Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 6142–6152
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Semi-prequasi-invex type multiobjective optimization and generalized fractional programming problems

Zai-Yun Peng[a], Ke-Ke Li[b], Jian-Ting Zhou[c,*]

[a] *College of Mathematics and Statistics, Chongqing JiaoTong University, Chongqing 400074, P. R. China.*
[b] *School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, P. R. China.*
[c] *College of Civil Engineering, Chongqing JiaoTong University, Chongqing 400074, P. R. China.*

Communicated by S. S. Chang

Abstract

In this paper, we mainly discuss some applications of semi-prequasi-invex type functions for multiobjective optimization and generalized nonlinear programming problems. Some optimality results for semi-prequasi-invex type multiobjective optimization problem are given, then some optimality necessary conditions under directional derivative and saddle point theories in semi-prequasi-invex type nonlinear programming problem are derived. Moreover, some duality theorems for the generalized nonlinear fractional programming problem with semi-prequasi-invexity are also obtained. Our results improve the corresponding ones in the literature.

Keywords: Semi-prequasi-invex functions, multiobjective optimization problem, nonlinear programming problem, generalized nonlinear fractional programming.
2010 MSC: 26B25, 90C26, 90C30.

1. Introduction

Convexity and generalized convexity play a crucial role in optimization theory. Therefore, researching on its applications is important in optimization theory. In recent decades, there have been many literatures studying on this subject (e.g., see [1–7, 9–14, 16]). Martin [6], Ben-Israel and Mond [2] established the characterizations for the classical invexity. In 1988, Weir and Mond[7] gave the definition of preinvex

*Corresponding author
Email addresses: pengzaiyun@126.com (Zai-Yun Peng), likeke135@163.com (Ke-Ke Li), jt-zhou@163.com (Jian-Ting Zhou)

Received 2016-09-17

functions, and discussed some applications in multiple objective optimization. Yang and Li presented some properties of preinvex functions and semistrictly preinvex functions in [12] and [13], respectively. In 2001, Yang et al. [14] introduced a class of prequasi-invexity, and some applications of prequasi-invex type functions in multiobjective optimization problem have been obtained. Luo et al. [4, 5] improved some of the results in [14] under weaker assumptions. In 2007, Antczak in [1] introduced an important generalized convex function named G-preinvex functions. Luo and Wu [3] discussed the relationships between G-preinvex functions and semistrictly G-preinvex functions. Yang and Chen proposed a class of semi-preinvexity in [11], and discussed applications of semi-preinvex functions in the pre-variational inequalities. A significant generalization of convex functions, so-called semi-prequasi-invex functions, was introduced by Yang in [9]. Recently, Zhao et al. [16] developed the criterion for semi-prequasi-invex functions. Xu [8] established four theorems of duality under suitable assumptions in fractional programming. Zhao [15] discussed a type of generalized convexity and other related ones and their applications in optimization theory.

Motivated by the results in [8, 13, 16] and mentioned above, in this paper, we mainly study some optimality and saddle point theories for multiobjective optimization and generalized nonlinear programming problems under semi-prequasi-invexity. We establish some optimality conditions and saddle point theorems for nonlinear programming problem (P_2) and multiobjective optimization problem (MP), respectively. Moreover, by employing the alternative theorem, we derive some duality results for generalized nonlinear fractional programming problem (FP) with semi-prequasi-invex type functions. Our results improve the corresponding ones in [8, 11, 15, 16].

2. Preliminaries

In this section, we first recall some concepts about semi-prequasi-invex functions.

Definition 2.1 ([9, 11])**.** A set $K \subseteq R^n$ is said to be semi-connected if there exists a vector function $\eta : K \times K \times [0, 1] \to K$, such that

$$x,\, y \in K,\, \lambda \in [0,\, 1] \Rightarrow y + \lambda\eta(x,\, y,\, \lambda) \in K.$$

Remark 2.2. If $K_i \subseteq R^n$ $(i \in I)$ is a family of semi-connected sets with respect to the same vector function $\eta : K \times K \times [0, 1] \to K$, then, their intersection $\bigcap_{i\in I} K_i$ is also a semi-connected set.

The following class of semi-prequasi-invex functions were introduced by Yang [9].

Definition 2.3 ([9])**.** Let $K \subseteq R^n$ be a semi-connected set with respect to $\eta : K \times K \times [0, 1] \to K$. We say that $f : K \to R^n$ is semi-prequasi-invex if, for all $x, y \in K, \lambda \in [0, 1]$,

$$f(y + \lambda\eta(x,\, y,\, \lambda)) \le \max\{f(x),\, f(y)\}.$$

Definition 2.4 ([9, 16])**.** Let $K \subseteq R^n$ be a semi-connected set with respect to $\eta : K \times K \times [0, 1] \to$ K. Let $f : K \to R^n$. We say that f is semistrictly semi-prequasi-invex if, for all $x, y \in K,\ f(x) \ne f(y), \lambda \in (0, 1)$,

$$f(y + \lambda\eta(x,\, y,\, \lambda)) < \max\{f(x),\, f(y)\}.$$

Definition 2.5 ([9, 16])**.** Let $K \subseteq R^n$ be a semi-connected set with respect to $\eta : K \times K \times [0, 1] \to$ K. Let $f : K \to R^n$. We say that f is strictly semi-prequasi-invex if for all $x, y \in K, x \ne y, \lambda \in (0, 1)$,

