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# Properties and application of smooth function germs of orbit tangent space

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**Abstract**

The finite determinacy of smooth function germ is the key in approximating the nonlinear function with infinite terms by its finite terms. In this paper, we discuss the inclusion relations with a new equivalent form for function germs in orbit tangent spaces, and get an improved form of the finite $k$-determinacy of smooth function germ. As an application, the methods in judging the right equivalency of Whitney function family with codimension 8 are presented. 

*Keywords:* Diffeomorphism, function germ, Jacobian ideal, $\mathcal{R}$-equivalence.
*2010 MSC:* 58C25, 58K40.

## 1. Introduction

Mather gave the necessary and sufficient conditions for smooth function germs with no more than 4 codimension are finitely determined in [4]. His theorem is quite effective on determining low codimension function germs. That work is a foundation of the study on the theory of finite determinacy. However, his theorem does not work well on high codimension function germs, even if the Whitney function family $W_t(x, y) = xy(x - y)(x - ty)$, $t \in (1, +\infty)$.

In recent years, there are a large number of literatures on the study of finite determinacy problem of smooth function germs. Such as Wall showed the necessary and sufficient conditions for the finite determinacy of smooth mapping germs in [7]. Wilson et al. gave relationships between the relative stability and

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the finite relative determinacy of mapping germs in [1, 6]. In addition, Zou et al. studied the definition and determination of finite determinacy and infinite relative determinacy for smooth function germs with certain boundary conditions in [2, 5, 8].

Our work is a complement to the previous work mentioned above. We discuss the property that the function germs still satisfy finite $k$-determinacy under sufficiently small disturbance in orbit tangent space (Theorem 3.1). Furthermore, we present the judging method of right equivalency for Whitney function family with codimension 8 (Theorem 3.4 and Example 4.1). The applicability of Mather's finite $k$-determinacy theorem is improved by this work.

The structure of this paper is as follows. We present the basic notations and preliminaries in Section 2. In Section 3, the theorems of finite determinacy for smooth function germs are established. In Section 4, as an application of the main results, the right equivalency for Whitney function family is presented.

All undefined terms and symbols could be seen in [3].

## 2. Basic concepts and preliminaries

Let $E_n$ be a $C^\infty$ ring of function germs at $0 \in \mathbb{R}^n$, $M_n$ be the only maximal ideal in $E_n$, $M_n^k$ be the $k$-th power of $M_n$, and $J(f) = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \rangle_{E_n}$ be the Jacobian ideal of the function germ $f$. Here $(t, x) = (t, x_1, x_2, \dots, x_n) \in \mathbb{R} \times \mathbb{R}^n$.

**Definition 2.1.** Let $I_1 = \langle f_1, f_2, \dots, f_r \rangle_{E_n}$ and $I_2 = \langle g_1, g_2, \dots, g_r \rangle_{E_n}$ be finitely generated ideals in $E_n$. Two ideals $I_1$ and $I_2$ are $\mathcal{R}$-equivalent, if there exists an invertible matrix $[u_{ij}]_{r\times r}$ in $E_n$, such that

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{pmatrix} = [u_{ij}]_{r\times r} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_r \end{pmatrix}.$$

**Definition 2.2.** Let $f, g \in E_n$. Two function germs $f$ and $g$ are said to be isomorphic (i.e., $\mathcal{R}$-equivalence) if there exists a local diffeomorphism germ $\Phi : (\mathbb{R}^n, 0) \to \mathbb{R}^n$ such that $g = f \circ \Phi$.

**Definition 2.3.** Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a $C^\infty$ real function germ and $k$ a positive integer. We say $f$ is $k$-determined if all the Taylor polynomial germs, which have the same order $k$ with $f$ in $E_n$, are $\mathcal{R}$-equivalent to $f$.

