Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 6096–6112
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Hermite-Hadamard type inequalities for logarithmically B-preinvex functions

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Communicated by M. Imdad

Abstract

In this paper, we introduce the notion of logarithmically B-preinvex functions and establish certain new Hermite-Hadamard type inequalities for the functions whose derivatives in absolute value are logarithmically B-preinvex. Our results generalize several known results for the classes of logarithmically preinvex functions. Some estimates for the left and right hand side of the Hermite-Hadamard inequality are also obtained for a new class of differentiable logarithmically α-preinvex functions.

Keywords: Hermite-Hadamard inequality, logarithmically B-preinvex functions, convex functions.
2010 MSC: 26A51, 26D15, 26B25, 52A30.

1. Introduction

Let function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The inequality is known as Hermite-Hadamard inequality for convex functions.

Convex functions had remained an area of great interest due to its wide applications in many branches of mathematics and the other sciences e.g., engineering, economics and optimization theory. In 1986, Israel

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Received 2015-11-25

and Mond [2] gave a sufficient condition for a differentiable function to be an invex function. In 1987, Hanson and Mond [3] identified it as a general class of functions. In [13], Weir and Mond named this class as preinvex functions. They further identified that "Preinvex functions are convex like, however, preinvex functions have some interesting properties that are not generally shared by the wider class of convex like functions. For example, as for convex functions, every local minimum of a preinvex function is a global minimum and nonnegative linear combinations of preinvex functions are preinvex". In 1993, Suneja et al. [11] generalized the notion of preinvex functions by introducing B-preinvex functions. In [5, 6, 8, 9], Noor et al. generalized the preinvex functions to h-preinvex, logarithmically preinvex and logarithmically h-preinvex functions.

We, first, restate some basic preinvexity domains and related results.

Let S be a non-empty closed subset of $\mathbb{R}^n$, $f : S \to \mathbb{R}$ and $\eta : S \times S \to \mathbb{R}^n$ be continuous functions.

Definition 1.1. A set S is said to be invex with respect to $\eta(\cdot,\cdot)$ if for $x, y \in S$ and $t \in [0,1]$.

$$y + t\eta\left(x, y\right) \in S.$$

The invex set S is also called an η-connected set.

Definition 1.2. Let S be an invex set with respect to η. For every $x, y \in S$, the η-path P_{xv} joining the points x and $v = x + \eta(y, x)$ is defined as follows:

$$P_{xv} := \left\{z : z = x + t\eta(y, x) : t \in [0,1]\right\}.$$

Definition 1.3. A function f is said to be preinvex with respect to an arbitrary bi-function $\eta(\cdot,\cdot)$ if

$$f\left(y + t\eta\left(x, y\right)\right) \leq tf\left(x\right) + \left(1-t\right)f\left(y\right)$$

holds for all $x, y \in S$ and $t \in [0,1]$.

Remark 1.4. If $\eta(x, y) = x - y$, then preinvex function becomes convex function.

Mohan and Neogy [4] further investigated the properties of invex sets and preinvex functions. The condition C for preinvex functions given by Mohan and Neogy is stated as follows:

Definition 1.5. Let $S \subseteq \mathbb{R}^n$ be an invex subset with respect to bi-function η. Then, for any $x, y \in S$ and and $t \in [0,1]$, we have

$$\begin{aligned} \eta\left(y, y + t\eta\left(x, y\right)\right) &= -t\eta\left(x, y\right), \\ \eta\left(x, y + t\eta\left(x, y\right)\right) &= \left(1-t\right)\eta\left(x, y\right). \end{aligned}$$

We say that η satisfies condition C, if for any $x, y \in S$ and $t_1, t_2 \in [0,1]$, we have

$$\eta\left(y + t_1\eta\left(x, y\right), y + t_2\eta\left(x, y\right)\right) = \left(t_2 - t_1\right)\eta\left(x, y\right).$$

Definition 1.6. A non-negative function f on the invex set S is said to be logarithmically preinvex with respect to η, if

$$f(y + t\eta\left(x, y\right)) \leq [f(x)]^t[f(y)]^{1-t}$$

for all $x, y \in S$ and $t \in [0,1]$.

Remark 1.7. A function is logarithmically preinvex with respect to η, if $\log f$ is preinvex.

In [7], Noor proved the following analogue of Hermite-Hadamard inequality for preinvex and logarithmically preinvex functions.

Theorem 1.8. *Let $f : S = [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on a real interval S^0 and $a, b \in S^0$ with $a < a + \eta(b, a)$. Then, the following inequality holds*

$$f(a + \frac{\eta(b,a)}{2}) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a) + f(a + \eta(b,a))}{2} \leq \frac{f(a) + f(b)}{2}.$$

Theorem 1.9. *Let f be a log-preinvex function on the interval $[a, a + \eta(b, a)]$. Then,*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a) - f(b)}{\log f(a) - \log f(b)}.$$

For more results on preinvex and logarithmically preinvex functions see [1, 6, 10]. In [9], Noor et al. presented the Hermite-Hadamard inequalities for logarithmically h-preinvex functions. Logarithmically h-preinvex functions are defined as follows:

Definition 1.10. Let $h : J \to \mathbb{R}$, where $(0, 1) \subseteq J$ be an interval in $\mathbb{R}$ and let S be an invex set with respect to $\eta(\cdot, \cdot)$. A non-negative function $f : S \to (0, \infty)$ is called logarithmically h-preinvex with respect to $\eta(\cdot, \cdot)$, if

$$f(y + t\eta(x, y)) \leq [f(x)]^{h(t)}[f(y)]^{h(1-t)}$$

for all $x, y \in S$ and $t \in [0, 1]$.

In this paper, we introduce a new class of logarithmically B-preinvex functions and give some new estimates for Hermite-Hadamard type inequalities for the functions whose derivatives in absolute value are logarithmically B-preinvex.

2. Main results

We, first, define a generalized notion of logarithmically preinvex functions as follows.

Definition 2.1. Let $b_1, b_2 : [0, 1] \to \mathbb{R}$ be nonnegative functions. A non-negative function $f : S \to (0, \infty)$ on the invex set S is said to be logarithmically B-preinvex w.r.t. η if the inequality

$$f(y + t\eta(x, y)) \leq [f(x)]^{b_2(t)}[f(y)]^{b_1(t)},$$

holds for all $x, y \in S$ and $t \in [0, 1]$ such that $b_1(t), b_2(t) \geq 0,\ b_1(t) + b_2(t) \leq 1, b_1(0) = 1 = b_2(1)$.