$$f(y + \lambda\eta(x,\, y,\, \lambda)) < \max\{f(x),\, f(y)\}.$$

Example 2.6. This example illustrates the existence of semi-prequasi-invex function with respect to η : $K \times K \times [0, 1] \to K$ on the semi-connected set K. Let $K = R$, and

$$f(x) = \begin{cases} 1, & x > 0; \\ 0, & x \le 0, \end{cases}$$

$$\eta(x,\, y,\, \lambda) = \begin{cases} \lambda^2 x - \lambda y + \lambda^3, & x > 0, y > 0; \\ \lambda x - \lambda y + \frac{\lambda^2}{2}, & x \le 0, y \le 0; \\ -\lambda x^2 - \lambda y + 5\lambda, & x > 0, y \le 0; \\ \lambda x^3 - \lambda y - \lambda^3, & x \le 0, y > 0. \end{cases}$$

Obviously, K is a semi-connected set with respect to η, and $f(x)$ is a semi-prequasi-invex function.

3. Optimality conditions and saddle points for optimization problems

In this section, we first consider the following multiobjective optimization problem:

$$(MP) : \min f(x) = (f_1(x), \cdots, f_m(x))^T, \quad \text{s.t.} \quad x \in K,$$

where $f : K \to R^m$ is a vector-valued function and $K \subseteq R^n$ is a semi-connected set with respect to $\eta : K \times K \times [0,\, 1] \to K$, $K \subseteq R^n$.

Throughout this section, let

$$R^m_+ = \{x \in R^m \mid x = (x_1, \cdots, x_m),\, x_i \ge 0, 1 \le i \le m\},$$
$$R^m_{++} = \{x \in R^m \mid x = (x_1, \cdots, x_m),\, x_i > 0, 1 \le i \le m\}.$$

Firstly, we recall the definitions of efficient solutions and weakly efficient solutions.

Definition 3.1 ([14]). A point $\bar{x} \in K$ is called a global efficient solution of (MP), if there does not exist any point $y \in K$, such that

$$f(y) \in f(\overline{x}) - R^m_+ \backslash \{0\}.$$

A point $\bar{x} \in K$ is called a local efficient solution of (MP), if there is a neighborhood $N(\overline{x})$ of $\bar{x}$, such that there does not exist any point $y \in K \cap N(\overline{x})$, such that

$$f(y) \in f(\overline{x}) - R^m_+ \backslash \{0\}.$$

Definition 3.2 ([14]). A point $\bar{x} \in K$ is called a global weakly efficient solution of (MP), if there does not exist any point $y \in K$, such that

$$f(y) \in f(\overline{x}) - R^m_{++}.$$

A point $\bar{x} \in K$ is called a local weakly efficient solution of (MP), if there is a neighborhood $N(\overline{x})$ of $\bar{x}$, such that there does not exist any point $y \in K \cap N(\overline{x})$, s.t.

$$f(y) \in f(\overline{x}) - R^m_{++}.$$

Similar to the proof of Lemma 1 in [11] (using the same method with some suitable modifications), we can obtain Lemma 3.3 as follows.

Lemma 3.3. *Let K be a semi-connected set of R^n, and $f_i(x)$, $i = 1, \cdots, m$, be semi-prequasi-invex functions. Then exactly one of the following two systems is solvable:*

(i) *there exists $\bar{x} \in K$, s.t. $f_1(\bar{x}) < 0, \cdots, f_m(\bar{x}) < 0$;*

(ii) *there exists $\lambda \in R^m_+ \backslash \{0\}$, s.t. $\sum_{i=1}^{m} \lambda_i f_i(x) \ge 0 \quad \forall x \in K$.*

Theorem 3.4. *Let $K \subseteq R^n$ be a semi-connected set with respect to $\eta : K \times K \times [0,\, 1] \to K$, and $f_i(x)$, $i = 1, \cdots, m$, be semi-prequasi-invex functions with respect to the same η. If $x^* \in K$ is a global weakly efficient (efficient) solution of (MP), then there exists $\lambda \in R^m_+ \backslash \{0\}$, such that x^* is an optimal solution of the following scalar optimization problem:*

$$(P_\lambda) : \quad \min \ \lambda^T f(x), \quad s.t. \ x \in K, \lambda \in R^m_+ \backslash \{0\}.$$

Proof. Since $x^* \in K$ is a global weakly efficient solution of (MP), then, the systems that there exists $x \in K$, such that $f_i(x) - f_i(x^*) < 0$ $(i = 1, \cdots, m)$, have no solution. From Lemma 3.3, there exists $\lambda \in R^m_+ \backslash \{0\}$, $\lambda_i \geq 0$ $(i = 1, \cdots, m)$, s.t.

$$\sum_{i=1}^{m} \lambda_i (f_i(x) - f_i(x^*)) \geq 0, \quad \forall x \in K,$$

which implies that

$$\sum_{i=1}^{m} \lambda_i f_i(x) \geq \sum_{i=1}^{m} \lambda_i f_i(x^*), \quad \forall x \in K,$$

or

$$\lambda^T f(x) \geq \lambda^T f(x^*), \quad \forall x \in K,$$

where $\lambda = (\lambda_1, \cdots, \lambda_m) \geq 0$, with $\lambda_k > 0$, $k \in \{1, \cdots, m\}$.

