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A unified framework for the two-sets split common fixed point problem in Hilbert spaces

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Abstract

The two-sets split common fixed point problem of two uniformly Lipschitzian asymptotically pseudocontractive operators is considered. A unified framework for the study of this class of problems and class of operators is provided. An iterative algorithm is constructed and strong convergence analysis is given.

Keywords: Split common fixed point, asymptotically pseudocontractive operators, strong convergence, Hilbert spaces.
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1. Introduction

Let H_1 and H_2 be two real Hilbert spaces equipped with its inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $S \colon H_2 \to H_2$ and $T \colon H_1 \to H_1$ be two nonlinear operators. We use $\mathrm{Fix}(S)$ and $\mathrm{Fix}(T)$ to denote the fixed point sets of S and T, respectively. Let $A \colon H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. The two-sets split common fixed point problem requires to seek an element $x^* \in H_1$ satisfying

$$x^* \in \mathrm{Fix}(T) \quad \text{and} \quad Ax^* \in \mathrm{Fix}(S). \tag{1.1}$$

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We use Γ to denote the set of solutions of (1.1), that is,

$$\Gamma = \{x^* | x^* \in \text{Fix}(T), Ax^* \in \text{Fix}(S)\}.$$

Recently, the split common fixed point problem has attracted so much attention due to it is a generalization of the split feasibility problem and the convex feasibility problem: Ceng et al. [1]; Censor and Segal [2]; Chang et al. [3]; Cholamjiak and Shehu [4]; Dong et al. [5]; He and Du [6]; Mainge [7]; Moudafi [8, 9]; Tang et al. [10]; Wang and Xu [11]; Xu [12, 13]; Yao et al. [14–18].

First, we give some definitions related to the involved operators.

Definition 1.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T: C \to C$ be an operator. $T: C \to C$ is said to be

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

(ii) quasi-nonexpansive if $\|Tx - x^*\| \leq \|x - x^*\|$ for all $x \in C$ and $x^* \in \text{Fix}(T)$;

(iii) firmly nonexpansive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$ for all $x, y \in C$;

(iv) directed (or firmly quasi-nonexpansive) if $\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|Tx - x\|^2$ for all $x \in C$ and $x^* \in \text{Fix}(T)$;

(v) k-demicontractive if $\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|Tx - x\|^2$ where $k \in [0, 1)$ for all $x \in C$ and $x^* \in \text{Fix}(T)$;

(vi) pseudocontractive if $\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2$ for all $x, y \in C$;

(vii) quasi-pseudocontractive if $\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|Tx - x\|^2$ for all $x \in C$ and $x^* \in \text{Fix}(T)$.

Definition 1.2. An operator $T : C \to C$ is said to be L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$.

Next we recall some existing results regarding the split common fixed point problem in the literature. To solve the two-sets split common fixed point problem (1.1), Censor and Segal [2] constructed the following iterative algorithm in the finite dimensional Euclid spaces.

Algorithm 1.3.

Initialization: Let $x_0 \in \mathbb{R}^N$ be arbitrary.

Cycle iteration: For $n \geq 1$, assume the n-th iteration x_n is constructed, then define the $(n + 1)$-th iteration x_{n+1} via the following recursive form

$$x_{n+1} = T(x_n + \lambda A^*(S - I)Ax_n), \quad n \geq 1, \tag{1.2}$$

where S and T are directed operators and $\lambda \in (0, 2/\gamma)$ with γ being the spectral radius of the operator A^*A.

Subsequently, Censor and Segal [2] proved the following convergence result.

Theorem 1.4. *Assume that $I - S$ and $I - T$ are demiclosed at zero. If $\Gamma \neq \emptyset$, then the sequence x_n generated by* (1.2) *converges to a split common fixed point $x^* \in \Gamma$.*

In [8], Moudafi considered a relaxation version of algorithm (1.2) for the k-demicontractive operator in the infinite dimensional Hilbert spaces.

Algorithm 1.5.

Initialization: Let $x_0 \in H_1$ be arbitrary.

Cycle iteration: For $n \geq 1$, assume the n-th iteration x_n is constructed. Set $u_n = x_n + \lambda A^*(S - I)Ax_n$ and define the $(n + 1)$-th iteration x_{n+1} by the following form

$$x_{n+1} = (1 - \alpha_n)u_n + \alpha_n T(u_n), \quad n \geq 1, \tag{1.3}$$

where $\alpha_n \in (0, 1)$ and $\lambda \in (0, \frac{1-k}{\gamma})$ with γ being the spectral radius of the operator A^*A.

Moreover, Moudafi [8] demonstrated the strong convergence of (1.3) to a general case in which the involved operators are demicontractive.

Theorem 1.6. *Let $T\colon H_1 \to H_1$ and $S\colon H_2 \to H_2$ be demicontractive operators with constants β and μ, respectively. Assume that $I - S$ and $I - T$ are demiclosed at zero. If $\Gamma \neq \emptyset$, then the sequence x_n generated by (1.3) converges weakly to a split common fixed-point $x^* \in \Gamma$, provided that $\alpha_n \in (\delta, 1 - \beta - \delta)$ for a small enough $\delta > 0$.*

Subsequently, Yao et al. [16] further extended the above results to a more general class in which the involved operators are quasi-pseudocontractive operators and they introduced the following iteration.

Algorithm 1.7.

