Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 6153–6160
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Local fractional Fourier method for solving modified diffusion equations with local fractional derivative

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Communicated by X. J. Yang

Abstract

In this manuscript, in order to solve the boundary and initial value problem of modified diffusion equation with local fractional derivative, we present the local fractional Fourier series method. The method can easily convert the partial fractional differential equation into the ordinary fractional equation system. And several test examples are given to show the procedure and reliability of the proposed technique.

Keywords: Local fractional Fourier series, local fractional derivative, modified diffusion equation.
2010 MSC: 34B30, 35G30, 35R11.

1. Introduction

The diffusion equation is a partial differential equation which depicts density dynamics in a material undergoing diffusion. The modified diffusion equations are used to portray some processes exhibiting diffusive-like behavior, which have a broad range of applications in mathematical physics, integral system and fluid mechanics. Thence, lots of methods have been used to solve this type of equations, for example, the numerical method [2], functional constraints method [8], and higher-order time-stepping method [7]. In recent years, the fractional derivative with derivative of arbitrary orders have been developed to handle with problems in many areas, such as physics, applied mathematics, engineering and so forth [1, 3–6, 10]. Fractional derivative depicts memory and hereditary characters of materials and processes, while the classical derivative only describes individual-level perspective. The fractional derivative can be defined in many

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Received 2016-09-08

forms, such as the Caputo derivative, the Riemann-Liouville derivative, the Grünwald-Letnikov derivative and so on [11, 12, 14]. However, most of them do not deal with the non-differentiable functions defined on Cantor sets. The local fractional derivative [11, 12] describes the non-differential problems defined on Cantor sets, while the classical derivative and most fractional derivative deal with functions in Euclidean space. In those papers of Xiao-Jun Yang and other co-authors, the theory has been successfully applied in investigating equations in fractal-like media, for example, the Navier-Stokes equations, the Helmholtz equations and the diffusion equations. Many methods have also developed to deal with the local fractional differential equations, such as the local fractional function decomposition method [9], the differential transform approach [15], the local fractional the variational iteration transform method [13] and so on. In this paper, our aims are to present the local fractional Fourier method to solve the modified diffusion equations with local fractional derivative. The organization of the manuscript is as follows. For clarity of presentation, in Section 2, some preliminaries and notations of the local fractional calculus theory are recalled. In Section 3, the local fractional Fourier method for solving the modified diffusion equations with local fractional derivative is investigated. In Section 4, several examples are considered. Finally, in Section 5, the conclusions are given.

2. Preliminaries

In this section, we introduce some mathematical preliminaries of the local fractional calculus theory in fractal space for our subsequent development [11, 12].

Definition 2.1. Suppose that there is [12]

$$|u(x)-u(x_0)|<\varepsilon^\alpha \tag{2.1}$$

with $|x-x_0|<\delta$, for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in R$, then $u(x)$ is called local fractional continuous at $x=x_0$ and it is denoted by $\lim\limits_{x\to x_0} u(x)=u(x_0)$.

Definition 2.2. Suppose that the function $u(x)$ is satisfied the condition (2.1) for $x\in(a,b)$, it is called local fractional continuous on the interval (a,b), denoted by

$$u(x)\in C_\alpha(a,b).$$

Definition 2.3. In fractal space, let $u(x)\in C_\alpha(a,b)$, the local fractional derivative of $u(x)$ of order α at $x=x_0$ is given by (see [12])

$$D_x^{(\alpha)}u(x_0)=u^{(\alpha)}(x_0)=\frac{d^\alpha u(x)}{dx^\alpha}|_{x=x_0}=\lim_{x\to x_0}\frac{\triangle^\alpha(u(x)-u(x_0))}{(x-x_0)^\alpha},$$

where $\triangle^\alpha(u(x)-u(x_0))=\Gamma(1+\alpha)\triangle(u(x)-u(x_0))$.