Remark 2.2.

1. For $b_1(t) = 1 - t$ and $b_2(t) = t,$ the logarithmically B-preinvex function reduces to logarithmically preinvex function.
2. For $b_1(t) = (1 - t)^s$ and $b_2(t) = t^s,$ the logarithmically B-preinvex function reduces to s-logarithmically preinvex function [12].
3. For $b_1(t) = 1 - t^\alpha$ and $b_2(t) = t^\alpha,$ the logarithmically B-preinvex function reduces to a new class, logarithmically α-preinvex function which cannot be obtained as a special class of logarithmically h-preinvex functions [9].
4. For $b_1(t) = \frac{1}{1-t}$ and $b_2(t) = \frac{1}{t},$ the logarithmically B-preinvex function reduces to logarithmically Q-preinvex function [9].

Remark 2.3. A function is logarithmically B-preinvex with respect to η, if $\log f$ is B-preinvex [11].

We, now, give an example to show that a function can be logarithmically B-preinvex without being logarithmically h-preinvex.

Example 2.4. Let us define

$$f(x) = e^{-|x|},$$

and

$$\eta(x,y) = \begin{cases} x - y, & xy < 0, \\ y - x, & xy > 0. \end{cases}$$

Since, we know that a function is logarithmically B-preinvex, if $\log f$ is B-preinvex, we show that $\log f$ is B-preinvex without being h-preinvex. Now, if $b_1(t) = 1 - t^{\alpha}$ and $b_2(t) = t^{\alpha}$, $t \in (0,1)$, then

$$f(y + t\eta(x,y)) \leq t^{\alpha} f(x) + (1 - t^{\alpha}) f(y)$$

holds for all x, y and α i.e., $\log f$ is B-preinvex while if $h(t) = t^{\alpha}$, then we have for $x = 1, y = 1, t = \frac{1}{2}$ and $\alpha = \frac{1}{2}$

$$f(y + t\eta(x,y)) = f(1) = -1 > t^{\alpha} f(x) + (1-t)^{\alpha} f(y) = \frac{1}{\sqrt{2}} f(1) = -\sqrt{2},$$

which shows that $\log f$ is not h-preinvex.

The following Lemmas will be needed in the sequel.

Lemma 2.5 ([1])**.** *Let $f : S \to (0, \infty)$ be a differentiable mapping on $Int\,[a, a + \eta(b,a)] \subseteq S$ with $\eta(b,a) > 0$. If $f^{'} \in L[a, a + \eta(b,a)]$, then*

$$\frac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a) + f(a + \eta(b,a))}{2} = \frac{\eta(b,a)}{2} \int\limits_0^1 (1 - 2t) f^{'}(a + t\eta(b,a))\,dt.$$

Lemma 2.6 ([10])**.** *Let $f : S \to (0, \infty)$ be a differentiable mapping on $Int\,[a, a + \eta(b,a)] \subseteq S$ with $\eta(b,a) > 0$. If $f^{'} \in L[a, a + \eta(b,a)]$, then*

$$\begin{aligned} &\frac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(x)\,dx - f\left(a + \frac{\eta(b,a)}{2}\right) \\ &= \eta(b,a) \left[\int\limits_0^{\frac{1}{2}} t f^{'}(a + t\eta(b,a))\,dt + \int\limits_{\frac{1}{2}}^1 (1-t) f^{'}(a + t\eta(b,a))\,dt \right]. \end{aligned} \tag{2.1}$$

Now we give the generalized Hermite-Hadamard inequalities for logarithmically B-preinvex functions stated as follows:

Theorem 2.7. *Let f be a logarithmically B-preinvex function η satisfies condition C, then, for $\eta(b,a) > 0$, we have:*

$$\begin{aligned} \log\left(f\left(a + \frac{\eta(b,a)}{2}\right)\right) &\leq \frac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} \log f(x)\,dx \\ &\leq \left(\int\limits_0^1 b_1(t)\,dt\right) \log f(a) + \left(\int\limits_0^1 b_2(t)\,dt\right) \log f(b). \end{aligned}$$

Consequently,

$$f\left(a + \frac{\eta(b,a)}{2}\right) \leq \exp\left[\frac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} \log f(x)\,dx\right] \leq [f(a)]^{\int_0^1 b_1(t)dt} [f(b)]^{\int_0^1 b_2(t)dt}.$$

Proof. By using the definition of logarithmically B-preinvex function and condition C, we get

$$\begin{aligned} f\left(a+\frac{\eta(b,a)}{2}\right) &= f\left(a+(1-t)\eta(b,a)+\frac{\eta(a+t\eta(b,a),a+(1-t)\eta(b,a))}{2}\right) \\ &\le [f(a+(1-t)\eta(b,a))]^{b_1\left(\frac{1}{2}\right)}[f(a+t\eta(b,a))]^{b_2\left(\frac{1}{2}\right)}. \end{aligned} \tag{2.2}$$

By taking the logarithm on both sides of (2.2), we get

$$\begin{aligned} \log f\left(a+\frac{\eta(b,a)}{2}\right) &\le \log\left[[f(a+(1-t)\eta(b,a))]^{b_1\left(\frac{1}{2}\right)}[f(a+t\eta(b,a))]^{b_2\left(\frac{1}{2}\right)}\right] \\ &= b_1\left(\frac{1}{2}\right)\log[f(a+(1-t)\eta(b,a))]+b_2\left(\frac{1}{2}\right)\log[f(a+t\eta(b,a))]. \end{aligned} \tag{2.3}$$

By integrating on the both sides of (2.3) with respect to $t\in[0,1]$, we obtain

$$\begin{aligned} \log f\left(a+\frac{\eta(b,a)}{2}\right) &= b_1\left(\frac{1}{2}\right)\int_0^1 \log[f(a+(1-t)\eta(b,a))]\,dt \\ &\quad + b_2\left(\frac{1}{2}\right)\int_0^1 \log[f(a+t\eta(b,a))]\,dt, \end{aligned}$$

or

$$\begin{aligned} \log f\left(a+\frac{\eta(b,a)}{2}\right) &= \frac{\left(b_1\left(\frac{1}{2}\right)+b_2\left(\frac{1}{2}\right)\right)}{\eta(b,a)}\int_a^{a+\eta(b,a)} \log[f(x)]\,dx \\ &\le \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} \log[f(x)]\,dx. \end{aligned} \tag{2.4}$$

Now, by applying log on the definition of logarithmically B-preinvex function and integrating with respect to $t\in[0,1]$, we get

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} \log[f(x)]\,dx \le \left(\int_0^1 b_1(t)\,dt\right)\log f(a)+\left(\int_0^1 b_2(t)\,dt\right)\log f(b). \tag{2.5}$$

By combining (2.4) and (2.5), we have the required result. □

Remark 2.8. For respective choices of the functions $b_1(t)$ and $b_2(t)$ in Theorem 2.7 as given in Remark 2.2, we can get estimates for logarithmically preinvex, s-logarithmically preinvex, logarithmically α-preinvex and logarithmically Q-preinvex functions.