Initialization: Let $x_0 \in H_1$ be arbitrary.

Cycle iteration: For $n \geq 1$, assume the n-th iteration x_n is constructed, then define the $(n+1)$-th iteration x_{n+1} by the following manner

$$\begin{cases} v_n = x_n + \delta A^*[(1-\zeta_n)I + \zeta_n S((1-\eta_n)I + \eta_n S) - I]Ax_n, \\ u_n = \alpha_n f(x_n) + (I - \alpha_n B)v_n, \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n T u_n), n \geq 1, \end{cases} \tag{1.4}$$

where S, T are two quasi-pseudocontractive operators, B is a strong positive linear bounded operator and f is a contractive operator and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Remark 1.8. Note that the class of quasi-pseudocontractive operators properly includes the classes of quasi-nonexpansive operators, directed operators and demicontractive operators, is more desirable for example in fixed point methods in image recovery where in many cases, it is possible to map the set of images possessing a certain property to the fixed point set of a nonlinear quasi-nonexpansive operator.

The purpose of this paper is to give a unified framework for the two-sets split common fixed point problem. We will extend the above results to the class of uniformly Lipschitzian asymptotically pseudocontractive operators. We construct an iterative algorithm based on the algorithm (1.4) and demonstrate its strong convergence.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H.

Definition 2.1. An operator $T\colon C \to C$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and for all $n \geq 1$.

Definition 2.2. An operator $T\colon C \to C$ is called asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y, x - y\rangle \leq k_n \|x - y\|^2 \tag{2.1}$$

for all $x, y \in C$ and for all $n \geq 1$.

Remark 2.3. It is easy to check that (2.1) is equal to

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|(x - T^n x) - (y - T^n y)\|^2 \tag{2.2}$$

for all $x, y \in C$ and for all $n \geq 1$.

Definition 2.4. An operator T is said to be demiclosed if, for any sequence x_n which weakly converges to $\tilde{x}$, and if the sequence $T(x_n)$ strongly converges to z, then $T(\tilde{x}) = z$.

In any Hilbert space, the following conclusions hold:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ t \in [0,1], \tag{2.3}$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{2.4}$$

and

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle \tag{2.5}$$

for all $x, y \in H$.

Lemma 2.5 ([19])**.** *Let C be a nonempty bounded and closed convex subset of a real Hilbert space H. Let $T\colon C \to C$ be a uniformly L-Lipschitzian and asymptotically pseudocontraction. Then $I - T$ is demiclosed at zero.*

Lemma 2.6 ([12])**.** *Let $\{\zeta_n\} \subset [0,\infty)$, $\{\varsigma_n\} \subset (0,1)$ and $\{\varrho_n\}$ be three sequences such that*

$$\zeta_{n+1} \le (1-\varsigma_n)\zeta_n + \varrho_n \quad \forall n \ge 1.$$

Assume the following restrictions are satisfied

(i) $\sum_{n=1}^{\infty} \varsigma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\varrho_n}{\varsigma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\varrho_n| < \infty$.

Then $\lim_{n\to\infty} \zeta_n = 0$.

Lemma 2.7 ([18])**.** *Let $\{w_n\}$ be a sequence of real numbers. Assume $\{w_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \le w_{n_k+1}$ for all $k \ge 0$. For every $n \ge N_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \le n : w_{n_i} < w_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge N_0$

$$\max\{w_{\tau(n)}, w_n\} \le w_{\tau(n)+1}.$$

3. Main results

Let H_1 and H_2 be two real Hilbert spaces. Let $S\colon H_2 \to H_2$ be a uniformly L_1-Lipschitzian asymptotically pseudocontractive operator with coefficient $k_n^{(1)}$ and $T\colon H_1 \to H_1$ be a uniformly L_2-Lipschitzian asymptotically pseudocontractive operator with coefficient $k_n^{(2)}$. Let $f\colon H_1 \to H_1$ be a ρ-contraction. Let $A\colon H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $B\colon H_1 \to H_1$ be a strong positive linear bounded operator with coefficient $\xi > 2\rho$.

Our object is to solve the two-sets split common fixed point problem (1.1). First, we present the following algorithm.

Algorithm 3.1.

Initialization: Let $x_0 \in H_1$ be arbitrary.

Cycle iteration: For $n \ge 1$, assume the n-th iteration x_n is constructed, then define the $(n+1)$-th iteration x_{n+1} via the following iterative scheme

$$\begin{cases} y_n = [(1-\zeta_n)I + \zeta_n S^n((1-\eta_n)I + \eta_n S^n)]Ax_n, \\ v_n = x_n + \delta A^*(y_n - Ax_n), \\ u_n = \alpha_n f(x_n) + (I - \alpha_n B)v_n, \\ z_n = (1-\gamma_n)u_n + \gamma_n T^n u_n, \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T^n z_n, n \ge 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\zeta_n\}$, and $\{\eta_n\}$ are five real number sequences in $(0, 1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Proposition 3.2. *Let H be a real Hilbert space. Let $T: H \to H$ be a uniformly L-Lipschitzian asymptotically pseudocontractive operator with coefficient k_n. If $0 < \zeta < \eta < \frac{1}{\sqrt{k_n^2+L^2+k_n^2}}$ for all $n \geq 1$, then*

$$\|(1-\zeta)x + \zeta T^n((1-\eta)I + \eta T^n)x - x^\dagger\|^2 \leq [1 + 2(k_n - 1)\zeta + 2(k_n - 1)(2k_n - 1)\eta\zeta]\|x - x^\dagger\|^2$$

for all $x \in H$ and $x^\dagger \in \mathrm{Fix(T)}$.