Obviously, the order α of local fractional derivative is equal to the dimension of Cantor set. Local fractional derivative of high order and local fractional partial derivative of high order are defined respectively in the following forms (see [11, 12]):

$$u^{(k\alpha)}(x)=\overbrace{D_x^{(\alpha)}...D_x^{(\alpha)}}^{\text{k times}}u(x),$$

$$\frac{\partial^{k\alpha}}{\partial x^{k\alpha}}u(x,y)=\overbrace{\frac{\partial^\alpha}{\partial x^\alpha}...\frac{\partial^\alpha}{\partial x^\alpha}}^{\text{k times}}u(x,y).$$

Definition 2.4. In fractal space, the Mittage-Leffler function, sine function, and cosine function are respectively defined by (see [11, 12]):

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}, x \in R, 0 < \alpha \leq 1,$$

$$\sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]}, x \in R, 0 < \alpha \leq 1,$$

$$\cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k\alpha}}{\Gamma(1+2k\alpha)}, x \in R, 0 < \alpha \leq 1.$$

Definition 2.5. Let $u(x)$ be the $2l$ periodic function. For $k \in Z$ and $u(x) \in C_\alpha(-\infty, +\infty)$, the local fractional Fourier series of $u(x)$ is defined as following (see [11]):

$$u(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos_\alpha \frac{(\pi k x)^\alpha}{l^\alpha} + b_k \sin_\alpha \frac{(\pi k x)^\alpha}{l^\alpha}],$$

where

$$a_k = \frac{1}{l^\alpha} \int_{-l}^{l} u(x) \cos_\alpha \frac{(\pi k x)^\alpha}{l^\alpha} (dx)^\alpha,$$

$$b_k = \frac{1}{l^\alpha} \int_{-l}^{l} u(x) \sin_\alpha \frac{(\pi k x)^\alpha}{l^\alpha} (dx)^\alpha$$

are all the local fractional Fourier coefficients.

3. Local fractional Fourier series method

In this section, we shall present the process of the local fractional Fourier series method to derive exact solution of some modified diffusion equations. We consider the following local fractional modified diffusion equation on fractal set:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + C \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} + R[u(x,t)] = g(x,t), \tag{3.1}$$

with boundary and initial conditions

$$u(o,t) = u(l,t) = 0,$$

where the parameter C is a constant, R is a linear or nonlinear operator, $g(x,t)$ is a source term and $l > 0$. Now we address the solution of the Eq. (3.1). Expand $u(x,t)$ and $g(x,t)$ to be odd functions of period $(2l)^\alpha$ in terms of the variable t, respectively (in the following, let $u(x,t)$ also denote its expanded odd function of period $(2l)^\alpha$ for convenience). In light of the local fractional Fourier series expansion theory, the Fourier series of concerning functions can be expressed by

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha}, \\ g(x,t) = \sum_{n=1}^{\infty} g_n(t) \sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha}, \\ R[u(x,t)] = \sum_{n=1}^{\infty} A_n(t) \sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha} + \sum_{n=1}^{\infty} B_n(t) \cos_\alpha \frac{(\pi n x)^\alpha}{l^\alpha}, \end{cases} \tag{3.2}$$

where the corresponding coefficients $u_n(t), g_n(t), A_n(t), B_n(t)$ are determined, respectively, by the following equation system

$$\begin{cases} u_n(t) = \dfrac{2^\alpha}{l^\alpha}\displaystyle\int_0^l u(x,t)\sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha}(dx)^\alpha, \\ g_n(t) = \dfrac{2^\alpha}{l^\alpha}\displaystyle\int_0^l g(x,t)\sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha}(dx)^\alpha, \\ A_n(t) = \dfrac{1}{l^\alpha}\displaystyle\int_{-l}^l R[u(x,t)]\sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha}(dx)^\alpha, \\ B_n(t) = \dfrac{1}{l^\alpha}\displaystyle\int_{-l}^l R[u(x,t)]\cos_\alpha \frac{(\pi n x)^\alpha}{l^\alpha}(dx)^\alpha. \end{cases} \tag{3.3}$$

Substituting the Eq. (3.3) into Eq. (3.1) and assuming the differential is permitted, we rewrite Eq. (3.1) as follows:

$$\begin{aligned} \sum_{n=1}^{\infty} u_n^{(\alpha)}(t)\sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha} - Cn^{2\alpha}(\frac{\pi}{l})^{2\alpha}\sum_{n=1}^{\infty} u_n(t)\sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha} + \sum_{n=1}^{\infty} A_n(t)\sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha} \\ + \sum_{n=1}^{\infty} B_n(t)\cos_\alpha \frac{(\pi n x)^\alpha}{l^\alpha} = \sum_{n=1}^{\infty} g_n(t)\sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha}. \end{aligned} \tag{3.4}$$

Comparing the same coefficients on both sides of Eq. (3.4) with respect to $\sin_\alpha \frac{(\pi n x)^\alpha}{l^\alpha}$ and $\cos_\alpha \frac{(\pi n x)^\alpha}{l^\alpha}$ respectively, we obtain the following equation system

$$\begin{cases} u_n^{(\alpha)}(t) + A_n(t) - Cn^{2\alpha}(\frac{\pi}{l})^{2\alpha}u_n(t) = g_n(t), \\ B_n(t) = 0. \end{cases} \tag{3.5}$$

Indeed, via the local fractional Fourier series transformation, the partial differential Eq. (3.1) is converted into the ordinary differential equation system (3.5). Obviously, it is more easier to solve system (3.5) than Eq. (3.1).