Note that $\lambda \in R^m_+ \backslash \{0\}$, then x^* is an optimal solution of $\min\{\lambda^T f(x)\}$, s.t. $x \in K$, $\lambda \in R^m_+ \backslash \{0\}$. This completes the proof. □

Next, we recall some definitions of directional derivative (for more details, see [8]).

Definition 3.5. Let $K \subseteq R^n$ be a semi-connected set with respect to $\eta : K \times K \times [0, 1] \to K$, $f(x) : K \to R^n$. If the following limit exists for $x, y \in K$, denoted by $f^+(P_{x,y}(0))$,

$$f^+(P_{x,y}(0)) = \lim_{\theta \downarrow 0} \frac{f(y + \theta\eta(x, y, \theta))}{\theta},$$

then, $f^+(P_{x,y}(0))$ is called the right directional derivative of $f(x)$ at y along the path $y + \theta\eta(x, y, \theta)$.

Definition 3.6. Let $K \subseteq R^n$ be a semi-connected set with respect to $\eta : K \times K \times [0, 1] \to K$, $f(x) : K \to R^n$. For $x, y \in K$, if there exists $\{\theta_i\} \subseteq [0, 1]$, $\lim_{i\to\infty} \theta_i = 0$, such that the following limit exists, denoted by $\xi(f, x, y)$,

$$\xi(f, x, y) = \lim_{\theta_i \downarrow 0} \frac{f(y + \theta_i\eta(x, y, \theta_i))}{\theta_i},$$

then, $\xi(f, x, y)$ is called a right directional limit of $f(x)$ at y along the path $y + \theta\eta(x, y, \theta)$. $M(f, x, y)$ denote all right directional limits of $f(x)$ at y along the path $y + \theta\eta(x, y, \theta)$, that is,

$$M(f, x, y) = \{\xi(f, x, y) \mid \exists \{\theta_i\} \subseteq [0, 1], \lim_{i\to\infty} \theta_i = 0, \text{ s.t. } \xi(f, x, y) = \lim_{\theta_i \downarrow 0} \frac{f(y + \theta_i\eta(x, y, \theta_i))}{\theta_i}\}.$$

Now, we consider the following nonlinear programming problem with inequality constraints.

$$(P_2): \quad \min f(x),$$
$$g_i(x) \leq 0, \ i \in J = \{1, \cdots, m\}, \quad x \in K,$$

where K is a subset of R^n, f, g_i $(i \in J)$ are real-valued functions on K, and $D = \{x \in K \mid g_i(x) \leq 0, \ i \in J\}$ denotes the feasible set of (P_2).

Theorem 3.7. *Let $K \subseteq R^n$ be a semi-connected set with respect to $\eta : K \times K \times [0, 1] \to K$, assume $f(x) : K \to R$, $g_i(x) : K \to R$, $i = 1, \cdots, m$, are semi-prequasi-invex functions on K with respect to the same vector valued function $\eta(x, y, \theta)$. If $\bar{x}$ is an optimal solution of (P_2), and the right directional derivatives of $f(x)$, $g_i(x)$, $i = 1, \cdots, m$, at $\bar{x}$ along the path $\overline{x} + \theta\eta(x, \overline{x}, \theta)$ exist for all $x \in K$. Then, there exists vector $(\lambda, \mu) \in (R_+ \times R^m_+) \backslash \{0\}$, such that*

$$\lambda f^+(P_{x,\overline{x}}(0)) + \sum_{i=1}^{m} \mu_i g^+(P_{x,\overline{x}}(0)) \geq 0, \qquad \sum_{i=1}^{m} \mu_i g_i(\overline{x}) = 0.$$

Proof. By the condition that $\bar{x}$ is an optimal solution of (P_2), it follows that the following systems have no solution on K.

$$f(x) - f(\overline{x}) < 0,$$
$$g_i(x) < 0,\ i = 1, \cdots, m.$$

By $f_i(x) : K \to R,\ g_i(x) : K \to R,\ i = 1, \cdots, m$ are semi-prequasi-invex functions, and Lemma 3.3, there exists vector $(\lambda, \mu) \in (R_+ \times R_+^m)\backslash\{0\}$, such that

$$\lambda(f(x) - f(\overline{x})) + \sum_{i=1}^{m} \mu_i g_i(x) \geq 0, \quad \forall x \in K. \tag{3.1}$$

Taking $x = \overline{x}$ into (3.1), then we have $\sum_{i=1}^{m} \mu_i g_i(\overline{x}) \geq 0$. Meanwhile, we derive from $\mu \geq 0,\ g_i(\overline{x}) \leq 0,\ i = 1, \cdots, m$ that $\sum_{i=1}^{m} \mu_i g_i(\overline{x}) \leq 0$. Thus,

$$\sum_{i=1}^{m} \mu_i g_i(\bar{x}) = 0. \tag{3.2}$$

From K is a semi-connected set with respect to $\eta(x, y, \theta)$, we derive that for all $x \in K$,

$$\overline{x} + \theta\eta(x,\ \overline{x},\ \theta) \in K, \quad \forall \theta \in [0,\ 1].$$