Proof. Since $x^\dagger \in \mathrm{Fix}(T)$, we have from (2.2) that

$$\begin{aligned} \|T^n((1-\eta)I + \eta T^n)x - x^\dagger\|^2 &\leq (2k_n - 1)\|(1-\eta)(x - x^\dagger) + \eta(T^n x - x^\dagger)\|^2 \\ &\quad + \|(1-\eta)x + \eta T^n x - T^n((1-\eta)x + \eta T^n x)\|^2, \end{aligned} \tag{3.2}$$

and

$$\|T^n x - x^\dagger\|^2 \leq (2k_n - 1)\|x - x^\dagger\|^2 + \|T^n x - x\|^2 \tag{3.3}$$

for all $x \in H$.

Since T is uniformly L-Lipschitzian and $x - ((1-\eta)x + \eta T^n x) = \eta(x - T^n x)$, we derive

$$\|T^n x - T^n((1-\eta)x + \eta T^n x)\| \leq \eta L\|x - T^n x\|. \tag{3.4}$$

From (2.3) and (3.3), we have

$$\begin{aligned} \|(1-\eta)(x - x^\dagger) + \eta(T^n x - x^\dagger)\|^2 &= (1-\eta)\|x - x^\dagger\|^2 + \eta\|T^n x - x^\dagger\|^2 - \eta(1-\eta)\|x - T^n x\|^2 \\ &\leq (1-\eta)\|x - x^\dagger\|^2 + \eta((2k_n - 1)\|x - x^\dagger\|^2 + \|T^n x - x\|^2) \\ &\quad - \eta(1-\eta)\|x - T^n x\|^2 \\ &= [1 + 2(k_n - 1)\eta]\|x - x^\dagger\|^2 + \eta^2\|T^n x - x\|^2. \end{aligned} \tag{3.5}$$

In view of (2.2) and (3.4), we get

$$\begin{aligned} &\|(1-\eta)x + \eta T^n x - T^n((1-\eta)x + \eta T^n x)\|^2 \\ &\quad = \|(1-\eta)(x - T^n((1-\eta)I + \eta T^n)x) + \eta(T^n x - T^n((1-\eta)I + \eta T^n)x)\|^2 \\ &\quad = (1-\eta)\|x - T^n((1-\eta)I + \eta T^n)x\|^2 + \eta\|T^n x - T^n((1-\eta)I + \eta T^n)x\|^2 \\ &\qquad - \eta(1-\eta)\|x - T^n x\|^2 \\ &\quad \leq (1-\eta)\|x - T^n((1-\eta)I + \eta T^n)x\|^2 - \eta(1 - \eta - \eta^2 L^2)\|x - T^n x\|^2. \end{aligned} \tag{3.6}$$

By (3.2), (3.5), and (3.6), we obtain

$$\begin{aligned} \|T^n((1-\eta)I + \eta T^n)x - x^\dagger\|^2 &\leq (2k_n - 1)[1 + 2(k_n - 1)\eta]\|x - x^\dagger\|^2 + (2k_n - 1)\eta^2\|x - T^n x\|^2 \\ &\quad + (1-\eta)\|x - T^n((1-\eta)x + \eta T^n x)\|^2 \\ &\quad - \eta(1 - \eta - \eta^2 L^2)\|x - T^n x\|^2 \\ &= (2k_n - 1)[1 + 2(k_n - 1)\eta]\|x - x^\dagger\|^2 \\ &\quad + (1-\eta)\|x - T^n((1-\eta)I + \eta T^n)x\|^2 \\ &\quad - \eta(1 - 2k_n\eta - \eta^2 L^2)\|x - Tx\|^2. \end{aligned} \tag{3.7}$$

Since $\eta < \frac{1}{\sqrt{k_n^2+L^2}+k_n}$, we deduce that $1 - 2k_n\eta - \eta^2 L^2 > 0$. According to (3.7), we get

$$\begin{aligned} \|T^n((1-\eta)I + \eta T^n)x - x^\dagger\|^2 &\leq (2k_n - 1)[1 + 2(k_n - 1)\eta]\|x - x^\dagger\|^2 \\ &\quad + (1-\eta)\|x - T^n((1-\eta)x + \eta T^n x)\|^2 \end{aligned} \tag{3.8}$$

for all $x \in H$ and $x^\dagger \in \mathrm{Fix}(T)$.