Theorem 2.9. *Let $f:S\to(0,\infty)$ be a differentiable function and $f'\in L[a,a+\eta(b,a)]$. If $\left|f'\right|$ is logarithmically B-preinvex on $[a,a+\eta(b,a)]\subseteq S$, $b_1(t)+b_2(t)=1$ and $\eta(b,a)>0$, then*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-\frac{f(a)+f(a+\eta(b,a))}{2}\right| \le \frac{\eta(b,a)}{2}\left|f'(a)\right|\int_0^1 |1-2t|\left|\frac{f'(b)}{f'(a)}\right|^{b_2(t)}\,dt. \tag{2.6}$$

Proof. Since $\eta(b,a) > 0$, thus by using Lemma 2.5 and the properties of modulus function, it follows that

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| \leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\left|f'(a+t\eta(b,a))\right|dt. \tag{2.7}$$

By applying the definition of logarithmically B-preinvex function on $\left|f'\right|$, we have from (2.7),

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| \leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\left|f'(a)\right|^{b_1(t)}\left|f'(b)\right|^{b_2(t)}dt. \tag{2.8}$$

By using the condition $b_1(t)+b_2(t)=1$ on (2.8), we have the required inequality (2.6). □

Theorem 2.10. *Let $f : S \to (0,\infty)$ be a differentiable function and $f' \in L[a, a+\eta(b,a)]$. If $\left|f'\right|$ is logarithmically B-preinvex on $[a, a+\eta(b,a)] \subseteq S$, $b_1(t)+b_2(t)=1$ and $\eta(b,a) > 0$, then*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{\eta(b,a)}{2}\right)\right| \leq \eta(b,a)\left|f'(a)\right| \tag{2.9}$$
$$\times\left[\int_0^{\frac{1}{2}} t\left|\frac{f'(b)}{f'(a)}\right|^{b_2(t)}dt + \int_{\frac{1}{2}}^1 (1-t)\left|\frac{f'(b)}{f'(a)}\right|^{b_2(t)}dt\right].$$

Proof. By applying modulus and its properties on (2.1) and using the definition of Logarithmically B-preinvex function, the inequality (2.9) follows similar to (2.6). □

Theorem 2.11. *Let $f : S \to (0,\infty)$ be a differentiable function and $f' \in L[a, a+\eta(b,a)]$. If $\left|f'\right|^q$ is logarithmically B-preinvex on $[a, a+\eta(b,a)] \subseteq S$ for $q > 1$, $b_1(t)+b_2(t)=1$ and $\eta(b,a) > 0$, then*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| \leq \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\int_0^1 |1-2t|\left|\frac{f'(b)}{f'(a)}\right|^{qb_2(t)}dt\right]^{\frac{1}{q}}. \tag{2.10}$$

Proof. Given $\eta(b,a) > 0$. By using Lemma 2.5, the properties of modulus function and the Hölder's inequality, it follows that

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right|$$
$$\leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t|^{1-\frac{1}{q}}|1-2t|^{\frac{1}{q}}\left|f'(a+t\eta(b,a))\right|dt \tag{2.11}$$
$$\leq \frac{\eta(b,a)}{2}\left[\int_0^1 |1-2t|\,dt\right]^{1-\frac{1}{q}}\left[\int_0^1 |1-2t|\left|f'(a+t\eta(b,a))\right|^q dt\right]^{\frac{1}{q}}.$$

By applying the definition of logarithmically B-preinvex function on $\left|f'\right|^q$, we have from (2.11) that

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-\frac{f(a)+f(a+\eta(b,a))}{2}\right|\le\frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\times\left[\int_0^1|1-2t|\left|f'(a)\right|^{qb_1(t)}\left|f'(b)\right|^{qb_2(t)}dt\right]^{\frac{1}{q}}. \tag{2.12}$$

By using the condition $b_1(t)+b_2(t)=1$ on (2.12), we have the required inequality (2.10). $\square$

Theorem 2.12. *Let $f:S\to(0,\infty)$ be a differentiable function and $f'\in L[a,a+\eta(b,a)]$. If $\left|f'\right|^q$ is logarithmically B-preinvex on $[a,a+\eta(b,a)]\subseteq S$ for $q>1$, $b_1(t)+b_2(t)=1$ and $\eta(b,a)>0$, then*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-f\left(a+\frac{\eta(b,a)}{2}\right)\right| \le\frac{\eta(b,a)}{2^{3-\frac{3}{q}}}\left|f'(a)\right|\times\left[\left(\int_0^{\frac{1}{2}}t\left|\frac{f'(b)}{f'(a)}\right|^{qb_2(t)}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(1-t)\left|\frac{f'(b)}{f'(a)}\right|^{qb_2(t)}dt\right)^{\frac{1}{q}}\right]. \tag{2.13}$$

Proof. Bu using Lemma 2.6, the proof is similar to Theorem 2.11. $\square$

Theorem 2.13. *Let $f:S\to(0,\infty)$ be a differentiable function and $f'\in L[a,a+\eta(b,a)]$. If $\left|f'\right|^q$ is logarithmically B-preinvex on $[a,a+\eta(b,a)]\subseteq S$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, $b_1(t)+b_2(t)=1$ and $\eta(b,a)>0$, then*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-\frac{f(a)+f(a+\eta(b,a))}{2}\right| \le\frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\int_0^1\left|\frac{f'(b)}{f'(a)}\right|^{qb_2(t)}dt\right]^{\frac{1}{q}}. \tag{2.14}$$