This fact together with (3.1) yields

$$\lambda(f(\overline{x} + \theta\eta(x,\ \overline{x},\ \theta)) - f(\overline{x})) + \sum_{i=1}^{m} \mu_i g_i(\overline{x} + \theta\eta(x,\ \overline{x},\ \theta) \geq 0.$$

Combining (3.2) and the above inequality yields

$$\frac{\lambda(f(\overline{x} + \theta\eta(x,\ \overline{x},\ \theta)) - f(\overline{x}))}{\theta} + \sum_{i=1}^{m} \mu_i \frac{g_i(\overline{x} + \theta\eta(x,\ \overline{x},\ \theta) - g_i(\overline{x})}{\theta} \geq 0,\ \ \forall \theta > 0.$$

By the arbitrariness of $\theta > 0$ and the existence of the right directional derivatives of $f(x),\ g_i(x),\ i = 1, \cdots, m$, at $\bar{x}$ along the path $\overline{x} + \theta\eta(x,\ \overline{x},\ \theta)$, we obtain that

$$\lambda f^+(P_{x,\overline{x}}(0)) + \sum_{i=1}^{m} \mu_i g^+(P_{x,\overline{x}}(0)) \geq 0 \ \text{ for all } \ x \in K.$$

This completes the proof. □

Remark 3.8. Theorem 3.7 improves and generalizes Theorem 3.1.2 in [15] from the semi-preinvexity case to the semi-prequasi-invexity case.

In order to research the property of problem (P_2), we give the following definition of Lagrangian function $L(x, \mu)$ and saddle point.

$$L(x, \mu) = f(x) + \sum_{i=1}^{m} \mu_i g_i(x) : K \times R_+^m \to R,\ K \subseteq R^n.$$

Definition 3.9 ([15])**.** A point $(\overline{x},\ \overline{\mu}) \in K \times R_+^m$ is said to be a saddle point for Lagrangian function $L(x,\ \mu)$ if the following condition is satisfied:

$$L(\overline{x},\ \mu) \leq L(\overline{x},\ \overline{\mu}) \leq L(x,\ \overline{\mu}), \quad \forall x \in K,\ \mu \in R_+^m.$$

Theorem 3.10. *Let $K \subseteq R^n$ be a semi-connected set with respect to $\eta : K \times K \times [0, 1] \to K$, assume $f(x) : K \to R$, $g_i(x) : K \to R$, $i = 1, \cdots, m$, are semi-prequasi-invex functions on K with respect to $\eta(x, y, \theta)$. If $\bar{x}$ is an optimal solution of (P_2), and there exists $x' \in K$, such that $g_i(x') < 0$, $i = 1, \cdots, m$, then, there exists a vector $\overline{\mu} \in R^m_+$, such that*

$$L(\overline{x}, \mu) \leq L(\overline{x}, \overline{\mu}) \leq L(x, \overline{\mu}), \quad \forall x \in K,\ \mu \in R^m_+,$$

where

$$L(x, \mu) = f(x) + \sum_{i=1}^{m} \mu_i g_i(x) : K \times R^m_+ \to R.$$

Proof. By the condition that $\bar{x}$ is an optimal solution of (P_2), it follows that the following two systems exclude each other on K.

$$f(x) - f(\overline{x}) < 0,$$
$$g_i(x) < 0,\ i = 1, \cdots, m.$$

The semi-prequasi-invexity of $f(x)$, $g_i(x)$, $i = 1, \cdots, m$, on K with respect to the same $\eta(x, y, \theta)$ and Lemma 3.3, implies that there exists $(\lambda, \beta) \in R_+ \times R^m_+$ satisfying

$$\lambda(f(x) - f(\overline{x})) + \sum_{i=1}^{m} \beta_i g_i(x) \geq 0,\ \forall x \in K. \tag{3.3}$$

Taking $x = \overline{x}$ into (3.3), we have $\sum_{i=1}^{m} \beta_i g_i(\overline{x}) \geq 0$. However, $\beta_i \geq 0$, $g_i(\overline{x}) \leq 0$, $i = 1, \cdots, m$ imply that $\sum_{i=1}^{m} \beta_i g_i(\overline{x}) \leq 0$. Consequently,

$$\lambda \sum_{i=1}^{m} \beta_i g_i(\overline{x}) = 0. \tag{3.4}$$

Next we prove that $\lambda > 0$. Otherwise, there must be $\lambda = 0$, $\beta \geq 0$, $\beta \neq 0$, taking them into (3.3), we have

$$\sum_{i=1}^{m} \beta_i g_i(x) \geq 0, \quad \forall x \in K. \tag{3.5}$$

Especially, taking $x = x'$ in (3.5), it follows that $\sum_{i=1}^{m} \beta_i g_i(x') \geq 0$, which contradicts the fact that $\beta \geq 0$, $\beta \neq 0$, and $g_i(x) < 0$, for all $i = 1, \cdots, m$. Therefore, $\lambda > 0$. Then, dividing (3.3), (3.4) by λ, respectively, we obtain

$$f(x) + \sum_{i=1}^{m} \overline{\mu}_i g_i(x) \geq f(\overline{x}), \tag{3.6}$$

$$\sum_{i=1}^{m} \overline{\mu}_i g_i(\overline{x}) = 0, \tag{3.7}$$

where $\overline{\mu}_i = \beta_i / \lambda$.