Combine (2.3) and (3.8) to get

$$\begin{aligned}\|(1-\zeta)x+\zeta T^n((1-\eta)x+\eta T^n x)-x^\dagger\|^2 &= \|(1-\zeta)(x-x^\dagger)+\zeta(T^n((1-\eta)x+\eta T^n x)-x^\dagger)\|^2\\ &= (1-\zeta)\|x-x^\dagger\|^2+\zeta\|T^n((1-\eta)x+\eta T^n x)-x^\dagger\|^2\\ &\quad -\zeta(1-\zeta)\|T^n((1-\eta)x+\eta T^n x)-x\|^2\\ &\le \zeta(2k_n-1)[1+2(k_n-1)\eta]\|x-x^\dagger\|^2+(1-\zeta)\|x-x^\dagger\|^2\\ &\quad +\zeta(1-\eta)\|x-T^n((1-\eta)x+\eta T^n x)\|^2\\ &\quad -\zeta(1-\zeta)\|T^n((1-\eta)x+\eta T^n x)-x\|^2\\ &= [1+2(k_n-1)\zeta+2(k_n-1)(2k_n-1)\eta\zeta]\|x-x^\dagger\|^2\\ &\quad +\zeta(\zeta-\eta)\|T((1-\eta)x+\eta Tx)-x\|^2.\end{aligned}$$

This together with $\zeta < \eta$ implies that

$$\|(1-\zeta)x+\zeta T^n((1-\eta)x+\eta T^n x)-x^\dagger\|^2 \le [1+2(k_n-1)\zeta+2(k_n-1)(2k_n-1)\eta\zeta]\|x-x^\dagger\|^2. \tag{3.9}$$

This completes the proof. □

Theorem 3.3. *Suppose the following conditions are satisfied:*

(C1) : $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) : $0 < a_1 < \beta_n < c_1 < \gamma_n < b_1 < \frac{1}{\sqrt{[k_n^{(2)}]^2+L_2^2}+k_n^{(2)}}$;

(C3) : $0 < a_2 < \zeta_n < c_2 < \eta_n < b_2 < \frac{1}{\sqrt{[k_n^{(1)}]^2+L_1^2}+k_n^{(1)}}$;

(C4) : $\sum_{n=1}^{\infty}(k_n^{(1)}-1) < +\infty$, $\sum_{n=1}^{\infty}(k_n^{(2)}-1) < +\infty$ *and* $\lim_{n\to\infty}\frac{k_n^{(1)}-1}{\alpha_n} = \lim_{n\to\infty}\frac{k_n^{(2)}-1}{\alpha_n} = 0$.

Then the sequence $\{x_n\}$ *generated by algorithm* (3.1) *converges strongly to* $x^* = P_\Gamma(f+I-B)x^*$.

Proof. First, note that $x^* = P_\Gamma(f+I-B)x^*$ is unique. Since $Ax^* \in \mathrm{Fix}(S)$, from (3.9), we get

$$\begin{aligned}\|y_n-Ax^*\|^2 &= \|[(1-\zeta_n)I+\zeta_n S^n((1-\eta_n)I+\eta_n S^n)]Ax_n-Ax^*\|^2\\ &= \|[(1-\zeta_n)I+\zeta_n S^n((1-\eta_n)I+\eta_n S^n)]Ax_n\\ &\quad -[(1-\zeta_n)I+\zeta_n S^n((1-\eta_n)I+\eta_n S^n)]Ax^*\|^2\\ &\le [1+2(k_n^{(1)}-1)\zeta_n+2(k_n^{(1)}-1)(2k_n^{(1)}-1)\eta_n\zeta_n]\|Ax_n-Ax^*\|^2.\end{aligned} \tag{3.10}$$

By the condition (C4), without loss of generality, we may assume that $\sup_n k_n^{(1)} \le 2$ and $\sup_n k_n^{(2)} \le 2$ for all $n \ge 1$.

Applying (3.8), we deduce

$$\begin{aligned}\|T^n z_n-x^*\|^2 &= \|T^n((1-\gamma_n)u_n+\gamma_n T^n u_n)-x^*\|^2\\ &\le (2k_n^{(2)}-1)[1+2(k_n^{(2)}-1)\gamma_n]\|u_n-x^*\|^2+(1-\gamma_n)\|u_n-T^n z_n\|^2.\end{aligned}$$

This together with (2.3) and (3.9) imply that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(1-\beta_n)u_n+\beta_n T^n z_n-x^*\|^2\\ &= (1-\beta_n)\|u_n-x^*\|^2+\beta_n\|T^n z_n-x^*\|^2-\beta_n(1-\beta_n)\|u_n-T^n z_n\|^2\\ &\le \left\{1+2(k_n^{(2)}-1)\beta_n[1+(2k_n^{(2)}-1)\gamma_n]\right\}\|u_n-x^*\|^2-\beta_n(\gamma_n-\beta_n)\|u_n-T^n z_n\|^2\\ &\le \left\{1+2(k_n^{(2)}-1)\beta_n[1+(2k_n^{(2)}-1)\gamma_n]\right\}\|u_n-x^*\|^2\\ &\le [1+8(k_n^{(2)}-1)]\|u_n-x^*\|^2.\end{aligned} \tag{3.11}$$

From (3.1), we have

$$\begin{aligned}\|u_n - x^*\| &= \|\alpha_n(f(x_n) - Bx^*) + (I - \alpha_n B)(v_n - x^*)\| \\ &\le \alpha_n\|f(x_n) - Bx^*\| + \|I - \alpha_n B\|\|v_n - x^*\| \\ &\le \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - Bx^*\| + (1 - \alpha_n\xi)\|v_n - x^*\| \\ &\le \alpha_n\rho\|x_n - x^*\| + \alpha_n\|f(x^*) - Bx^*\| + (1 - \alpha_n\xi)\|v_n - x^*\|.\end{aligned} \tag{3.12}$$

Utilizing equality (2.4), we get

$$\begin{aligned}\|v_n - x^*\|^2 &= \|x_n - x^* + \delta A^*(y_n - Ax_n)\|^2 \\ &= \|x_n - x^*\|^2 + \delta^2\|A^*(y_n - Ax_n)\|^2 + 2\delta\langle x_n - x^*, A^*(y_n - Ax_n)\rangle.\end{aligned} \tag{3.13}$$