Proof. Given $\eta(b,a)>0$. By using Lemma 2.5, the properties of modulus and the Hölder's inequality, it follows that

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-\frac{f(a)+f(a+\eta(b,a))}{2}\right| \le\frac{\eta(b,a)}{2}\left[\int_0^1|1-2t|^p\,dt\right]^{\frac{1}{p}}\times\left[\int_0^1\left|f'(a+t\eta(b,a))\right|^q dt\right]^{\frac{1}{q}}. \tag{2.15}$$

By applying the definition of logarithmically B-preinvex function on $\left|f'\right|^q$ and simplifying, we have from (2.15) that

$$
\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-\frac{f(a)+f(a+\eta(b,a))}{2}\right| \\
&\leq \eta(b,a)\left[\int_0^1\left|\frac{1}{2}-t\right|^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'(a)\right|^{qb_1(t)}\left|f'(b)\right|^{qb_2(t)}dt\right]^{\frac{1}{q}} \\
&=\eta(b,a)\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)^p dt+\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)^p dt\right]^{\frac{1}{p}} \qquad (2.16)\\
&\quad\times\left[\int_0^1\left|f'(a)\right|^{qb_1(t)}\left|f'(b)\right|^{qb_2(t)}dt\right]^{\frac{1}{q}} \\
&=\frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left[\int_0^1\left|f'(a)\right|^{qb_1(t)}\left|f'(b)\right|^{qb_2(t)}dt\right]^{\frac{1}{q}}.
\end{aligned}
$$

By using the condition $b_1(t)+b_2(t)=1$ on (2.16), we have the required inequality (2.14). □

Theorem 2.14. *Let $f: S\to(0,\infty)$ be a differentiable function and $f'\in L[a,a+\eta(b,a)]$. If $\left|f'\right|^q$ is logarithmically B-preinvex on $[a,a+\eta(b,a)]\subseteq S$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, $b_1(t)+b_2(t)=1$ and $\eta(b,a)>0$, then*

$$
\begin{aligned}
\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-f\left(a+\frac{\eta(b,a)}{2}\right)\right| &\leq \frac{\eta(b,a)}{2^{1+\frac{1}{p}}}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right| \\
&\quad\times\left[\left(\int_0^{\frac{1}{2}}\left|\frac{f'(b)}{f'(a)}\right|^{qb_2(t)}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left|\frac{f'(b)}{f'(a)}\right|^{qb_2(t)}dt\right)^{\frac{1}{q}}\right]. \qquad (2.17)
\end{aligned}
$$

Proof. By using Lemma 2.6, the proof is similar to Theorem 2.13. □

The following theorem establishes new inequalities for two logarithmically B-preinvex functions.

Theorem 2.15. *Let $f,g:[a,a+\eta(b,a)]\to(0,\infty)$ be logarithmically B-preinvex on $Int[a,a+\eta(b,a)]$ and $a,b\in Int[a,a+\eta(b,a)]$ with $b_1(t)+b_2(t)=1$ and $\eta(b,a)>0$. Then, the following inequality holds:*

$$
\frac{2}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,g(x)\,dx\leq[f(a)]^2\int_0^1\left[\frac{f(b)}{f(a)}\right]^{2b_2(t)}dt+[g(a)]^2\int_0^1\left[\frac{g(b)}{g(a)}\right]^{2b_2(t)}dt.
$$

Proof. Let f,g be logarithmically B-preinvex functions. Then,

$$
\begin{aligned}
f(a+t\eta(b,a))&\leq[f(a)]^{b_1(t)}[f(b)]^{b_2(t)},\\
g(a+t\eta(b,a))&\leq[g(a)]^{b_1(t)}[g(b)]^{b_2(t)}.
\end{aligned}
$$

Now,

$$
\int_a^{a+\eta(b,a)} f(x)\,g(x)\,dx=\eta(b,a)\int_0^1 f(a+t\eta(b,a))\,g(a+t\eta(b,a))\,dt
$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1 \left[\{f(a+t\eta(b,a))\}^2+\{g(a+t\eta(b,a))\}^2\right]$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1 \left[\left[[f(a)]^{b_1(t)}[f(b)]^{b_2(t)}\right]^2+\left[[g(a)]^{b_1(t)}[g(b)]^{b_2(t)}\right]^2\right]dt$$

$$= \frac{\eta(b,a)}{2}\left[[f(a)]^2\int_0^1\left[\frac{f(b)}{f(a)}\right]^{2b_2(t)}dt+[g(a)]^2\int_0^1\left[\frac{g(b)}{g(a)}\right]^{2b_2(t)}dt\right].$$

Thus, we have the required result. □

Remark 2.16. For respective choices of the functions $b_1(t)$ and $b_2(t)$ in Theorems 2.9–2.14 as given in Remark 2.2, we recapture estimates for logarithmically preinvex given in [10]. Moreover, some new estimates can also be obtained for the logarithmically α-preinvex functions.

Now we give the results for logarithmically α-preinvex functions.

Theorem 2.17. *Let $f: S\to(0,\infty)$ be a differentiable function and $f'\in L[a,a+\eta(b,a)]$. If $\left|f'\right|$ is logarithmically α-preinvex on $[a,a+\eta(b,a)]\subseteq S$, and $\eta(b,a)>0$, then*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx-\frac{f(a)+f(a+\eta(b,a))}{2}\right|\leq\frac{\eta(b,a)}{2}\left|f'(a)\right| M(\alpha,t),$$

where

$$M(\alpha,t)=\begin{cases}\frac{1}{2}, & \text{for } \mu=1,\\ \frac{1}{(\alpha\ln\mu)^2}\left[4\mu^{\frac{\alpha}{2}}+\mu^{\alpha}(\alpha\ln\mu-2)-(\alpha\ln\mu+2)\right], & \text{for } \mu<1,\\ \left(\frac{\alpha}{\ln\mu}\right)^2\left[4\mu^{\frac{1}{2\alpha}}+\mu^{\frac{1}{\alpha}}\left(\frac{1}{\alpha}\ln\mu-2\right)-\left(\frac{1}{\alpha}\ln\mu+2\right)\right], & \text{for } \mu>1,\end{cases}$$

and μ is defined as:

$$\mu=\left|\frac{f'(b)}{f'(a)}\right|.$$

Proof. For $b_1(t)=1-t^\alpha$, $b_2(t)=t^\alpha$ in (2.8), we have

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx-\frac{f(a)+f(a+\eta(b,a))}{2}\right|\leq\frac{\eta(b,a)}{2}\left|f'(a)\right|\int_0^1|1-2t|\left|\frac{f'(b)}{f'(a)}\right|^{t^\alpha}dt. \tag{2.18}$$