Clearly, (3.6) and (3.7) imply that $L(x, \overline{\mu}) \geq L(\overline{x}, \overline{\mu})$. Because of $\mu^T g(\overline{x}) \leq 0$ for all $\mu \in R^m_+$, we have

$$L(\overline{x}, \overline{\mu}) = f(\overline{x}) + \sum_{i=1}^{m} \overline{\mu}_i g_i(\overline{x}) \geq f(\overline{x}) + \sum_{i=1}^{m} \mu_i g_i(\overline{x}) = L(\overline{x}, \mu).$$

The proof is complete. □

Remark 3.11. Theorem 3.10 is a true generalization of Theorem 3.1.5 of [15], in which the semi-preinvexity is extended to the semi-prequasi-invexity.

4. Duality in generalized nonlinear fractional programming

In this section, we shall study the applications of semi-prequasi-invex type functions in generalized nonlinear fractional programming (FP), and we also demonstrate that the same results or even general ones than [8] and [15] can be obtained under the semi-prequasi-invexity assumptions.

Throughout this section, let $\|\cdot\|$ denote l_1-norm.

Consider the following generalized nonlinear fractional programming problem:

$$(FP): \quad \overline{\theta} = \inf_{x\in S} \max_{1\leq i\leq p} \{\frac{f_i(x)}{g_i(x)}\},$$

where $f_i(x): K \to R$, $g_i(x): K \to R$ for all $x \in K$, $g_i(x) > 0$ $(i = 1, \cdots, p)$, $h_j(x): K \to R$ $(j = 1, \cdots, m)$, $K \subseteq R^n$, and $S = \{x \in K : h_j(x) \leq 0,\ j = 1, \cdots, m\} \neq \varnothing$. Furthermore, the feasible set $S \neq \varnothing$, implies that $\overline{\theta} < +\infty$. Throughout this section, unless otherwise is specified, we use the following notations.

$$F(x) = (f_1(x), \cdots, f_p(x))^T,$$
$$G(x) = (g_1(x), \cdots, g_p(x))^T,$$
$$H(x) = (h_1(x), \cdots, h_p(x))^T.$$

To investigate the dual for (FP), let us first recall some definitions and lemmas about problem (FP) (for more details, see [8] and [15]).

Definition 4.1. For $x \in K$, $\mu \in R^p_+$, $\|\mu\| = 1$, and $v \in R^m_+$, we denote

$$GL(x, \mu, v) = \frac{\mu^T F(x) + v^T H(x)}{\mu^T G(x)},$$
$$GK(x, v) = \max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)} + \sum_{j=1}^{m} v_j \max_{1\leq i\leq p} \frac{h_j(x)}{g_i(x)}.$$

Then, we define

$$\phi_1(\mu, v) = \inf_{x\in K} GL(x, \mu, v),$$
$$\phi_2(v) = \inf_{x\in K} GK(x, v),$$

and two duals of the problem (FP):

$$(FD_1): \quad \sup_{\mu\in R^p_+\setminus\{0\}, v\in R^m_+} \phi_1(\mu, v),$$
$$(FD_2): \quad \sup_{v\in R^m_+} \phi_2(v).$$

In the sequel, we cite the following three lemmas (for more details, see [8] and [15]), which declare a weak duality relationship between (FD_1) and (FP), (FD_2) and (FP).

Lemma 4.2. *Let $x \in S$, then for any $\mu \in R^p_+, \|\mu\| = 1$ and $v \in R^m_+$, we have*

$$\phi_1(\mu, v) \leq \max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)},$$
$$\phi_2(v) \leq \max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)}.$$

Lemma 4.3. *Let $v(FD_i)$, $i \in \{1, 2\}$, denote the optimal value of (FD_i), $i \in \{1, 2\}$, if $v(FD_1) = \overline{\theta}$, then $v(FD_2) = \overline{\theta}$.*

Remark 4.4. Obviously, if $\overline{\theta} = -\infty$, then $v(FD_1) = v(FD_2) = -\infty$. So we focus on the case when $+\infty > \overline{\theta} > -\infty$.

Lemma 4.5. *If $\bar{x}$ is an optimal solution of (FP), then $\bar{x}$ is a weakly efficient solution of the system (TFP_1), where*

$$(TFP_1): \quad \min(F(x) - \overline{\theta}G(x))$$
$$H(x) \leq 0,\, x \in K.$$

Now, we give two duality results and a saddle point theorem to (FP).

