{2}}e^{-c(t-\tau)}\|\partial_x^k(u^2-\varpi^2)(\tau)\|_{L^2}d\tau \\ &\le C\mathscr{E}_1X(t)\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+t)^{-1+\varepsilon}d\tau + C\mathscr{E}_1X(t)\int_0^{\frac{t}{2}}e^{-c(t-\tau)}(1+t)^{-\frac{5}{4}-\frac{k}{2}+\varepsilon}d\tau \\ &\le C\mathscr{E}_1X(t)(1+t)^{-\frac{3}{4}-\frac{k}{2}+\varepsilon} \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} J_{42} &\le C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^k(u^2-\varpi^2)(\tau)\|_{L^2}d\tau + C\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}\|\partial_x^k(u^2-\varpi^2)(\tau)\|_{L^2}d\tau \\ &\le C\mathscr{E}_1X(t)\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}-\frac{k}{2}+\varepsilon}d\tau + C\mathscr{E}_1X(t)\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}(1+t)^{-\frac{5}{4}-\frac{k}{2}+\varepsilon}d\tau \\ &\le C\mathscr{E}_1X(t)(1+t)^{-\frac{3}{4}-\frac{k}{2}+\varepsilon}. \end{aligned} \tag{3.35}$$

Lemma 3.2, Gagliardo-Nirenberg inequality and (3.12) give the estimates

$$\|\varpi^2(\tau)\|_{L^1}\le C\|\varpi(\tau)\|_{L^2}^2\le C\mathscr{E}_1^2(1+\tau)^{-\frac{1}{2}} \tag{3.36}$$

and

$$\|\partial_x^k\varpi^2(\tau)\|_{L^2}\le C\|\varpi(\tau)\|_{L^\infty}\|\partial_x^k\varpi(\tau)\|_{L^2}\le C\mathscr{E}_1^2(1+\tau)^{-\frac{3}{4}-\frac{k}{2}}. \tag{3.37}$$

Owing to (3.10) and (3.36), (3.37), we deduce that

$$\begin{aligned} J_{51} &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}}\|\varpi^2(\tau)\|_{L^1}d\tau + C\int_0^{\frac{t}{2}}e^{-c(t-\tau)}\|\partial_x^k\varpi^2(\tau)\|_{L^2}d\tau \\ &\le C\mathscr{E}_1^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{4}-\frac{k}{2}}(1+\tau)^{-\frac{1}{2}}d\tau + C\mathscr{E}_1^2\int_0^{\frac{t}{2}}e^{-c(t-\tau)}(1+\tau)^{-\frac{3}{4}-\frac{k}{2}}d\tau \\ &\le C\mathscr{E}_1^2(1+t)^{-\frac{3}{4}-\frac{k}{2}}. \end{aligned} \tag{3.38}$$

Equations (3.10) and (3.37) give the estimates

$$\begin{aligned} J_{52} &\le C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-1}\|\partial_x^k\varpi^2(\tau)\|_{L^2}d\tau + C\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}\|\partial_x^k\varpi^2(\tau)\|_{L^2}d\tau \\ &\le C\mathscr{E}_1^2\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-1}(1+\tau)^{-\frac{3}{4}-\frac{k}{2}}d\tau + C\mathscr{E}_1^2\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}(1+\tau)^{-\frac{3}{4}-\frac{k}{2}}d\tau \\ &\le C\mathscr{E}_1^2(1+t)^{-\frac{3}{4}-\frac{k}{2}+\varepsilon}. \end{aligned} \tag{3.39}$$

Using (3.20), (3.36) and (3.37), we have

$$\begin{aligned} J_{61} &\leq C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{7}{4}-\frac{k}{2}}\|\varpi^2(\tau)\|_{L^1}d\tau + C\int_0^{\frac{t}{2}}e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_x^k\varpi^2(\tau)\|_{L^2}d\tau \\ &\leq C\mathscr{E}_1^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{7}{4}-\frac{k}{2}}(1+\tau)^{-\frac{1}{2}}d\tau + C\mathscr{E}_1^2\int_0^{\frac{t}{2}}e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{4}-\frac{k}{2}}d\tau \\ &\leq C\mathscr{E}_1^2(1+t)^{-\frac{5}{4}-\frac{k}{2}}. \end{aligned} \tag{3.40}$$