Let $\mu=\left|\frac{f'(b)}{f'(a)}\right|$ in (2.18), we have three cases:

Case 1: For $\mu=1$, (2.18) takes the form

$$\begin{aligned}\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx-\frac{f(a)+f(a+\eta(b,a))}{2}\right| &\leq\frac{\eta(b,a)}{2}\left|f'(a)\right|\int_0^1|1-2t|\,dt\\ &\leq\frac{\eta(b,a)}{2}\left|f'(a)\right|\left[\int_0^{\frac{1}{2}}(1-2t)\,dt+\int_{\frac{1}{2}}^1(2t-1)\,dt\right]\\ &\leq\frac{\eta(b,a)}{4}\left|f'(a)\right|.\end{aligned}\tag{2.19}$$

Case 2: For $\mu < 1$, we have $\mu^{t^\alpha} \leq \mu^{\alpha t}$. Thus, (2.18) becomes

$$
\begin{aligned}
&\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a) + f(a+\eta(b,a))}{2} \right| \\
&\leq \frac{\eta(b,a)}{2} \left| f'(a) \right| \int_0^1 |1-2t|\, \mu^{\alpha t} dt \\
&\leq \frac{\eta(b,a)}{2} \left| f'(a) \right| \left[\int_0^{\frac{1}{2}} (1-2t)\, \mu^{\alpha t} dt + \int_{\frac{1}{2}}^1 (2t-1)\, \mu^{\alpha t} dt \right] \\
&\leq \frac{\eta(b,a)}{2} \left| f'(a) \right| \left[-\frac{1}{\alpha \ln \mu} + \frac{2}{(\alpha \ln \mu)^2} \left(\mu^{\frac{\alpha}{2}} - 1 \right) + \frac{\mu^\alpha}{\alpha \ln \mu} - \frac{2}{(\alpha \ln \mu)^2} \left(\mu^\alpha - \mu^{\frac{\alpha}{2}} \right) \right] \\
&\leq \frac{\eta(b,a)}{2(\alpha \ln \mu)^2} \left| f'(a) \right| \left[4\mu^{\frac{\alpha}{2}} + \mu^\alpha (\alpha \ln \mu - 2) - (\alpha \ln \mu + 2) \right].
\end{aligned}
\tag{2.20}
$$

Case 3: For $\mu > 1$, we have $\mu^{t^\alpha} \leq \mu^{\frac{t}{\alpha}}$. Thus, (2.18) becomes

$$
\begin{aligned}
&\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a) + f(a+\eta(b,a))}{2} \right| \\
&\leq \frac{\eta(b,a)}{2} \left| f'(a) \right| \int_0^1 |1-2t|\, \mu^{\frac{t}{\alpha}} dt \\
&\leq \frac{\eta(b,a)}{2} \left| f'(a) \right| \left[\int_0^{\frac{1}{2}} (1-2t)\, \mu^{\frac{t}{\alpha}} dt + \int_{\frac{1}{2}}^1 (2t-1)\, \mu^{\frac{t}{\alpha}} dt \right] \\
&\leq \frac{\eta(b,a)}{2} \left| f'(a) \right| \left[-\frac{1}{\frac{1}{\alpha} \ln \mu} + \frac{2}{\left(\frac{1}{\alpha} \ln \mu\right)^2} \left(\mu^{\frac{1}{2\alpha}} - 1 \right) + \frac{\mu^{\frac{1}{\alpha}}}{\frac{1}{\alpha} \ln \mu} - \frac{2}{\left(\frac{1}{\alpha} \ln \mu\right)^2} \left(\mu^{\frac{1}{\alpha}} - \mu^{\frac{1}{2\alpha}} \right) \right] \\
&\leq \frac{\eta(b,a)}{2} \left(\frac{\alpha}{\ln \mu} \right)^2 \left| f'(a) \right| \left[-\frac{1}{\alpha} \ln \mu + 2 \left(\mu^{\frac{1}{2\alpha}} - 1 \right) + \frac{1}{\alpha} \ln \mu \mu^{\frac{1}{\alpha}} - 2 \left(\mu^{\frac{1}{\alpha}} - \mu^{\frac{1}{2\alpha}} \right) \right] \\
&\leq \frac{\eta(b,a)}{2} \left(\frac{\alpha}{\ln \mu} \right)^2 \left| f'(a) \right| \left[4\mu^{\frac{1}{2\alpha}} + \mu^{\frac{1}{\alpha}} \left(\frac{1}{\alpha} \ln \mu - 2 \right) - \left(\frac{1}{\alpha} \ln \mu + 2 \right) \right].
\end{aligned}
\tag{2.21}
$$

Therefore, (2.19), (2.20) and (2.21) are the required inequalities. □

Theorem 2.18. *Let $f : S \to (0, \infty)$ be a differentiable function and $f' \in L[a, a+\eta(b,a)]$. If $\left| f' \right|$ is logarithmically α-preinvex on $[a, a+\eta(b,a)] \subseteq S$, and $\eta(b,a) > 0$, then*

$$
\begin{aligned}
\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a + \frac{\eta(b,a)}{2} \right) \right| &\leq \eta(b,a) \left| f'(a) \right| \left[\int_0^{\frac{1}{2}} t \left| \frac{f'(b)}{f'(a)} \right|^{t^\alpha} dt + \int_{\frac{1}{2}}^1 (1-t) \left| \frac{f'(b)}{f'(a)} \right|^{t^\alpha} dt \right] \\
&\leq \eta(b,a) \left| f'(a) \right| M(\alpha, t),
\end{aligned}
$$

where

$$M(\alpha,t)=\begin{cases}\frac{1}{2^2}, & \text{for } \mu=1,\\ \frac{1}{(\alpha\ln\mu)^2}\left[-2\mu^{\frac{\alpha}{2}}+\mu^{\alpha}+1\right], & \text{for } \mu<1,\\ \left(\frac{\alpha}{\ln\mu}\right)^2\left[-2\mu^{\frac{1}{2\alpha}}+\mu^{\frac{1}{\alpha}}+1\right], & \text{for } \mu>1,\end{cases}$$

and μ *is defined as*

$$\mu=\left|\frac{f'(b)}{f'(a)}\right|.$$

Proof. For $b_1(t)=1-t^{\alpha}$, $b_2(t)=t^{\alpha}$ in (2.15), we have

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-f\left(a+\frac{\eta(b,a)}{2}\right)\right|\le\eta(b,a)\left|f'(a)\right| \times\left[\int_0^{\frac{1}{2}} t\left|\frac{f'(b)}{f'(a)}\right|^{t^{\alpha}}dt+\int_{\frac{1}{2}}^{1}(1-t)\left|\frac{f'(b)}{f'(a)}\right|^{t^{\alpha}}dt\right]. \tag{2.22}$$