Theorem 4.6. *Let $K \subseteq R^n$ be a nonempty semi-connected set with respect to $\eta : K \times K \times [0, 1] \to K$, assume $f_i(x) - \overline{\theta}\, g_i(x)\,(i = 1, \cdots, p)$, $h_j(x)$ $(j = 1, \cdots, m)$ are semi-prequasi-invex functions on K with respect to the same $\eta(x, y, \theta)$ and there exists $x' \in K$, such that $H(x') < 0$. Then, (FD_1) must have an optimal solution $(\overline{\mu}, \overline{v})$, with $v(FD_1) = v(FD_2) = \overline{\theta}$.*

Proof. For all $x \in S$, since $\max\limits_{1\leq i\leq p}\{\frac{f_i(x)}{g_i(x)}\} \geq \overline{\theta}$, we have the following systems that have no solution.

$$\max_{1\leq i\leq p}\{f_i(x) - \overline{\theta}\, g_i(x)\} < 0,$$
$$H(x) \leq 0,\; x \in K.$$

This implies that the following systems also have no solution.

$$f_i(x) - \overline{\theta}\, g_i(x) < 0,\, i = 1, \cdots, p,$$
$$h_j(x) < 0,\; i = 1, \cdots, m,\, x \in K.$$

Note that $f_i(x) - \overline{\theta}\, g_i(x),\; h_j(x)\,(i = 1, \cdots, p,\, j = 1, \cdots, m)$ are semi-prequasi-invex functions on K with respect to the same $\eta(x, y, \theta)$. This fact together with Lemma 3.3 yields that there exist $\overline{\mu} \in R^p_+$, $\overline{v} \in R^m_+$, $(\overline{\mu}, \overline{v}) \neq 0$ such that

$$\overline{\mu}^T(F(x) - \overline{\theta}G(x)) + \overline{v}^T H(x) \geq 0 \;\text{ for all}\, x \in K,$$

or

$$\overline{\mu}^T F(x) - \overline{\theta}\overline{\mu}^T G(x) + \overline{v}^T H(x) \geq 0 \;\text{ for all}\, x \in K. \tag{4.1}$$

Since $(\overline{\mu}, \overline{v}) \neq 0$, $H(x') < 0$, there must be $\overline{\mu} \neq 0$. Without loss of generality, we set $\|\overline{\mu}\| = 1$, then, we get $\overline{\mu}^T G(x) > 0$. Hence, from (4.1) we can deduce that

$$\frac{\overline{\mu}^T F(x) + \overline{v}^T H(x)}{\overline{\mu}^T G(x)} \geq \overline{\theta} \;\text{ for all}\, x \in K. \tag{4.2}$$

Therefore, by (4.2), Lemmas 4.2 and 4.3 we get the conclusion. □

Theorem 4.7. *Let $K \subseteq R^n$ be a nonempty semi-connected set with respect to $\eta : K \times K \times [0, 1] \to K$. Suppose $\max_{1\leq i\leq p}\{f_i(x) - \overline{\theta}\, g_i(x)\}$, $h_j(x)$ $(j = 1, \cdots, m)$, are semi-prequasi-invex functions on K with respect to the same $\eta(x, y, \theta)$, and there exists $x' \in K$, s.t. $H(x') < 0$. Then, (FD_2) must have an optimal solution $\bar{\mu}$, with $v(FD_2) = \overline{\theta}$.*

Proof. For all $x \in S$, since $\max\limits_{1\leq i\leq p}\{\frac{f_i(x)}{g_i(x)}\} \geq \overline{\theta}$, we have the following systems with no solution.

$$\max_{1\leq i\leq p}\{f_i(x) - \overline{\theta}\, g_i(x)\} < 0,$$
$$H(x) < 0,\; x \in K.$$

By the semi-prequasi-invexity of $\max\limits_{1\leq i\leq p}\{f_i(x)-\overline{\theta}g_i(x)\}$, $h_j(x)\,(j=1,\cdots,m)$ and Lemma 3.3, using the same proof in Theorem 4.6, it holds that there exists $\overline{\mu}\in R^m_+$, such that

$$\max_{1\leq i\leq p}\{f_i(x)-\overline{\theta}g_i(x)\}+\overline{\mu}^T H(x)\geq 0 \text{ for all } x\in K. \tag{4.3}$$

Then, for any fixed $x\in K$, let $s\triangleq s(x)\in\{1,\cdots,p\}$, such that

$$\max_{1\leq i\leq p}\{f_i(x)-\overline{\theta}g_i(x)\}=f_s(x)-\overline{\theta}g_s(x). \tag{4.4}$$

Note that $g_s(x)>0$ for all $x\in K$. This fact together with (4.3) and (4.4) leads to

$$\begin{aligned}0&\leq\frac{f_s(x)}{g_s(x)}-\overline{\theta}+\frac{\overline{\mu}^T H(x)}{g_s(x)}\\ &\leq\max_{1\leq i\leq p}\{\frac{f_i(x)}{g_i(x)}\}+\sum_{j=1}^m\mu_j\max_{1\leq i\leq p}\{\frac{h_j(x)}{g_i(x)}\}-\overline{\theta}\\ &=GK(x,\overline{\mu})-\overline{\theta}.\end{aligned}$$

Combining the above inequality with Lemma 4.2 and the definition of $\overline{\theta}$ yields

$$v(FD_2)=\overline{\theta},\ \phi_2(\overline{\mu})=\inf_{x\in K}GK(x,\overline{\mu})=\overline{\theta}.$$

Therefore $\overline{\mu}$ is an optimal solution of (FD_2) and thus completes the proof. □

Remark 4.8. Obviously, convexity and semi-preinvexity are special cases of semi-prequasi-invexity, thus, Theorem 4.7 generalizes Theorem 3.4 in [8] and Theorem 3.4.2 in [15].