By (3.20) and (3.37), we may obtain

$$\begin{aligned} J_{62} &\leq C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{2}}\|\partial_x^k\varpi^2\|_{L^2}d\tau + C\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_x^k\varpi^2\|_{L^2}d\tau \\ &\leq C\mathscr{E}_1^2\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{3}{4}-\frac{k}{2}}d\tau + C\mathscr{E}_1^2\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{4}-\frac{k}{2}}d\tau \\ &\leq C\mathscr{E}_1^2(1+t)^{-\frac{3}{4}-\frac{k}{2}}. \end{aligned} \tag{3.41}$$

Combining (3.25)-(3.41) yields

$$X(t) \leq C\mathscr{E}_1 + C\mathscr{E}_1X(t) + C\mathscr{E}_1^2 + C\mathscr{E}_1^3.$$

This inequality can be solved as $X(t) \leq C\mathscr{E}_1$ if $\mathscr{E}_1$ is sufficiently small. This completes the proof of Theorem 1.1. □

4. Asymptotic profile of solution to (1.1), (1.2) with $\sigma = 1$

We may rewrite (3.10) as

$$\varpi(t) = \mathcal{G}_1(t)*\varpi_0^+ + \frac{\phi''(0)}{4}\int_0^t\mathcal{G}_1(t-\tau)*\partial_x\varpi^2(\tau)d\tau + \mathcal{G}_2(t)*\varpi_0^- - \frac{\phi''(0)}{4}\int_0^t\mathcal{G}_2(t-\tau)*\partial_x\varpi^2(\tau)d\tau, \tag{4.1}$$

where

$$\begin{cases} \mathcal{G}_1(t) = \mathcal{F}^{-1}\{e^{(-\frac{\nu}{2}\xi^2+i\xi)t}\} = \dfrac{1}{\sqrt{2\pi\nu t}}e^{-\frac{(x+1)^2}{2\nu t}}, \\ \mathcal{G}_2(t) = \mathcal{F}^{-1}\{e^{(-\frac{\nu}{2}\xi^2-i\xi)t}\} = \dfrac{1}{\sqrt{2\pi\nu t}}e^{-\frac{(x-1)^2}{2\nu t}}, \end{cases} \tag{4.2}$$

and

$$\varpi_0^+ = \frac{1}{2}f + \frac{1}{2}g, \quad \varpi_0^- = \frac{1}{2}f - \frac{1}{2}g. \tag{4.3}$$

We need the following decay properties for $\mathcal{G}_1(t)*$ and $\mathcal{G}_2(t)*$.

Lemma 4.1 ([7]). *Let $1 \leq q \leq p \leq \infty$ and $0 \leq j \leq k$. Then we have*

$$\|\partial_x^k\mathcal{G}_1(t)*\varphi\|_{L^p} \leq Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k-j}{2}}\|\partial_x^j\varphi\|_{L^q} \tag{4.4}$$

and

$$\|\partial_x^k\mathcal{G}_2(t)*\varphi\|_{L^p} \leq Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k-j}{2}}\|\partial_x^j\varphi\|_{L^q}. \tag{4.5}$$

Lemma 4.2 ([7]). *Let $1 \leq p \leq 2$, $0 \leq j \leq k$ and $0 \leq l \leq k$. Then*

$$\|\partial_x^k\mathcal{G}_1(t)*\varphi\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k-j}{2}}\|\partial_x^j\varphi\|_{L^p} + Ce^{-ct}t^{-\frac{k-l}{2}}\|\partial_x^l\varphi\|_{L^2} \tag{4.6}$$

and

$$\|\partial_x^k\mathcal{G}_2(t)*\varphi\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k-j}{2}}\|\partial_x^j\varphi\|_{L^p} + Ce^{-ct}t^{-\frac{k-l}{2}}\|\partial_x^l\varphi\|_{L^2}.$$

When $\int_{-\infty}^{\infty} \varphi(x)dx = 0$, the above decay properties may be improved as follows.