**Proposition 2.4.** *Let*

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x) \frac{\partial}{\partial x_i}$$

*be a* $C^\infty$ *vector field on an open neighborhood of* $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}^n$, $t \in [0, 1]$. *Then there exists an open set* $U$ *containing* $[0, 1] \times \{0\}$, *which makes the following differential equations have a unique solution.*

$$\begin{cases} \dfrac{d\Phi_1(t,x)}{dt} = X_1(\Phi_1(t,x), \dots, \Phi_n(t,x)), \\ \dfrac{d\Phi_2(t,x)}{dt} = X_2(\Phi_1(t,x), \dots, \Phi_n(t,x)), \\ \qquad\qquad \vdots \\ \dfrac{d\Phi_n(t,x)}{dt} = X_n(\Phi_1(t,x), \dots, \Phi_n(t,x)), \end{cases}$$

*with the initial condition*

$$\begin{pmatrix} \Phi_1(0,x) \\ \Phi_2(0,x) \\ \vdots \\ \Phi_n(0,x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

*Here,* $\Phi_t : (U_0, x) \to (U_t, \Phi(t,x))$ *is a local diffeomorphism* (*see* [3]).

## 3. Finite determination of high codimension smooth function germs

In this section, we present our main results and proofs.

**Theorem 3.1.** *Let* $h(x) \in M_n^{k+1}$ *be sufficiently small,* $\tau \in [0,1]$*. Then* $M_n^{k+1} \subset M_n^2 \cdot J(f)$ *if and only if* $M_n^{k+1} \subset M_n^2 \cdot J(f + \tau h)$*.*

*Proof.* Notice that

$$\begin{aligned} M_n^2 \cdot J(f) =& \langle x_1^2, x_1x_2, \ldots, x_1x_n, x_2^2, \ldots, x_2x_n, x_3^2, \ldots, x_3x_n, \ldots, x_{n-1}^2, x_{n-1}x_n, x_n^2\rangle_{E_n} \\ & \cdot \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\rangle_{E_n} \\ =& \langle x_1^2 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_1^2 \cdot \frac{\partial f}{\partial x_n}, x_1x_2 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_1x_2 \cdot \frac{\partial f}{\partial x_n}, \ldots, x_1x_n \cdot \frac{\partial f}{\partial x_1}, \ldots, x_1x_n \cdot \frac{\partial f}{\partial x_n}, \\ & x_2^2 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_2^2 \cdot \frac{\partial f}{\partial x_n}, x_2x_3 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_2x_3 \cdot \frac{\partial f}{\partial x_n}, \ldots, x_2x_n \cdot \frac{\partial f}{\partial x_1}, \ldots, x_2x_n \cdot \frac{\partial f}{\partial x_n}, \\ & x_3^2 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_3^2 \cdot \frac{\partial f}{\partial x_n}, \ldots, x_3x_n \cdot \frac{\partial f}{\partial x_1}, \ldots, x_3x_n \cdot \frac{\partial f}{\partial x_n}, \ldots, x_{n-1}^2 \cdot \frac{\partial f}{\partial x_1}, \ldots, \\ & x_{n-1}^2 \cdot \frac{\partial f}{\partial x_n}, x_{n-1}x_n \cdot \frac{\partial f}{\partial x_1}, \ldots, x_{n-1}x_n \cdot \frac{\partial f}{\partial x_n}, x_n^2 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_n^2 \cdot \frac{\partial f}{\partial x_n}\rangle_{E_n}, \end{aligned}$$