In the sequel, we discuss the saddle point for $GK(x,\mu)$.

Theorem 4.9. *Let $K\subseteq R^n$ be a nonempty semi-connected set with respect to $\eta: K\times K\times[0,1]\to K$. Suppose $\max_{1\leq i\leq p}\{f_i(x)-\overline{\theta}\,g_i(x)\}$, $h_j(x)\,(j=1,\cdots,m)$, are semi-prequasi-invex functions on K with respect to the same $\eta(x,y,\theta)$. If $\bar{x}$ is an optimal solution of (FP), and there exists $x'\in K$, s.t. $H(x')<0$. Then, there exists $\overline{\mu}\in R^p_+$, such that $(\overline{x},\overline{\mu})$ is a saddle point of $GK(\overline{x},\overline{\mu})$ on $K\times R^m_+$, that is, for all $x\in K$, for all $\mu\in R^m_+$, we have*

$$GK(\overline{x},\mu)\leq GK(\overline{x},\overline{\mu})\leq GK(x,\overline{\mu}),$$

where

$$GK(x,\mu)=\max_{1\leq i\leq p}\{\frac{f_i(x)}{g_i(x)}\}+\sum\nolimits_{j=1}^m\mu_j\max_{1\leq i\leq p}\{\frac{h_j(x)}{g_i(x)}\}.$$

Proof. We first consider the following semi-prequasi-invexity programming problem,

$$\begin{aligned}(TFP_2):\quad &\min\max_{1\leq i\leq p}\{f_i(x)-\overline{\theta}g_i(x)\},\\ &\text{s.t. } H(x)\leq 0,\ x\in K.\end{aligned}$$

Let

$$HL(x,\mu)=\max_{1\leq i\leq p}\{f_i(x)-\overline{\theta}g_i(x)\}+\mu^T H(x).$$

One can easily check that $\bar{x}$ is an optimal solution of (TPF_2). By the fact that (TPF_2) is a semi-prequasi-invexity programming and Theorem 3.7, we obtain that there exists $\mu\in R^m_+$, such that

$$\begin{aligned}\max_{1\leq i\leq p}\{f_i(\overline{x})-\overline{\theta}g_i(\overline{x})\}+\xi^T H(\overline{x})&\leq\max_{1\leq i\leq p}\{f_i(\overline{x})-\overline{\theta}g_i(\overline{x})\}+\overline{\mu}^T H(\overline{x})\\ &\leq\max_{1\leq i\leq p}\{f_i(x)-\overline{\theta}g_i(x)\}+\overline{\mu}^T H(x)\ \text{ for all } x\in K,\ \xi\in R^m_+,\end{aligned} \tag{4.5}$$

and

$$\sum_{j=1}^{m} \overline{\mu}_j h_j(x) = 0. \tag{4.6}$$

Note that the problem $\overline{\theta} = \max\limits_{1\leq i\leq p}\{\frac{f_i(\overline{x})}{g_i(\overline{x})}\}$ is equivalent to the problem $\max\limits_{1\leq i\leq p}\{f_i(\overline{x})-\overline{\theta}g_i(\overline{x})\}$. This fact together with (4.6) yields

$$\begin{aligned} 0 &= \max_{1\leq i\leq p}\{f_i(\overline{x}) - \overline{\theta}g_i(\overline{x})\} + \mu^T H(\overline{x}) \\ &= \max_{1\leq i\leq p}\{\frac{f_i(\overline{x})}{g_i(\overline{x})}\} - \overline{\theta} + \mu^T H(\overline{x}) \\ &= \max_{1\leq i\leq p}\{\frac{f_i(\overline{x})}{g_i(\overline{x})}\} + \sum_{j=1}^{m} \overline{\mu}_j \max_{1\leq i\leq p}\{\frac{h_j(\overline{x})}{g_i(\overline{x})}\} - \overline{\theta} \\ &= GK(\overline{x}, \overline{\mu}) - \overline{\theta}. \end{aligned} \tag{4.7}$$

Then, taking $\xi_j = \mu_j[\min\limits_{1\leq i\leq p}\{\frac{1}{g_i(\overline{x})}\}]$ into (4.5), we have

$$\begin{aligned} 0 &\geq \max_{1\leq i\leq p}\{f_i(\overline{x}) - \overline{\theta}g_i(\overline{x})\} + \sum_{j=1}^{m} \mu_j \max_{1\leq i\leq p}\{\frac{h_j(\overline{x})}{g_i(\overline{x})}\} \\ &= \max_{1\leq i\leq p}\{\frac{f_i(\overline{x})}{g_i(\overline{x})}\} + \sum_{j=1}^{m} \mu_j \max_{1\leq i\leq p}\{\frac{h_j(\overline{x})}{g_i(\overline{x})}\} - \overline{\theta} \\ &= GK(\overline{x}, \mu) - \overline{\theta} \quad \text{for all}\, \mu \in R^m_+. \end{aligned} \tag{4.8}$$