Using the fact that A is a linear operator with its adjoint A^*, we have

$$\begin{aligned}\langle x_n - x^*, A^*(y_n - Ax_n)\rangle &= \langle A(x_n - x^*), y_n - Ax_n\rangle \\ &= \langle Ax_n - y_n, y_n - Ax_n\rangle + \langle y_n - Ax^*, y_n - Ax_n\rangle \\ &= \langle y_n - Ax^*, y_n - Ax_n\rangle - \|y_n - Ax_n\|^2.\end{aligned} \tag{3.14}$$

Apply (2.4) to obtain

$$\langle y_n - Ax^*, y_n - Ax_n\rangle = \frac{1}{2}(\|y_n - Ax^*\|^2 + \|y_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2). \tag{3.15}$$

From (3.10), (3.14), and (3.15), we get

$$\begin{aligned}\langle x_n - x^*, A^*(y_n - Ax_n)\rangle &= \frac{1}{2}(\|y_n - Ax^*\|^2 + \|y_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2) - \|y_n - Ax_n\|^2 \\ &\le \frac{1}{2}\Big\{[1 + 2(k_n^{(1)} - 1)\zeta_n + 2(k_n^{(1)} - 1)(2k_n^{(1)} - 1)\eta_n\zeta_n]\|Ax_n - Ax^*\|^2 \\ &\quad + \|y_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2\Big\} - \|y_n - Ax_n\|^2 \\ &= -\frac{1}{2}\|y_n - Ax_n\|^2 + (k_n^{(1)} - 1)\zeta_n[1 + (2k_n^{(1)} - 1)\eta_n]\|Ax_n - Ax^*\|^2.\end{aligned} \tag{3.16}$$

By (3.13) and (3.16), we derive

$$\begin{aligned}\|v_n - x^*\|^2 &= \|x_n - x^* + \delta A^*(y_n - Ax_n)\|^2 \\ &\le \delta^2\|A\|^2\|y_n - Ax_n\|^2 + \|x_n - x^*\|^2 - \delta\|y_n - Ax_n\|^2 \\ &\quad + 2\delta[(k_n^{(1)} - 1)\zeta_n + (k_n^{(1)} - 1)(2k_n^{(1)} - 1)\eta_n\zeta_n]\|Ax_n - Ax^*\|^2 \\ &\le \|x_n - x^*\|^2 + (\delta^2\|A\|^2 - \delta)\|y_n - Ax_n\|^2 \\ &\quad + 2\delta\|A\|^2(k_n^{(1)} - 1)\zeta_n[1 + (2k_n^{(1)} - 1)\eta_n]\|x_n - x^*\|^2 \\ &\le [1 + 8(k_n^{(1)} - 1)]\|x_n - x^*\|^2.\end{aligned} \tag{3.17}$$

It follows that

$$\|v_n - x^*\| = \|x_n - x^* + \delta A^*(y_n - Ax_n)\| \le [1 + 4(k_n^{(1)} - 1)]\|x_n - x^*\|. \tag{3.18}$$

Substituting (3.18) into (3.12) to deduce

$$\begin{aligned}\|u_n - x^*\| &\le \alpha_n\rho\|x_n - x^*\| + \alpha_n\|f(x^*) - Bx^*\| + (1 - \alpha_n\xi)\|v_n - x^*\| \\ &\le \alpha_n\|f(x^*) - Bx^*\| + [1 - (\xi - \rho)\alpha_n]\|x_n - x^*\| + 4(k_n^{(1)} - 1)\|x_n - x^*\| \\ &\le [1 + 4(k_n^{(1)} - 1)]\max\left\{\|x_n - x^*\|, \frac{\|f(x^*) - Bx^*\|}{\xi - \rho}\right\}.\end{aligned} \tag{3.19}$$

From (3.11) and (3.19), we get

$$\begin{aligned}\|x_{n+1}-x^*\| &\le [1+4(k_n^{(2)}-1)]\|u_n-x^*\|\\ &\le [1+4(k_n^{(1)}-1)][1+4(k_n^{(2)}-1)]\max\left\{\|x_n-x^*\|,\frac{\|f(x^*)-Bx^*\|}{\xi-\rho}\right\}\\ &\le \prod_{i=1}^{n}[1+4(k_i^{(1)}-1)]\prod_{i=1}^{n}[1+4(k_i^{(2)}-1)]\max\left\{\|x_0-x^*\|,\frac{\|f(x^*)-Bx^*\|}{\xi-\rho}\right\}.\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded by the conditions

$$\sum_{n=1}^{\infty}(k_n^{(1)}-1)<\infty, \qquad \sum_{n=1}^{\infty}(k_n^{(2)}-1)<\infty.$$

Next, we consider two possible cases:

Case 1: there exists n_0 such that the sequence $\{\|x_n-x^*\|\}_{n\ge n_0}$ is decreasing.

Case 2: for any n_0, there exists integer $m\ge n_0$ such that $\|x_m-x^*\|\le\|x_{m+1}-x^*\|$.

More precisely, regarding the situation when $\{\|x_n-x^*\|\}$ is monotonous at infinity (Case 1) and bounded (hence convergent), we prove that its only possible limit is zero.