Lemma 4.3 ([7]). *Let $1 \leq p \leq \infty$, $0 \leq \beta \leq 1$, and $k \geq 0$ be an integer. Assume that $\varphi \in L^1_\beta$ and $\int_{-\infty}^{\infty} \varphi(x)dx = 0$. Then we have*

$$\|\partial_x^k \mathcal{G}_1(t) * \varphi\|_{L^p} \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k+\beta}{2}} \|\varphi\|_{L^1_\beta} \tag{4.7}$$

and

$$\|\partial_x^k \mathcal{G}_2(t) * \varphi\|_{L^p} \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k+\beta}{2}} \|\varphi\|_{L^1_\beta}. \tag{4.8}$$

In what follows, we consider the self-similar solution of the viscous Burgers equation

$$y_t + \big(\frac{1}{2}y^2\big)_x = \frac{\nu}{2} y_{xx}.$$

Note that the self-similar solution is a solution of the form $y = \frac{1}{\sqrt{t}}\Phi(\frac{x}{\sqrt{t}})$. Let $y = \frac{1}{\sqrt{t}}\Phi(\frac{x}{\sqrt{t}}; K)$ be the self-similar solution that satisfies $\int_{-\infty}^{\infty} \Phi(x; K)dx = K$. It is well-known that $\Phi(x; K)$ is given explicitly as (see [7])

$$\Phi(x; M) = \sqrt{\frac{\nu}{2}} \frac{(e^{\frac{K}{\nu}} - 1)e^{-\frac{x^2}{2\nu}}}{\sqrt{\pi} + (e^{\frac{K}{\nu}} - 1)\int_{\frac{x}{\sqrt{2\nu}}}^{\infty} e^{-y^2}dy}.$$

We define a function $v(x, t)$ by

$$v(x, t) = \frac{1}{2b_1} \frac{1}{\sqrt{t+1}} \Phi\Big(\frac{x - a_1(t + 1)}{\sqrt{t+1}}; 2b_1K\Big), \tag{4.9}$$

where a_1 and $b_1 \neq 0$ be real constant. Then $v(x, t)$ is a nonlinear diffusion wave which travels at the speed a_1 in the x direction and has the mass K. It satisfies

$$\begin{cases} v_t + a_1 v_x + b_1(v^2)_x = \dfrac{\nu}{2} v_{xx}, \quad v(x, 0) = v_0(x), \\ \displaystyle\int_{-\infty}^{\infty} v(x, t)dx = \int_{-\infty}^{\infty} v_0(x)dx = K, \end{cases} \tag{4.10}$$

where $v_0(x) = \frac{1}{2b_1}\Theta(x - a_1; 2b_1K)$. Then we define the nonlinear diffusion waves $v = v_\pm(x, t)$ corresponding to (4.1) by the formula (4.9) with the following parameters:

$$a_1 = \mp 1, \qquad b_1 = \mp \frac{\phi''(0)}{4}, \qquad K = K_\pm := \int_{-\infty}^{\infty} \varpi_0^\pm(x)dx, \tag{4.11}$$

where $\varpi_0^\pm(x)$ are defined in (4.3). We note that the diffusion waves $v = v_\pm(x, t)$ travel at the different speeds ∓ 1 in the x direction, respectively, and satisfy (4.10) with the above parameters. In particular, the corresponding initial data $v_0(x) = v_0^\pm(x)$ are given by $v_0^\pm(x) = \frac{1}{2b_1}\Theta(x \pm 1; 2b_1K_\pm)$ with $b_1 = \mp\frac{\phi''(0)}{4}$ and satisfy the relations

$$\int_{-\infty}^{\infty} v_0^\pm(x)dx = K_\pm = \int_{-\infty}^{\infty} \varpi_0^\pm(x)dx.$$

Obviously, the nonlinear diffusion waves satisfy

$$\begin{cases} v_+(t) = \mathcal{G}_1(t) * v_0^+ + \frac{\phi''(0)}{4}\int_0^t \mathcal{G}_1(t - \tau) * \partial_x(v_+^2)(\tau)d\tau, \\ v_-(t) = \mathcal{G}_2(t) * v_0^- - \frac{\phi''(0)}{4}\int_0^t \mathcal{G}_2(t - \tau) * \partial_x(v_-^2)(\tau)d\tau, \end{cases} \tag{4.12}$$

where $G_1(x, t), G_2(x, t)$ are given by (4.2).