In the sequel, for any $x \in K$, let $s \triangleq s(x) \in \{1, \cdots, p\}$ such that

$$\max_{1\leq i\leq p}\{f_i(x) - \overline{\theta}g_i(x)\} = f_s(x) - \overline{\theta}g_s(x).$$

By $g_s(x) > 0$ for all $x \in K$ and (4.5), we have

$$\begin{aligned} 0 &\leq \frac{\max\limits_{1\leq i\leq p}\{f_i(x) - \overline{\theta}g_i(x)\} + \mu^T H(x)}{g_i(x)} \\ &= \frac{f_s(x)}{g_s(x)} + \sum_{j=1}^{m} \overline{\mu}_j \frac{h_j(\overline{x})}{g_i(\overline{x})} - \overline{\theta} \\ &\leq \max_{1\leq i\leq p}\{\frac{f_i(x)}{g_i(x)}\} + \sum_{j=1}^{m} \overline{\mu}_j \max_{1\leq i\leq p}\{\frac{h_j(x)}{g_i(x)}\} - \overline{\theta} \\ &= GK(x, \overline{\mu}) - \overline{\theta} \quad \text{for all}\, x \in K. \end{aligned} \tag{4.9}$$

By virtue of (4.7)-(4.9), we obtain that

$$GK(\overline{x}, \mu) \leq GK(\overline{x}, \overline{\mu}) \leq GK(x, \overline{\mu}) \quad \text{for all}\, x \in K,\ \mu \in R^m_+.$$

Hence, the proof is complete. □

Acknowledgment

The first author is supported by the National Natural Science Foundation of China (11301571), the Basic and Advanced Research Project of Chongqing (2015jcyjA00025), the China Postdoctoral Science Foundation funded project (2015M580774,2016T90837) and the Program for University Innovation Team of Chongqing (CXTDX201601022). The second author is supported by the Chongqing Graduate Research Innovative Training Program (CYS16144). The third author was partially supported by the National Outstanding Youth Science Fund Project of China (51425801) and the National Natural Science Foundation of China (51278512).

References

[1] T. Antczak, *New optimality conditions and duality results of G type in differentiable mathematical programming*, Nonlinear Anal., **66** (2007), 1617–1632. 1

[2] A. Ben-Israel, B. Mond, *What is invexity?*, J. Austral. Math. Soc. Ser. B, **28** (1986), 1–9. 1

[3] H.-Z. Luo, H.-X. Wu, *On the relationships between G-preinvex functions and semistrictly G-preinvex functions*, J. Comput. Appl. Math., **222** (2008), 372–380. 1

[4] H.-Z. Luo, H.-X. Wu, Y.-H. Zhu, *Remarks on criteria of prequasi-invex functions*, Appl. Math. J. Chinese Univ. Ser. B, **19** (2004), 335–341. 1

[5] H.-Z. Luo, Z. K. Xu, *On characterizations of prequasi-invex functions*, J. Optim. Theory Appl., **120** (2004), 429–439. 1

[6] D. H. Martin, *The essence of invexity*, J. Optim. Theory Appl., **47** (1985), 65–76. 1

[7] T. Weir, B. Mond, *Pre-invex functions in multiple objective optimization*, J. Math. Anal. Appl., **136** (1988), 29–38. 1

[8] Z. K. Xu, *Duality in generalized nonlinear fractional programming*, J. Math. Anal. Appl., **169** (1992), 1–9. 1, 3, 4, 4, 4.8

[9] X.-M. Yang, *Problems of semi-pteinvexity and multiobjective programming*, (Chinese) J. Chongqing Normal Univ. Nat. Sci., **1** (1994), 1–5. 1, 2.1, 2, 2.3, 2.4, 2.5

[10] X.-M. Yang, *A note on preinvexity*, J. Ind. Manag. Optim., **10** (2014), 1319–1321.

[11] X. Q. Yang, G. Y. Chen, *A class of nonconvex functions and pre-variational inequalities*, J. Math. Anal. Appl., **169** (1992), 359–373. 1, 2.1, 3

[12] X.-M. Yang, D. Li, *On properties of preinvex functions*, J. Math. Anal. Appl., **256** (2001), 229–241. 1

[13] X.-M. Yang, D. Li, *Semistrictly preinvex functions*, J. Math. Anal. Appl., **258** (2001), 287–308. 1

[14] X.-M. Yang, X. Q. Yang, K. L. Teo, *Characterizations and applications of prequasi-invex functions*, J. Optim. Theory Appl., **110** (2001), 645–668. 1, 3.1, 3.2

[15] Y.-X. Zhao, *A type of generalized convexity and applications in optimization theory*, (In Chinese) Master Degree Thesis. Jinhua: Zhejiang Normal University, (2005). 1, 3.8, 3.9, 3.11, 4, 4, 4.8

[16] Y.-X. Zhao, X.-G. Meng, H. Qiao, S.-Y. Wang, L. Coladas Uria, *Characterizations of semi-prequasi-invexity*, J. Syst. Sci. Complex., **27** (2014), 1008–1026. 1, 2.4, 2.5