Let $\mu=\left|\frac{f'(b)}{f'(a)}\right|$ in (2.22), we have three cases:

Case 1: For $\mu=1$, (2.22) takes the form

$$\begin{aligned}\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-f\left(a+\frac{\eta(b,a)}{2}\right)\right|&\le\eta(b,a)\left|f'(a)\right|\left[\int_0^{\frac{1}{2}} t\,dt+\int_{\frac{1}{2}}^{1}(1-t)\,dt\right]\\&\le\frac{\eta(b,a)}{4}\left|f'(a)\right|.\end{aligned} \tag{2.23}$$

Case 2: For $\mu<1$, we have $\mu^{t^{\alpha}}\le\mu^{\alpha t}$. Thus, (2.22) becomes

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-f\left(a+\frac{\eta(b,a)}{2}\right)\right|\\&\le\eta(b,a)\left|f'(a)\right|\left[\int_0^{\frac{1}{2}} t\mu^{\alpha t}dt+\int_{\frac{1}{2}}^{1}(1-t)\,\mu^{\alpha t}dt\right]\\&\le\eta(b,a)\left|f'(a)\right|\left[\frac{\mu^{\frac{\alpha}{2}}}{2\alpha\ln\mu}-\frac{1}{(\alpha\ln\mu)^2}\left(\mu^{\frac{\alpha}{2}}-1\right)\right.\\&\quad\left.-\frac{\mu^{\frac{\alpha}{2}}}{2\alpha\ln\mu}+\frac{1}{(\alpha\ln\mu)^2}\left(\mu^{\alpha}-\mu^{\frac{\alpha}{2}}\right)\right]\\&\le\eta(b,a)\left|f'(a)\right|\left[\frac{\mu^{\alpha}-\mu^{\frac{\alpha}{2}}-\mu^{\frac{\alpha}{2}}+1}{(\alpha\ln\mu)^2}\right]\\&\le\frac{\eta(b,a)\left|f'(a)\right|}{(\alpha\ln\mu)^2}\left[-2\mu^{\frac{\alpha}{2}}+\mu^{\alpha}+1\right].\end{aligned} \tag{2.24}$$

Case 3: For $\mu > 1$, we have $\mu^{t^{\alpha}} \leq \mu^{\frac{t}{\alpha}}$. Thus, (2.22) becomes

$$\begin{aligned}
\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{\eta(b,a)}{2}\right)\right| &\leq \eta(b,a)\left|f'(a)\right|\left[\int_{0}^{\frac{1}{2}} t\mu^{\frac{t}{\alpha}}dt + \int_{\frac{1}{2}}^{1}(1-t)\,\mu^{\frac{t}{\alpha}}dt\right] \\
&\leq \eta(b,a)\left|f'(a)\right| \\
&\quad \times \left[-\frac{1}{\left(\frac{1}{\alpha}\ln\mu\right)^{2}}\left(\mu^{\frac{1}{2\alpha}}-1\right)+\frac{1}{\left(\frac{1}{\alpha}\ln\mu\right)^{2}}\left(\mu^{\frac{1}{\alpha}}-\mu^{\frac{1}{2\alpha}}\right)\right] \qquad (2.25) \\
&\leq \eta(b,a)\left|f'(a)\right|\left[\frac{\mu^{\frac{1}{\alpha}}-\mu^{\frac{1}{2\alpha}}-\mu^{\frac{1}{2\alpha}}+1}{\left(\frac{1}{\alpha}\ln\mu\right)^{2}}\right] \\
&\leq \frac{\alpha^{2}\eta(b,a)\left|f'(a)\right|}{(\ln\mu)^{2}}\left[-2\mu^{\frac{1}{2\alpha}}+\mu^{\frac{1}{\alpha}}+1\right].
\end{aligned}$$

Therefore, (2.23), (2.24) and (2.25) are the required inequalities. □

Theorem 2.19. *Let $f : S \to (0,\infty)$ be a differentiable function and $f' \in L[a, a+\eta(b,a)]$. If $\left|f'\right|^{q}$ is logarithmically α-preinvex on $[a, a+\eta(b,a)] \subseteq S$ for $q > 1$, and $\eta(b,a) > 0$, then*

$$\begin{aligned}
\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| &\leq \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\int_{0}^{1}|1-2t|\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right]^{\frac{1}{q}} \\
&\leq \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right| M(t,\alpha,q),
\end{aligned}$$

where

$$M(t,\alpha,q) = \begin{cases} \frac{1}{2^{\frac{1}{q}}}, & \text{for } \mu = 1, \\ \frac{1}{(\alpha q\ln\mu)^{2}}\left[4\mu^{\frac{\alpha q}{2}}+\mu^{\alpha q}(\alpha q\ln\mu-2)-(\alpha q\ln\mu+2)\right]^{\frac{1}{q}}, & \text{for } \mu < 1, \\ \frac{\alpha^{2}}{(q\ln\mu)^{2}}\left[4\mu^{\frac{q}{2\alpha}}+\mu^{\frac{q}{\alpha}}\left(\frac{q}{\alpha}\ln\mu-2\right)-\left(\frac{q}{\alpha}\ln\mu+2\right)\right]^{\frac{1}{q}}, & \text{for } \mu > 1, \end{cases}$$

and μ is defined as

$$\mu = \left|\frac{f'(b)}{f'(a)}\right|.$$

Proof. For $b_1(t) = 1-t^{\alpha}$, $b_2(t) = t^{\alpha}$ in (2.16), we have

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| \\
&\leq \frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\int_{0}^{1}|1-2t|\left|f'(a)\right|^{q(1-t^{\alpha})}\left|f'(b)\right|^{qt^{\alpha}}dt\right]^{\frac{1}{q}} \qquad (2.26) \\
&\leq \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\int_{0}^{1}|1-2t|\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right]^{\frac{1}{q}}.
\end{aligned}$$