The nonlinear diffusion waves have the following decay estimates, which has been established in [5, 7].

Lemma 4.4. *Let $K_1 = \max\{|K_\pm|\}$ be suitably small. Then we have:*

$$\|\partial_x^k v_\pm(t)\|_{L^p} \le CK_1(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}$$

and

$$\|\partial_x^k (v_+ v_-)(t)\|_{L^p} \le CM_1^2 e^{-ct}, \tag{4.13}$$

where $1 \le p \le \infty$ and $k \ge 0$. Here we note that $K_1 \le C\|(f,g)\|_{L^1} \le CE_1$.

To prove Theorem 1.2, we also need to discuss the interaction between two diffusion waves in the different fields. Let $v(x,t)$ be the nonlinear diffusion wave defined by (4.9). We consider the problem

$$\psi_t + a_2\psi_x + b_2(v^2)_x = \frac{\nu}{2}\psi_{xx}, \qquad \psi(x,0) = 0, \tag{4.14}$$

where a_2, b_2 and ν are real constants with $\nu > 0$. Note that the problem (4.14) is equivalent to the integral formula

$$\psi(t) = -b_2 \int_0^t \mathfrak{S}_1(t-\tau) * \partial_x(v^2)(\tau) d\tau,$$

where $\mathfrak{S}_1(x,t) = \frac{1}{\sqrt{2\pi\nu t}} e^{-\frac{(x-a_2 t)^2}{2\nu t}}$ is a modified heat kernel. The solution ψ of (4.14) satisfies the following decay estimate.

Lemma 4.5 ([6, 10])**.** *Let $1 \le p \le \infty$ and $k \ge 0$. Let $v(x,t)$ be the nonlinear diffusion wave defined by (4.9). Assume that $|K|$ is suitably small and that $a_1 \ne a_2$. Then the solution ψ of the problem (4.14) satisfies the decay estimate*

$$\|\partial_x^k \psi(t)\|_{L^p} \le C|K|^2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}-\frac{k}{2}}. \tag{4.15}$$

Based on the preliminaries, we give the proof of Theorem 1.2.

Proof. To prove (1.10), we set $\omega = \varpi - v_+ - v_-$ and define

$$\mathscr{N}(t) = \sum_{k=0}^{s} \sup_{0\le\tau\le t} (1+\tau)^{\frac{1}{2}+\frac{k}{2}} \|\partial_x^k \omega(\tau)\|_{L^2}.$$

It is not difficult to check

$$\varpi^2 = \omega^2 + 2\omega(v_+ + v_-) + v_+^2 + v_-^2 + 2v_+v_- = \omega(\varpi + v_+ + v_-) + v_+^2 + v_-^2 + 2v_+v_-.$$

By the above equality and (4.1), (4.12), it holds that

$$\begin{aligned}
\omega = {} & \mathcal{G}_1(t) * (v_0^+ - \varpi_0^+) + \mathcal{G}_2(t) * (v_0^- - \varpi_0^-) \\
& + \frac{\phi''(0)}{4} \int_0^t \mathcal{G}_1(t-\tau) * \partial_x \left((\varpi + v_+ + v_-)\omega\right) d\tau + \frac{\phi''(0)}{2} \int_0^t \mathcal{G}_1(t-\tau) * \partial_x(v_+v_-) d\tau \\
& + \frac{\phi''(0)}{4} \int_0^t \mathcal{G}_1(t-\tau) * \partial_x(v_-^2) d\tau - \frac{\phi''(0)}{4} \int_0^t \mathcal{G}_2(t-\tau) * \partial_x \left((\varpi + v_+ + v_-)\omega\right) d\tau \\
& - \frac{\phi''(0)}{2} \int_0^t \mathcal{G}_2(t-\tau) * \partial_x(v_+v_-) d\tau - \frac{\phi''(0)}{4} \int_0^t \mathcal{G}_2(t-\tau) * \partial_x(v_+^2) d\tau.
\end{aligned} \tag{4.16}$$