In Case 1, we assume there exists an integer $n_0>0$ such that $\{\|x_n-x^*\|\}$ is decreasing for all $n\ge n_0$. In this case, we know that $\lim_{n\to\infty}\|x_n-x^*\|$ exists. From (3.11), (3.12), and (3.17), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|u_n-x^*\|^2+8(k_n^{(2)}-1)\|u_n-x^*\|^2\\ &\le [\alpha_n\rho\|x_n-x^*\|+\alpha_n\|f(x^*)-Bx^*\|+(1-\alpha_n\xi)\|v_n-x^*\|]^2+8(k_n^{(2)}-1)\|u_n-x^*\|^2\\ &= \alpha_n^2(\rho\|x_n-x^*\|+\|f(x^*)-Bx^*\|)^2+2\alpha_n(1-\alpha_n\xi)(\rho\|x_n-x^*\|\\ &\quad +\|f(x^*)-Bx^*\|)\times\|v_n-x^*\|+(1-\alpha_n\xi)^2\|v_n-x^*\|^2+8(k_n^{(2)}-1)\|u_n-x^*\|^2\\ &\le M(\alpha_n+k_n^{(2)}-1)+(1-\alpha_n\xi)\|v_n-x^*\|^2\\ &\le M(\alpha_n+k_n^{(2)}-1)+8(1-\alpha_n\xi)(k_n^{(1)}-1)\|x_n-x^*\|^2+(1-\alpha_n\xi)\|x_n-x^*\|^2\\ &\quad +(1-\alpha_n\xi)(\delta^2\|A\|^2-\delta)\|y_n-Ax_n\|^2\\ &\le (1-\alpha_n\xi)(\delta^2\|A\|^2-\delta)\|y_n-Ax_n\|^2+\|x_n-x^*\|^2+M(\alpha_n+k_n^{(1)}-1+k_n^{(2)}-1)\\ &\le M(\alpha_n+k_n^{(1)}-1+k_n^{(2)}-1)+\|x_n-x^*\|^2,\end{aligned} \tag{3.20}$$

where $M>0$ is a constant such that

$$\sup_n\left\{(\rho\|x_n-x^*\|+\|f(x^*)-Bx^*\|)(3\|x_n-x^*\|+\|f(x^*)-Bx^*\|)+10\|x_n-x^*\|^2+16\|u_n-x^*\|^2\right\}\le M.$$

Hence,

$$(1-\alpha_n\xi)(\delta-\delta^2\|A\|^2)\|y_n-Ax_n\|^2\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+M(\alpha_n+k_n^{(1)}-1+k_n^{(2)}-1).$$

Since $\lim_{n\to\infty}\|x_n-x^*\|$ exists, $\alpha_n\to 0$, $k^{(1)}\to 1$, and $k^{(2)}\to 1$, we deduce

$$\lim_{n\to\infty}\|y_n-Ax_n\|=0. \tag{3.21}$$

Therefore,

$$\lim_{n\to\infty}\|Ax_n-S^n((1-\eta_n)I+\eta_nS^n)Ax_n\|=0.$$

Observe that

$$\begin{aligned}\|Ax_n-S^nAx_n\| &\le \|Ax_n-S^n((1-\eta_n)I+\eta_nS^n)Ax_n\|+\|S^n((1-\eta_n)I+\eta_nS^n)Ax_n-S^nAx_n\|\\ &\le \|Ax_n-S^n((1-\eta_n)I+\eta_nS^n)Ax_n\|+L_1\eta_n\|Ax_n-S^nAx_n\|.\end{aligned}$$

It follows that

$$\|Ax_n - S^n Ax_n\| \leq \frac{1}{1 - L_1\eta_n}\|Ax_n - S^n((1-\eta_n)I + \eta_n S^n)Ax_n\|.$$

Thus,

$$\lim_{n\to\infty} \|Ax_n - S^n Ax_n\| = 0. \tag{3.22}$$

Note that

$$\begin{aligned}\|u_n - x_n\| &= \|\delta A^*(y_n - Ax_n) + \alpha_n(Bx_n + \delta BA^*(y_n - Ax_n) - f(x_n))\| \\ &\leq \delta\|A\|\|y_n - Ax_n\| + \alpha_n\|Bx_n + \delta BA^*(y_n - Ax_n) - f(x_n)\|.\end{aligned}$$

This together with (3.21) implies that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.23}$$

Combining (3.11) with (3.20), we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \left\{1 + 2(k_n^{(2)} - 1)\beta_n[1 + (2k_n^{(2)} - 1)\gamma_n]\right\}\|u_n - x^*\|^2 - \beta_n(\gamma_n - \beta_n)\|u_n - T^n z_n\|^2 \\ &\leq \left\{1 + 2(k_n^{(2)} - 1)\beta_n[1 + (2k_n^{(2)} - 1)\gamma_n]\right\}[\|x_n - x^*\|^2 \\ &\quad + M(\alpha_n + k_n^{(1)} - 1)] - \beta_n(\gamma_n - \beta_n)\|u_n - T^n z_n\|^2 \\ &\leq \left\{1 + 2(k_n^{(2)} - 1) + 2\gamma_n(2k_n^{(2)} - 1)(k_n^{(2)} - 1)\right\}\|x_n - x^*\|^2 \\ &\quad + \left\{1 + 2(k_n^{(2)} - 1)\beta_n[1 + (2k_n^{(2)} - 1)\gamma_n]\right\}M(\alpha_n + k_n^{(1)} - 1) - \beta_n(\gamma_n - \beta_n)\|u_n - T^n z_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\beta_n(\gamma_n - \beta_n)\|u_n - T^n z_n\|^2 &\leq \left\{1 + 2(k_n^{(2)} - 1) + 2\gamma_n(2k_n^{(2)} - 1)(k_n^{(2)} - 1)\right\}\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + \left\{1 + 2(k_n^{(2)} - 1)\beta_n[1 + (2k_n^{(2)} - 1)\gamma_n]\right\}M(\alpha_n + k_n^{(1)} - 1).\end{aligned}$$