$$\begin{aligned} M_n^2 \cdot J(f+\tau h) =& \langle x_1^2, \ldots, x_1x_n, x_2^2, \ldots, x_2x_n, \ldots, x_{n-1}^2, x_{n-1}x_n, x_n^2\rangle_{E_n} \\ & \cdot \langle \frac{\partial (f+\tau h)}{\partial x_1}, \frac{\partial (f+\tau h)}{\partial x_2}, \ldots, \frac{\partial (f+\tau h)}{\partial x_n}\rangle_{E_n} \\ =& \langle x_1^2 \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, x_1^2 \cdot \frac{\partial (f+\tau h)}{\partial x_n}, \ldots, x_1x_n \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, x_1x_n \cdot \frac{\partial (f+\tau h)}{\partial x_n}, \\ & x_2^2 \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, x_2^2 \cdot \frac{\partial (f+\tau h)}{\partial x_n}, \ldots, \\ & x_2x_n \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, x_2x_n \cdot \frac{\partial (f+\tau h)}{\partial x_n}, x_3^2 \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, \\ & x_3^2 \cdot \frac{\partial (f+\tau h)}{\partial x_n}, \ldots, x_3x_n \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, x_3x_n \cdot \frac{\partial (f+\tau h)}{\partial x_n}, \ldots, x_{n-1}^2 \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, \\ & x_{n-1}^2 \cdot \frac{\partial (f+\tau h)}{\partial x_n}, x_{n-1}x_n \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, x_{n-1}x_n \cdot \frac{\partial (f+\tau h)}{\partial x_n}, x_n^2 \cdot \frac{\partial (f+\tau h)}{\partial x_1}, \ldots, \\ & x_n^2 \cdot \frac{\partial (f+\tau h)}{\partial x_n}\rangle_{E_n} \\ =& \langle x_1^2 \cdot \frac{\partial f}{\partial x_1} + x_1^2 \cdot \frac{\partial (\tau h)}{\partial x_1}, \ldots, x_1^2 \cdot \frac{\partial f}{\partial x_n} + x_1^2 \cdot \frac{\partial (\tau h)}{\partial x_n}, \ldots, x_1x_n \cdot \frac{\partial f}{\partial x_1} + x_1x_n \cdot \frac{\partial (\tau h)}{\partial x_1}, \ldots, \\ & x_1x_n \cdot \frac{\partial f}{\partial x_n} + x_1x_n \cdot \frac{\partial (\tau h)}{\partial x_n}, x_2^2 \cdot \frac{\partial f}{\partial x_1} + x_2^2 \cdot \frac{\partial (\tau h)}{\partial x_1}, \ldots, x_2^2 \cdot \frac{\partial f}{\partial x_n} + x_2^2 \cdot \frac{\partial (\tau h)}{\partial x_n}, \ldots, \\ & x_2x_n \cdot \frac{\partial f}{\partial x_1} + x_2x_n \cdot \frac{\partial (\tau h)}{\partial x_1}, \ldots, x_2x_n \cdot \frac{\partial f}{\partial x_n} + x_2x_n \cdot \frac{\partial (\tau h)}{\partial x_n}, x_3^2 \cdot \frac{\partial f}{\partial x_1} + x_3^2 \cdot \frac{\partial (\tau h)}{\partial x_1}, \ldots, \\ & x_3^2 \cdot \frac{\partial f}{\partial x_n} + x_3^2 \cdot \frac{\partial (\tau h)}{\partial x_n}, \ldots, x_3x_n \cdot \frac{\partial f}{\partial x_1} + x_3x_n \cdot \frac{\partial (\tau h)}{\partial x_1}, \ldots, x_3x_n \cdot \frac{\partial f}{\partial x_n} + x_3x_n \cdot \frac{\partial (\tau h)}{\partial x_n}, \ldots, \end{aligned}$$

$$x_{n-1}^2 \cdot \frac{\partial f}{\partial x_1} + x_{n-1}^2 \cdot \frac{\partial(\tau h)}{\partial x_1}, \dots, x_{n-1}^2 \cdot \frac{\partial f}{\partial x_n} + x_{n-1}^2 \cdot \frac{\partial(\tau h)}{\partial x_n},$$
$$x_{n-1}x_n \cdot \frac{\partial f}{\partial x_1} + x_{n-1}x_n \cdot \frac{\partial(\tau h)}{\partial x_1}, \dots,$$
$$x_{n-1}x_n \cdot \frac{\partial f}{\partial x_n} + x_{n-1}x_n \cdot \frac{\partial(\tau h)}{\partial x_n}, x_n^2 \cdot \frac{\partial f}{\partial x_1} + x_n^2 \cdot \frac{\partial(\tau h)}{\partial x_1}, \dots, x_n^2 \cdot \frac{\partial f}{\partial x_n} + x_n^2 \cdot \frac{\partial(\tau h)}{\partial x_n}\rangle_{E_n}.$$