By applying ∂_x^k to (4.16) and taking the L^2 norm, by Minkowski's inequality, we have

$$\begin{aligned}
\|\partial_x^k \omega(t)\|_{L^2} \le {} & \|\partial_x^k \mathcal{G}_1(t) * (v_0^+ - \varpi_0^+)\|_{L^2} + \|\partial_x^k \mathcal{G}_2(t) * (v_0^- - \varpi_0^-)\|_{L^2} \\
& + |\frac{\phi''(0)}{4}| \int_0^t \|\partial_x^k \mathcal{G}_1(t-\tau) * \partial_x \left((\varpi + v_+ + v_-)\omega\right)\|_{L^2} d\tau
\end{aligned}$$

$$
\begin{aligned}
&+|\frac{\phi''(0)}{2}|\int_0^t \|\partial_x^k\mathcal{G}_1(t-\tau)*\partial_x(v_+v_-)\|_{L^2}d\tau+\|\frac{\phi''(0)}{4}\int_0^t \partial_x^k\mathcal{G}_1(t-\tau)*\partial_x(v_-^2)d\tau\|_{L^2}\\
&+|\frac{\phi''(0)}{4}|\int_0^t \|\partial_x^k\mathcal{G}_2(t-\tau)*\partial_x\left((\varpi+v_++v_-)\omega\right)\|_{L^2}d\tau\\
&+|\frac{\phi''(0)}{2}|\int_0^t \|\partial_x^k\mathcal{G}_2(t-\tau)*\partial_x(v_+v_-)d\tau\|_{L^2}+\|\frac{\phi''(0)}{4}\int_0^t \partial_x^k\mathcal{G}_2(t-\tau)*\partial_x(v_+^2)d\tau\|_{L^2}\\
=\;&\mathbb{I}+\mathbb{J}+\mathbb{K}+\mathbb{L}+\mathbb{P}+\mathbb{Q}+\mathbb{S}+\mathbb{T}.
\end{aligned}
$$

For the term $\mathbb{I}$, noting that $\int_{\mathbb{R}}\left(v_0^+-\varpi_0^+\right)dx=0$, it follows from (4.7) with $p=2$ and $\beta=\frac{1}{2}$ that

$$\mathbb{I}\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}(\tilde{\mathscr{E}}_1+\mathscr{E}_2),$$

where $\mathscr{E}_2=\|(f,g)\|_{L^1_{\frac{1}{2}}}$.

Similarly, we obtain

$$\mathbb{J}\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}(\tilde{\mathscr{E}}_1+\mathscr{E}_2).$$

In what follows, we estimate the nonlinear term $\mathbb{K}$. We divide $\mathbb{K}$ into two parts and write $\mathbb{K}=\mathbb{K}_1+\mathbb{K}_2$, where $\mathbb{K}_1$ and $\mathbb{K}_2$ are corresponding to the time intervals $[0,\frac{t}{2}]$ and $[\frac{t}{2},t]$, respectively. We estimate the term $\mathbb{K}_1$ by using (4.6) with $p=1, j=0$ and $l=k$. Then we arrive

$$
\begin{aligned}
\mathbb{K}_1\le\;& C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|(\varpi+v_++v_-)\omega\|_{L^1}d\tau\\
&+C\int_0^{\frac{t}{2}}e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_x^k((\varpi+v_++v_-)\omega)\|_{L^2}d\tau\\
\le\;& C\tilde{\mathscr{E}}_1\mathscr{N}(t)\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-\frac{3}{4}}d\tau+C\tilde{\mathscr{E}}_1\mathscr{N}(t)\int_0^{\frac{t}{2}}e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1-\frac{k}{2}}d\tau\\
\le\;& C\tilde{\mathscr{E}}_1\mathscr{N}(t))(1+t)^{-\frac{1}{2}-\frac{k}{2}}.
\end{aligned}
\tag{4.17}
$$