So,

$$\lim_{n\to\infty} \|u_n - T^n z_n\| = 0. \tag{3.24}$$

Since $x_{n+1} - u_n = \beta_n(T^n z_n - u_n)$, we get

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

It follows from (3.23) that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0, \tag{3.25}$$

and

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.26}$$

Observe that

$$\|u_n - T^n u_n\| \leq \|u_n - T^n z_n\| + \|T^n z_n - T^n u_n\| \leq \|u_n - T^n z_n\| + L_2\gamma_n\|u_n - T^n u_n\|.$$

Thus,

$$\|u_n - T^n u_n\| \leq \frac{1}{1 - L_2\gamma_n}\|u_n - T^n z_n\|.$$

This together with (3.24) implies that

$$\lim_{n\to\infty} \|u_n - T^n u_n\| = 0. \tag{3.27}$$

Since T is uniformly L_2-Lipschitzian, we can derive

$$\begin{aligned}\|u_{n+1} - Tu_{n+1}\| &\le \|u_{n+1} - T^{n+1}u_{n+1}\| + \|T^{n+1}u_{n+1} - T^{n+1}u_n\| + \|T^{n+1}u_n - Tu_{n+1}\| \\ &\le \|u_{n+1} - T^{n+1}u_{n+1}\| + L_2\|u_{n+1} - u_n\| + L_2\|T^n u_n - u_{n+1}\| \\ &\le \|u_{n+1} - T^{n+1}u_{n+1}\| + 2L_2\|u_{n+1} - u_n\| + L_2\|T^n u_n - u_n\|.\end{aligned} \tag{3.28}$$

By (3.25), (3.27), and (3.28), we have immediately that

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0.$$

From (3.1) and (3.21), we have

$$\lim_{n\to\infty} \|v_n - x_n\| = 0.$$

Since S is uniformly L_1-Lipschitzian, we can derive

$$\begin{aligned}\|Ax_{n+1} - SAx_{n+1}\| &\le \|Ax_{n+1} - S^{n+1}Ax_{n+1}\| + \|S^{n+1}Ax_{n+1} - S^{n+1}Ax_n\| \\ &\quad + \|S^{n+1}Ax_n - SAx_{n+1}\| \\ &\le \|Ax_{n+1} - S^{n+1}Ax_{n+1}\| + L_1\|Ax_{n+1} - Ax_n\| + L_1\|S^n Ax_n - Ax_{n+1}\| \\ &\le \|Ax_{n+1} - S^{n+1}Ax_{n+1}\| + 2L_1\|Ax_{n+1} - Ax_n\| + L_1\|S^n Ax_n - Ax_n\| \\ &\le \|Ax_{n+1} - S^{n+1}Ax_{n+1}\| + 2L_1\|A\|\|x_{n+1} - x_n\| + L_1\|S^n Ax_n - Ax_n\|.\end{aligned} \tag{3.29}$$

By (3.22), (3.26), and (3.29), we get

$$\lim_{n\to\infty} \|Ax_{n+1} - SAx_{n+1}\| = 0.$$

Next, we show that $\limsup_{n\to\infty}\langle f(x^*) - Bx^*, u_n - x^*\rangle \le 0$. Choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - Bx^*, u_n - x^*\rangle = \lim_{i\to\infty}\langle f(x^*) - Bx^*, u_{n_i} - x^*\rangle.$$

Since the sequence $\{u_{n_i}\}$ is bounded, we can choose a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $u_{n_{i_j}} \rightharpoonup z$. For the sake of convenience, we assume (without loss of generality) that $u_{n_i} \rightharpoonup z$. And, hence $Au_{n_i} \rightharpoonup Az$. Then, apply Lemma 2.5 to deduce $Az \in \mathrm{Fix}(S)$ and $z \in \mathrm{Fix}(T)$. That is to say, $z \in \Gamma$.

Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - Bx^*, u_n - x^*\rangle &= \lim_{i\to\infty}\langle f(x^*) - Bx^*, u_{n_i} - x^*\rangle \\ &= \lim_{i\to\infty}\langle f(x^*) - Bx^*, z - x^*\rangle \\ &\le 0.\end{aligned} \tag{3.30}$$