Here, $\tau \in [0,1]$ , $h(x) \in M_n^{k+1}$ is sufficiently small. So both $x_i^2 \cdot \tau \cdot \frac{\partial h}{\partial x_i}$ and $x_i \cdot x_j \cdot \tau \cdot \frac{\partial h}{\partial x_i}$ are sufficiently small, $i, j = 1, 2, \dots, \frac{n^2 \cdot (n+1)}{2}$. By Definition 2.1, we get

$$M_n^2 \cdot J(f + \tau h) = M_n^2 \cdot J(f).$$

Then $M_n^{k+1} \subset M_n^2 \cdot J(f)$ if and only if $M_n^{k+1} \subset M_n^2 \cdot J(f + \tau h)$. □

By Theorem 3.1, we can get the following corollary.

**Corollary 3.2.** *Let $M_n^{k+1} \subset M_n^2 \cdot J(f)$. If $h(x) \in M_n^{k+1}$ is sufficiently small, then the algebraic equation*

$$-h(x) = \sum_{i=1}^{n} X_i(x) \cdot (\frac{\partial f(x)}{\partial x_i} + t \cdot \frac{\partial h(x)}{\partial x_i}), \quad X_i(x) \in M_n^2, \quad i = 1, 2, \dots, n, \quad t \in [0,1]$$

*is solvable.*

*Proof.* For $t \in [0,1]$, $t \cdot h(x)$ is sufficiently small since $h(x) \in M_n^{k+1}$ is sufficiently small. $M_n^{k+1} \subset M_n^2 \cdot J(f)$. According to Theorem 3.1, there exist $h(x) \in M_n^{k+1} \subset M_n^2 \cdot J(f)$ and $X_i(x) \in M_n^2$ $(i = 1, 2, \dots, n)$, such that

$$-h(x) = \sum_{i=1}^{n} X_i(x) \cdot (\frac{\partial f(x)}{\partial x_i} + t \cdot \frac{\partial h(x)}{\partial x_i}) \in M_n^2 \cdot J(f + t \cdot h).$$

That is, the algebraic equation

$$-h(x) = \sum_{i=1}^{n} X_i(x) \cdot (\frac{\partial f(x)}{\partial x_i} + t \cdot \frac{\partial h(x)}{\partial x_i}), \quad X_i(x) \in M_n^2, \quad i = 1, 2, \dots, n, \quad t \in [0,1]$$

has a solution. □

**Lemma 3.3.** *Let $F(t,x) = f(x) + t \cdot h(x)$ be a function germ, where $t \in [0,1]$, $h(x) \in M_n^{k+1}$ and $h(x)$ is sufficiently small. Then there exists a vector field*

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial}{\partial x_i},$$

*such that $X \cdot F = 0$.*

*Proof.* By Corollary 3.2, there exist $X_i(x) \in M_n^2$ $(i = 1, 2, \dots, n)$ satisfying that the following algebraic equation has a solution

$$-h(x) = \sum_{i=1}^{n} X_i(x) \cdot (\frac{\partial f(x)}{\partial x_i} + t \cdot \frac{\partial h(x)}{\partial x_i}).$$

Hence, for $F(t,x) = f(x) + t \cdot h(x)$, there exists a vector field

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial}{\partial x_i}$$

such that

$$X \cdot F = \frac{\partial F}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial F}{\partial x_i} = \frac{\partial (f(x) + t \cdot h(x))}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial (f(x) + t \cdot h(x))}{\partial x_i}$$
$$= h(x) + \sum_{i=1}^{n} X_i(x) \cdot (\frac{\partial f(x)}{x_i} + t \cdot \frac{\partial h(x)}{\partial x_i}) = 0,$$

if $t \in [0,1]$ and $h(x) \in M_n^{k+1}$ is sufficiently small. □

**Theorem 3.4.** *Let $f \in E_n$ and $M_n^{k+1} \subset M_n^2 \cdot J(f)$. Then the function germ $g$ is $\mathcal{R}$-equivalent to the function germ $f$, if $g - f \in M_n^{k+1}$ and $j^k g - j^k f \in P_n^k$ are sufficiently small.*