Let $\mu = \left|\frac{f'(b)}{f'(a)}\right|$ in (2.26), we have three cases:

Case 1: For $\mu = 1$, (2.26) takes the form

$$\begin{aligned}\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| &\le \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\int_0^1 |1-2t|\,dt\right]^{\frac{1}{q}} \\ &\le \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\int_0^{\frac{1}{2}}(1-2t)\,dt + \int_{\frac{1}{2}}^1 (2t-1)\,dt\right]^{\frac{1}{q}} \\ &\le \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\frac{1}{2}\right]^{\frac{1}{q}} \\ &\le \frac{\eta(b,a)}{4}\left|f'(a)\right|.\end{aligned}\tag{2.27}$$

Case 2: For $\mu < 1$, we have $\mu^{t^\alpha} \le \mu^{\alpha t}$. Thus, (2.26) becomes

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| \\ &\le \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\int_0^1 |1-2t|\,\mu^{\alpha q t}dt\right]^{\frac{1}{q}} \\ &\le \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\int_0^{\frac{1}{2}}(1-2t)\,\mu^{\alpha q t}dt + \int_{\frac{1}{2}}^1 (2t-1)\,\mu^{\alpha q t}dt\right]^{\frac{1}{q}} \\ &\le \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[-\frac{1}{\alpha q\ln\mu} + \frac{2}{(\alpha q\ln\mu)^2}\left(\mu^{\frac{\alpha q}{2}}-1\right)\right. \\ &\quad\left. + \frac{\mu^{\alpha q}}{\alpha q\ln\mu} - \frac{2}{(\alpha q\ln\mu)^2}\left(\mu^{\alpha q}-\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}} \\ &\le \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\frac{1}{(\alpha q\ln\mu)^2}\left(4\mu^{\frac{\alpha q}{2}} + \mu^{\alpha q}(\alpha q\ln\mu - 2) - (\alpha q\ln\mu + 2)\right)\right]^{\frac{1}{q}}.\end{aligned}\tag{2.28}$$

Case 3: For $\mu > 1$, we have $\mu^{qt^\alpha} \le \mu^{\frac{qt}{\alpha}}$. Thus, (2.26) becomes

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| \\ &\le \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\int_0^1 |1-2t|\,\mu^{\frac{qt}{\alpha}}dt\right]^{\frac{1}{q}} \\ &\le \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\int_0^{\frac{1}{2}}(1-2t)\,\mu^{\frac{qt}{\alpha}}dt + \int_{\frac{1}{2}}^1 (2t-1)\,\mu^{\frac{qt}{\alpha}}dt\right]^{\frac{1}{q}} \\ &\le \frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[-\frac{1}{\frac{q}{\alpha}\ln\mu} + \frac{2}{\left(\frac{q}{\alpha}\ln\mu\right)^2}\left(\mu^{\frac{q}{2\alpha}}-1\right)\right.\end{aligned}\tag{2.29}$$

$$+\frac{\mu^{\frac{q}{\alpha}}}{\frac{q}{\alpha}\ln\mu}-\frac{2}{\left(\frac{q}{\alpha}\ln\mu\right)^{2}}\left(\mu^{\frac{q}{\alpha}}-\mu^{\frac{q}{2\alpha}}\right)\Bigg]^{\frac{1}{q}}$$
$$\leq\frac{\eta(b,a)}{2^{2-\frac{1}{q}}}\left|f'(a)\right|\left[\frac{\alpha^{2}}{(q\ln\mu)^{2}}\left(4\mu^{\frac{q}{2\alpha}}+\mu^{\frac{q}{\alpha}}\left(\frac{q}{\alpha}\ln\mu-2\right)-\left(\frac{q}{\alpha}\ln\mu+2\right)\right)\right]^{\frac{1}{q}}.$$

Therefore, (2.27), (2.28) and (2.29) are the required inequalities. □

Theorem 2.20. *Let $f:S\to(0,\infty)$ be a differentiable function and $f'\in L[a,a+\eta(b,a)]$. If $\left|f'\right|^{q}$ is logarithmically α-preinvex on $[a,a+\eta(b,a)]\subseteq S$ for $q>1$, and $\eta(b,a)>0$, then*

$$\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx-f\left(a+\frac{\eta(b,a)}{2}\right)\right|\leq\frac{\eta(b,a)}{2^{3-\frac{3}{q}}}\left|f'(a)\right|\left[\left(\int_{0}^{\frac{1}{2}}t\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(1-t)\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\eta(b,a)}{2^{3-\frac{3}{q}}}\left|f'(a)\right|M(t,\alpha,q),$$

where

$$M(t,\alpha,q)=\begin{cases}\frac{1}{2^{\frac{3}{q}-1}}, & \text{for } \mu=1,\\ \left(\frac{\mu^{\frac{\alpha}{2}}}{2\alpha\ln\mu}-\frac{1}{(\alpha\ln\mu)^{2}}\left(\mu^{\frac{\alpha}{2}}-1\right)\right)^{\frac{1}{q}}+\left(-\frac{\mu^{\frac{\alpha}{2}}}{2\alpha\ln\mu}+\frac{1}{(\alpha\ln\mu)^{2}}\left(\mu^{\alpha}-\mu^{\frac{\alpha}{2}}\right)\right)^{\frac{1}{q}}, & \text{for } \mu<1,\\ \left(\frac{\mu^{\frac{1}{2\alpha}}}{\frac{2}{\alpha}\ln\mu}-\frac{1}{\left(\frac{1}{\alpha}\ln\mu\right)^{2}}\left(\mu^{\frac{1}{2\alpha}}-1\right)\right)^{\frac{1}{q}}+\left(-\frac{\mu^{\frac{1}{2\alpha}}}{\frac{2}{\alpha}\ln\mu}+\frac{1}{\left(\frac{1}{\alpha}\ln\mu\right)^{2}}\left(\mu^{\frac{1}{\alpha}}-\mu^{\frac{1}{2\alpha}}\right)\right)^{\frac{1}{q}}, & \text{for } \mu>1,\end{cases}$$

and μ is defined as:

$$\mu=\left|\frac{f'(b)}{f'(a)}\right|.$$

Proof. For $b_1(t)=1-t^{\alpha}$, $b_2(t)=t^{\alpha}$ in (2.13), we have

$$\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx-f\left(a+\frac{\eta(b,a)}{2}\right)\right|\leq\frac{\eta(b,a)}{2^{3-\frac{3}{q}}}\left|f'(a)\right|\left[\left(\int_{0}^{\frac{1}{2}}t\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(1-t)\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right)^{\frac{1}{q}}\right].$$