Owing to (4.6) with $p=2, j=k$ and $l=k$. This yields

$$
\begin{aligned}
\mathbb{K}_2\le\;& C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^k((\varpi+v_++v_-)\omega)\|_{L^2}d\tau\\
&+C\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_x^k((\varpi+v_++v_-)\omega)\|_{L^2}d\tau\\
\le\;& C\tilde{\mathscr{E}}_1\mathscr{N}(t)C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1-\frac{k}{2}}d\tau+C\tilde{\mathscr{E}}_1\mathscr{N}(t)\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1-\frac{k}{2}}d\tau\\
\le\;& C\tilde{\mathscr{E}}_1\mathscr{N}(t)(1+t)^{-\frac{1}{2}-\frac{k}{2}}.
\end{aligned}
\tag{4.18}
$$

Combining the estimates (4.17) and (4.18) yields

$$\mathbb{K}\le C\tilde{\mathscr{E}}_1\mathscr{N}(t)(1+t)^{-\frac{1}{2}-\frac{k}{2}}.$$

Similarly, we can prove

$$\mathbb{Q}\le C\tilde{\mathscr{E}}_1\mathscr{N}(t)(1+t)^{-\frac{1}{2}-\frac{k}{2}}.$$

It follows from (4.4) that

$$\mathbb{L}\le C\int_0^t(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(\|v_+v_-(\tau)\|_{L^1}+\|\partial_x^k(v_+v_-)(\tau)\|_{L^2})d\tau$$

$$
\begin{aligned}
&\le C\tilde{\mathscr{E}}_1^2 \int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} e^{-c\tau} d\tau \\
&\le \tilde{\mathscr{E}}_1^2 (1+t)^{-\frac{1}{2}-\frac{k}{2}},
\end{aligned}
$$

where we used (4.13) with $\le C\tilde{\mathscr{E}}_1$.

Similarly, by (4.5) and (4.13) with $\le C\tilde{\mathscr{E}}_1$, we have

$$\mathbb{S} \le C\tilde{\mathscr{E}}_1^2 (1+t)^{-\frac{1}{2}-\frac{k}{2}}.$$

By applying (4.15) to $\mathbb{P}$ and $\mathbb{T}$, we deduce that

$$\mathbb{P} \le C\tilde{\mathscr{E}}_1^2 (1+t)^{-\frac{1}{2}-\frac{k}{2}},$$

and

$$\mathbb{T} \le C\tilde{\mathscr{E}}_1^2 (1+t)^{-\frac{1}{2}-\frac{k}{2}}.$$

Therefore, we arrive at

$$\mathscr{N}(t) \le C\tilde{\mathscr{E}}_1 \mathscr{N}(t) + C(\tilde{\mathscr{E}}_1 + \tilde{\mathscr{E}}_1^2 + \mathscr{E}_2).$$

This inequality can be solved as $\mathscr{N}(t) \le C\mathscr{E}_2$ if $\tilde{\mathscr{E}}_1$ is sufficiently small. This completes the proof of Theorem 1.2. □

5. Asymptotic profile of solutions to (1.1), (1.2) with $\sigma \geq 2$

The aim of this section is to derive a simpler asymptotic profile of the solution u_L to the problem (2.1), (1.2). We now define $\bar{u}_L$ by

$$\bar{u}_L(t) = \mathcal{G}_0(t) * \partial_x g + \mathcal{H}_0(t) * f. \tag{5.1}$$

In what follows, we shall prove that $\bar{u}_L$ is a asymptotic profile of the linear solution u_L. In fact we have:

Lemma 5.1. *Let $s \ge 0$. Assume that $f \in H^s \cap L^1$ and $g \in H^{s+1} \cap L^1$, and put $\mathscr{E}_1 = \|u_0\|_{H^s\cap L^1} + \|g\|_{H^{s+1}\cap L^1}$. Let u_L be the linear solution and let $\bar{u}_L$ be defined by* (5.1)*. Then we have*

$$\|\partial_x^k (u_L - \bar{u}_L)(t)\|_{L^2} \le C\mathscr{E}_1 (1+t)^{-\frac{3}{4}-\frac{k}{2}}$$

for $0 \le k \le s$.