*Proof.* Let $g - f = h \in M_n^{k+1}$ and $F(t,x) = f(x) + t \cdot h(x)$, $t \in [0,1]$. For sufficiently small $h(x) \in M_n^{k+1}$ and $M_n^{k+1} \subset M_n^2 \cdot J(f)$, by Lemma 3.3, there exists a vector field

$$X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial}{\partial x_i}$$

such that $X \cdot F = 0$. That is,

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} X_i(x) \cdot \frac{\partial F}{\partial x_i} = 0.$$

By Proposition 2.4, we get

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} \frac{d\Phi_i(t,x)}{dt} \cdot \frac{\partial F}{\partial x_i} = 0,$$

this means

$$\frac{d}{dt}(F \circ \Phi(t,x)) = 0.$$

Thus, for any $t_1, t_2 \in [0,1]$, $t_1 \neq t_2$, we have $F \circ \Phi(t_1, x) = F \circ \Phi(t_2, x)$. Especially, when $t_1 = 0, t_2 = 1$, we have $F(0, \Phi(0,x)) = F(1, \Phi(1,x))$. By $F(t,x) = f(x) + t \cdot h(x)$, then

$$F(0,x) = f(x), \quad F(1,x) = f(x) + h(x) = g(x).$$

Hence, $g = f \circ \Phi(1,x)$. This implies that $g$ is isomorphic ($\mathcal{R}$-equivalent) to $f$. □

## 4. Example

As an application of Theorem 3.4, we state the following example.

**Example 4.1.** Let $W_t(x,y) = xy(x-y)(x-ty)$ be a two variable function family, $t \in (1, +\infty)$. For all $t_0, t_1 \in (1, +\infty)$, $t_0 \neq t_1$, $W_{t_1}(x,y)$ is $\mathcal{R}$-equivalent to $W_{t_0}(x,y)$.

*Proof.* Since $M_2^2 = \langle x^2, xy, y^2 \rangle_{E_2}$, $M_2^5 = \langle x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5 \rangle_{E_2} = \langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle_{E_2}$, and $W_{t_0} = x^3y - (t_0+1)x^2y^2 + t_0xy^3$, we have

$$J(W_{t_0}) = \langle 3x^2y - 2(t_0+1)xy^2 + t_0y^3, x^3 - 2(t_0+1)x^2y + 3t_0xy^3 \rangle_{E_{2+1}},$$

and

$$\begin{aligned} &M_2^2 \cdot J(W_{t_0}) \\ &= \langle 3x^4y - 2(t_0+1)x^3y^2 + t_0x^2y^3, x^5 - 2(t_0+1)x^4y + 3t_0x^3y^2, 3x^3y^2 - 2(t_0+1)x^2y^3 + t_0xy^4, \\ &\quad x^4y - 2(t_0+1)x^3y^2 + 3t_0x^2y^3, 3x^2y^3 - 2(t_0+1)xy^4 + t_0y^5, x^3y^2 - 2(t_0+1)x^2y^3 + 3t_0xy^4 \rangle_{E_{2+1}} \end{aligned}$$

$$
\begin{aligned}
=&\langle x^5-2(t_0+1)x^4y+3t_0x^3y^2, 3x^4y-2(t_0+1)x^3y^2+t_0x^2y^3, x^4y-2(t_0+1)x^3y^2+3t_0x^2y^3,\\
&3x^3y^2-2(t_0+1)x^2y^3+t_0xy^4, x^3y^2-2(t_0+1)x^2y^3+3t_0xy^4, 3x^2y^3-2(t_0+1)xy^4+t_0y^5\rangle_{E_{2+1}}\\
=&\langle f_1,f_2,f_3,f_4,f_5,f_6\rangle_{E_{2+1}}.
\end{aligned}
$$