Applying inequality (2.5), we have

$$\begin{aligned}\|u_n - x^*\|^2 &= \|(I - \alpha_n B)(v_n - x^*) + \alpha_n(f(x_n) - Bx^*)\|^2 \\ &\le (1 - \alpha_n\xi)\|v_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - Bx^*, u_n - x^*\rangle \\ &\le (1 - \alpha_n\xi)[1 + 8(k_n^{(1)} - 1)]\|x_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - Bx^*, u_n - x^*\rangle \\ &= (1 - \alpha_n\xi)[1 + 8(k_n^{(1)} - 1)]\|x_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - f(x^*), u_n - x^*\rangle \\ &\quad + 2\alpha_n\langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &\le (1 - \alpha_n\xi)[1 + 8(k_n^{(1)} - 1)]\|x_n - x^*\|^2 + 2\alpha_n\rho\|x_n - x^*\|\|u_n - x^*\| \\ &\quad + 2\alpha_n\langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &\le (1 - \alpha_n\xi)[1 + 8(k_n^{(1)} - 1)]\|x_n - x^*\|^2 + \alpha_n\rho\|x_n - x^*\|^2 + \alpha_n\rho\|u_n - x^*\|^2 \\ &\quad + 2\alpha_n\langle f(x^*) - Bx^*, u_n - x^*\rangle.\end{aligned}$$

It follows that

$$\|u_n - x^*\|^2 \le [1 - \frac{(\xi - 2\rho)\alpha_n}{1 - \alpha_n \rho}]\|x_n - x^*\|^2 + \frac{8(1 - \alpha_n \xi)(k_n^{(1)} - 1)}{1 - \alpha_n \rho}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n \rho}\langle f(x^*) - Bx^*, u_n - x^*\rangle.$$

Therefore,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \|u_n - x^*\|^2 + 8(k_n^{(2)} - 1)\|u_n - x^*\|^2 \\ &\le [1 - \frac{(\xi - 2\rho)\alpha_n}{1 - \alpha_n \rho}]\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n \rho}\langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &\quad + \frac{8(1 - \alpha_n \xi)(k_n^{(1)} - 1)}{1 - \alpha_n \rho}\|x_n - x^*\|^2 + 8(k_n^{(2)} - 1)\|u_n - x^*\|^2 \\ &\le [1 - \frac{(\xi - 2\rho)\alpha_n}{1 - \alpha_n \rho}]\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n \rho}\langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &\quad + M(k_n^{(1)} - 1 + k_n^{(2)} - 1). \end{aligned} \tag{3.31}$$

Applying Lemma 2.6 and (3.30) to (3.31), we deduce $x_n \to x^*$.

In Case 2 above, we know that, for any integer n_0, there exists another integer $p \ge n_0$ such that $\|x_p - x^*\| \le \|x_{p+1} - x^*\|$. Let n_0 be such that $\|x_{n_0} - x^*\| \le \|x_{n_0+1} - x^*\|$. Set $\omega_n = \{\|x_n - x^*\|\}$. Then, we have

$$\omega_{n_0} \le \omega_{n_0+1}.$$

Define an integer sequence $\{\tau_n\}$ for all $n \ge n_0$ as follows:

$$\tau(n) = \max\{l \in \mathbb{N} | n_0 \le l \le n, \omega_l \le \omega_{l+1}\}.$$

It is clear that $\tau(n)$ is a non-decreasing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = \infty$$

and

$$\omega_{\tau(n)} \le \omega_{\tau(n)+1}$$

for all $n \ge n_0$.

By the similar argument as that of Case 1, we can obtain

$$\lim_{n\to\infty} \|SAx_{\tau(n)} - Ax_{\tau(n)}\| = 0$$

and

$$\lim_{n\to\infty} \|u_{\tau(n)} - Tu_{\tau(n)}\| = 0.$$

This implies that

$$\omega_w(u_{\tau(n)}) \subset \Gamma.$$

Thus, we obtain

$$\limsup_{n\to\infty} \langle f(x^*) - Bx^*, u_{\tau(n)} - x^*\rangle \le 0. \tag{3.32}$$

Since $\omega_{\tau(n)} \le \omega_{\tau(n)+1}$, we have from (3.31) that

$$\begin{aligned} \omega_{\tau(n)}^2 &\le \omega_{\tau(n)+1}^2 \\ &\le [1 - \frac{(\xi - 2\rho)\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\rho}]\omega_{\tau(n)}^2 + \frac{2\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\rho}\langle f(x^*) - Bx^*, u_{\tau(n)} - x^*\rangle + M(k_{\tau(n)}^{(1)} - 1 + k_{\tau(n)}^{(2)} - 1). \end{aligned} \tag{3.33}$$

It follows that

$$\omega_{\tau(n)}^2 \leq \frac{2}{\xi - 2\rho}\langle f(x^*) - Bx^*, u_{\tau(n)} - x^*\rangle + \frac{M}{\xi - 2\rho}\Big(\frac{k_{\tau(n)}^{(1)} - 1}{\alpha_{\tau(n)}} + \frac{k_{\tau(n)}^{(2)} - 1}{\alpha_{\tau(n)}}\Big). \tag{3.34}$$

Combining (3.32) with (3.34), we have

$$\limsup_{n\to\infty} \omega_{\tau(n)} \leq 0,$$

and hence

$$\lim_{n\to\infty} \omega_{\tau(n)} = 0. \tag{3.35}$$

From (3.33), we deduce

$$\limsup_{n\to\infty} \omega_{\tau(n)+1}^2 \leq \limsup_{n\to\infty} \omega_{\tau(n)}^2.$$

This together with (3.35) implies that

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0.$$

Apply Lemma 2.7 to get

$$0 \leq \omega_n \leq \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}.$$

Therefore, $\omega_n \to 0$. That is, $x_n \to x^*$. This completes the proof. □

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