From (5.1), we arrive at

$$\hat{\bar{u}}_L = e^{(-\frac{\nu}{2}\xi^2 + i\xi)t}(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) + e^{(-\frac{\nu}{2}\xi^2 - i\xi)t}(\frac{1}{2}\hat{f} - \frac{1}{2}\hat{g}).$$

Let $\mathcal{G}_1(t)$ and $\mathcal{G}_2(t)$ be defined by (4.2), then

$$\bar{u}_L = \mathcal{G}_1(t) * (\frac{1}{2}f + \frac{1}{2}g) + \mathcal{G}_2(t) * (\frac{1}{2}f - \frac{1}{2}g).$$

Let $\rho(x,t), \varrho(x,t)$ be the solutions to the following problems

$$\partial_t \rho - \frac{\nu}{2}\partial_{xx}^2 \rho - \partial_x \rho = 0, \;\; \rho(x,0) = \frac{1}{2}(f+g)(x),$$

and

$$\partial_t \varrho - \frac{\nu}{2}\partial_{xx}^2 \varrho + \partial_x \varrho = 0, \;\; \varrho(x,0) = \frac{1}{2}(f-g)(x),$$

respectively. Set $M_\pm = \int_{-\infty}^{\infty} (\frac{1}{2}f \pm \frac{1}{2}g)(x)dx$. We call

$$(v_+ + v_-)(x,t) = M_+\mathcal{G}_1(x,t+1) + M_-\mathcal{G}_2(x,t+1) \tag{5.2}$$

is the superposition of the diffusion wave with the amounts M_+ and M_-.

Noting that $\bar{u}_L(x,t) = (\rho + \varrho)(x,t)$, therefore

$$\bar{u}_L - v_+ - v_- = \rho - v_+ + \varrho - v_-.$$

Then $\rho - v_+$ and $\varrho - v_-$ satisfy the following problem

$$\begin{cases} \partial_t(\rho - v_+) - \frac{\nu}{2}\partial_{xx}^2(\rho - v_+) - \partial_x(\rho - v_+) = 0, \\ (\rho - v_+)(x,0) = \frac{1}{2}(f+g)(x) - M_+G_1(x,1), \end{cases}$$

and

$$\begin{cases} \partial_t(\varrho - v_-) - \frac{\nu}{2}\partial_{xx}^2(\varrho - v_-) - \partial_x(\varrho - v_-) = 0, \\ (\varrho - v_-)(x,0) = \frac{1}{2}(f-g)(x) - M_-G_2(x,1), \end{cases}$$

respectively. By (4.4), (4.5) and (4.7), (4.8), it is not difficult to prove the following lemma.

Lemma 5.2. *Let $s \geq 0$. Assume that $f,g \in H^{s+1}\bigcap L_1^1$. Put $\mathscr{E}_1 = \|(f,g)\|_{H^{s+1}\bigcap L^1}$ and $\mathscr{E}_3 = \|(f,g)\|_{H^{s+1}\bigcap L_1^1}$. Let $v_\pm$ be the diffusion waves defined by* (5.2). *There exists a small positive constant δ_3 such that if $\mathscr{E}_1 \leq \delta_3$, then we have*

$$\|\partial_x^k(\rho - v_+)(t)\|_{L^2} \leq C\mathscr{E}_3(1+t)^{-\frac{3}{4}-\frac{k}{2}}$$

and

$$\|\partial_x^k(\varrho - v_-)(t)\|_{L^2} \leq C\mathscr{E}_3(1+t)^{-\frac{3}{4}-\frac{k}{2}}$$

for $0 \leq k \leq s$.

Proof of Theorems 1.3 *and* 1.4. Combining (1.8) and Lemmas 5.1 and 5.2, we immediately obtain Theorems 1.3 and 1.4. Thus we have completed the proof of Theorems 1.3 and 1.4. □

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