The rest of the proof is similar to Theorem 2.19. □

Theorem 2.21. *Let $f : S \to (0,\infty)$ be a differentiable function and $f' \in L[a, a+\eta(b,a)]$. If $\left|f'\right|^q$ is logarithmically α-preinvex on $[a, a+\eta(b,a)] \subseteq S$ for $q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$, and $\eta(b,a) > 0$, then*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| \le \frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\int_0^1\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right]^{\frac{1}{q}}$$
$$\le \frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right| M(t,\alpha,q),$$

where

$$M(t,\alpha,q) = \begin{cases} 1, & \text{for } \mu = 1, \\ \left[\frac{1}{\alpha q \ln\mu}\left(\mu^{\alpha q}-1\right)\right]^{\frac{1}{q}}, & \text{for } \mu < 1, \\ \left[\frac{\alpha}{q\ln\mu}\left(\mu^{\frac{q}{\alpha}}-1\right)\right]^{\frac{1}{q}}, & \text{for } \mu > 1, \end{cases}$$

and μ is defined as

$$\mu = \left|\frac{f'(b)}{f'(a)}\right|.$$

Proof. For $b_1(t) = 1-t^{\alpha}$, $b_2(t) = t^{\alpha}$ in (2.14), we have:

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| \le \eta(b,a)\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)^p dt + \int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)^p dt\right]^{\frac{1}{p}}$$
$$\times\left[\int_0^1\left|f'(a)\right|^{q-qt^{\alpha}}\left|f'(b)\right|^{qt^{\alpha}}dt\right]^{\frac{1}{q}} \tag{2.30}$$
$$\le \frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\int_0^1\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right]^{\frac{1}{q}}.$$

Let $\mu = \left|\frac{f'(b)}{f'(a)}\right|$ in (2.30), we have three cases:

Case 1: For $\mu = 1$, (2.30) takes the form

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| \le \frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\int_0^1 1.dt\right]^{\frac{1}{q}} \tag{2.31}$$
$$\le \frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|.$$

Case 2: For $\mu < 1$, we have $\mu^{qt^{\alpha}} \le \mu^{\alpha q t}$. Thus, (2.30) becomes

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| \le \frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\int_0^1 \mu^{\alpha q t}dt\right]^{\frac{1}{q}} \tag{2.32}$$
$$\le \frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\frac{1}{\alpha q\ln\mu}\left(\mu^{\alpha q}-1\right)\right]^{\frac{1}{q}}.$$

Case 3: For $\mu > 1$, we have $\mu^{qt^{\alpha}} \leq \mu^{\frac{qt}{\alpha}}$. Thus, (2.30) becomes

$$\begin{aligned}\left|\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(x)\,dx - \frac{f(a)+f(a+\eta(b,a))}{2}\right| &\leq \frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\int\limits_0^1 \mu^{\frac{qt}{\alpha}}\,dt\right]^{\frac{1}{q}} \\ &\leq \frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\frac{1}{\frac{q}{\alpha}\ln\mu}\left(\mu^{\frac{q}{\alpha}}-1\right)\right]^{\frac{1}{q}} \\ &\leq \frac{\eta(b,a)}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\frac{\alpha}{q\ln\mu}\left(\mu^{\frac{q}{\alpha}}-1\right)\right]^{\frac{1}{q}}.\end{aligned} \tag{2.33}$$

Therefore, (2.31), (2.32) and (2.33) are the required inequalities. □

Theorem 2.22. *Let $f : S \to (0,\infty)$ be a differentiable function and $f' \in L[a, a+\eta(b,a)]$. If $\left|f'\right|^q$ is logarithmically α-preinvex on $[a, a+\eta(b,a)] \subseteq S$ for $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, and $\eta(b,a) > 0$, then:*

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{\eta(b,a)}{2}\right)\right| \\ &\leq \frac{\eta(b,a)}{2^{1+\frac{1}{p}}}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\left(\int\limits_0^{\frac{1}{2}}\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right)^{\frac{1}{q}} + \left(\int\limits_{\frac{1}{2}}^1\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right)^{\frac{1}{q}}\right] \\ &\leq \frac{\eta(b,a)}{2^{1+\frac{1}{p}}}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right| M(t,\alpha,q),\end{aligned}$$

where

$$M(t,\alpha,q) = \begin{cases} 1, & \text{for } \mu = 1, \\ \left(\frac{1}{\alpha q\ln\mu}\left(\mu^{\frac{\alpha q}{2}}-1\right)\right)^{\frac{1}{q}} + \left(\frac{1}{\alpha q\ln\mu}\left(\mu^{\alpha q}-\mu^{\frac{\alpha q}{2}}\right)\right)^{\frac{1}{q}}, & \text{for } \mu < 1, \\ \left(\frac{\alpha}{q\ln\mu}\left(\mu^{\frac{q}{2\alpha}}-1\right)\right)^{\frac{1}{q}} + \left(\frac{\alpha}{q\ln\mu}\left(\mu^{\frac{q}{\alpha}}-\mu^{\frac{q}{2\alpha}}\right)\right)^{\frac{1}{q}}, & \text{for } \mu > 1, \end{cases}$$

and μ is defined by

$$\mu = \left|\frac{f'(b)}{f'(a)}\right|.$$

Proof. For $b_1(t) = 1 - t^{\alpha}$, $b_2(t) = t^{\alpha}$ in (2.17), we have

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{\eta(b,a)}{2}\right)\right| \\ &\leq \frac{\eta(b,a)}{2^{1+\frac{1}{p}}}\frac{1}{(p+1)^{\frac{1}{p}}}\left|f'(a)\right|\left[\left(\int\limits_0^{\frac{1}{2}}\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right)^{\frac{1}{q}} + \left(\int\limits_{\frac{1}{2}}^1\left|\frac{f'(b)}{f'(a)}\right|^{qt^{\alpha}}dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

The rest of the proof is similar to Theorem 2.21. □

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