This means

$$
\begin{cases}
g_1=f_1+2(t_0+1)g_2-3t_0g_3,\\
g_2=\dfrac{1}{3}f_2+2(t_0+1)g_3-t_0g_4,\\
g_3=g_2-\dfrac{1}{2(t_0+1)}f_3-3t_0g_5,\\
g_4=\dfrac{1}{3}g_3-\dfrac{1}{2(t_0+1)}f_4-3t_0g_5,\\
g_5=g_3-2(t_0+1)g_4+\dfrac{1}{3t_0}f_5,\\
g_6=3g_4-2(t_0+1)g_5+\dfrac{1}{t_0}f_6.
\end{cases}
$$

That is,

$$
\begin{cases}
f_1=g_1-2(t_0+1)g_2+3t_0g_3\\
f_2=3g_2-2(t_0+1)g_3+t_0g_4,\\
f_3=g_2-2(t_0+1)g_3+3t_0g_5,\\
f_4=3g_3-2(t_0+1)g_4+t_0g_5,\\
f_5=_3-2(t_0+1)g_4+3t_0g_5,\\
f_6=3g_4-2(t_0+1)g_5+t_0g_6.
\end{cases}
$$

The matrix representation of the above equations is

$$
\begin{pmatrix} f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6 \end{pmatrix}
=
\begin{pmatrix}
1 & -2(t_0+1) & 3t_0 & 0 & 0 & 0\\
0 & 3 & -2(t_0+1) & t_0 & 0 & 0\\
0 & 1 & -2(t_0+1) & 0 & 3t_0 & 0\\
0 & 0 & 3 & -2(t_0+1) & t_0 & 0\\
0 & 0 & 1 & -2(t_0+1) & 3t_0 & 0\\
0 & 0 & 0 & 3 & -2(t_0+1) & t_0
\end{pmatrix}
\begin{pmatrix} g_1\\ g_2\\ g_3\\ g_4\\ g_5\\ g_6 \end{pmatrix}.
$$

For $t_0\in(1,+\infty)$,

$$
\begin{vmatrix}
1 & -2(t_0+1) & 3t_0 & 0 & 0 & 0\\
0 & 3 & -2(t_0+1) & t_0 & 0 & 0\\
0 & 1 & -2(t_0+1) & 0 & 3t_0 & 0\\
0 & 0 & 3 & -2(t_0+1) & t_0 & 0\\
0 & 0 & 1 & -2(t_0+1) & 3t_0 & 0\\
0 & 0 & 0 & 3 & -2(t_0+1) & t_0
\end{vmatrix}
=324t_0^2(t_0-1)(4t_0+5)\neq 0.
$$

Therefore, the matrix $\begin{pmatrix}
1 & -2(t_0+1) & 3t_0 & 0 & 0 & 0\\
0 & 3 & -2(t_0+1) & t_0 & 0 & 0\\
0 & 1 & -2(t_0+1) & 0 & 3t_0 & 0\\
0 & 0 & 3 & -2(t_0+1) & t_0 & 0\\
0 & 0 & 1 & -2(t_0+1) & 3t_0 & 0\\
0 & 0 & 0 & 3 & -2(t_0+1) & t_0
\end{pmatrix}$ is invertible.

By Definition 2.1, we have $M_2^2\cdot J(W_{t_0})=M_2^5$.

Mather's theorem is quite effective for the finite determinacy of smooth function germ with codimension less than 4. Unfortunately, we can not draw the conclusion that $W_{t_0}(x,y)$ and $W_{t_1}(x,y)$ are $\mathcal{R}$-equivalent for all $t_0, t_1 \in (1,+\infty), t_0 \neq t_1$, since the codimension of function family $W_t(x,y)$ is 8. Here, as an application of our methods, we present the $\mathcal{R}$-equivalency of function family $W_t(x,y)$.

In fact, for any $t_0, t_1 \in (1,+\infty)$, $t_0 \neq t_1$, if $|t_0 - t_1|$ is sufficiently small, then

$$j^4 W_{t_1}(x,y) - j^4 W_{t_0}(x,y) = W_{t_1}(x,y) - W_{t_0}(x,y) = xy^2(x-y)(t_0 - t_1) \in P_2^4$$

is sufficiently small. By Theorem 3.4, $W_{t_1}(x,y)$ is $\mathcal{R}$-equivalent to $W_{t_0}(x,y)$. □

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