4. Illustrative examples

In order to illustrate the validity of the local fractional Fourier method in Section 3, we give the following several equations with the local fractional derivative.

Example 4.1. We consider the following modified diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} - \sin_\alpha(t^\alpha) = 0 \tag{4.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \dfrac{\partial^\alpha u(0,t)}{\partial t^\alpha} = \sin_\alpha(x^\alpha), \\ u(0,t) = u(x,\pi) = 0. \end{cases}$$

According to the expression of Eqs. (3.2), (3.3) and (4.1), we let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin_\alpha(nx)^\alpha, \tag{4.2}$$

and

$$g(x,t) = -\sin_\alpha(t^\alpha). \tag{4.3}$$

Substituting Eqs. (4.2) and (4.3) into the Eq. (4.1) and then comparing the coefficient of like $\sin_\alpha(nx)^\alpha$ of the transformed equation, we can deduce the following results

$$u_n^{2\alpha}(t) = n^{2\alpha} u_n(t), (n \neq 1) \tag{4.4}$$

and

$$u_1^{(\alpha)}(t) - u_1(t) - 1 = 0. \tag{4.5}$$

Analyzing Eq. (4.4), we can simplify the solution of Eq. (4.4) by imposing the following assumptions:

$$n^{2\alpha} u_n(t) = 0, (n \neq 1).$$

Moreover, solving Eq. (4.5), we can obtain

$$u_1(t) = E_\alpha(t^\alpha) - 1.$$

Thus, the exact solution of Eq. (4.1) is

$$u(x,t) = [E_\alpha(t^\alpha) - 1] \sin_\alpha(x^\alpha).$$

Example 4.2. We consider the following local fractional modified diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = 1 \tag{4.6}$$

with the following boundary and initial conditions

$$\begin{cases} \dfrac{\partial^\alpha u(x,0)}{\partial t^\alpha} = x^\alpha, \\ u(0,t) = u(\pi,t) = 0. \end{cases} \tag{4.7}$$

According to the expression of Eqs. (3.2), (3.3), and (4.7), we let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin_\alpha(nx)^\alpha, \tag{4.8}$$

and

$$g(x,t) = 1 = \sum_{n=1}^{\infty} \frac{2^\alpha[1-(-1)^n]}{n^\alpha} \sin_\alpha(nx)^\alpha. \tag{4.9}$$

Substituting Eqs. (4.8) and (4.9) into Eq. (4.6), and then comparing the coefficient of like $\sin_\alpha(nx)^\alpha$ of the transformed equation, we can obtain

$$u_n^{(\alpha)}(t) - n^{2\alpha} u_n(t) = \frac{2^\alpha[1-(-1)^n]}{n^\alpha}. \tag{4.10}$$

Solving Eq. (4.10), we can obtain

$$u_n(t) = \frac{2^\alpha[1-(-1)^n]}{n^{3\alpha}} + c_n E_\alpha(n^{2\alpha} t^\alpha). \tag{4.11}$$

In light of Eq. (4.11), Eq. (4.7) can be rewritten as

$$\frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = x^\alpha = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^\alpha}{n^\alpha} \sin_\alpha(nx)^\alpha. \tag{4.12}$$

By virtue of Eqs. (4.11), (4.12), and (4.7), we can obtain

$$c_n = (-1)^{n+1}\frac{2^\alpha}{n^{3\alpha}}. \tag{4.13}$$

Inserting Eq. (4.13) into Eq. (4.11), we can derive

$$u_n(t) = \frac{2^\alpha[1-(-1)^n]}{n^{3\alpha}} + (-1)^{n+1}\frac{2^\alpha}{n^{3\alpha}}E_\alpha(n^{2\alpha}t^\alpha).$$

Hence, the corresponding solution of Eq. (4.6) is given as follows:

$$u(x,t) = \sum_{n=1}^{\infty}[\frac{2^\alpha[1-(-1)^n]}{n^{3\alpha}} + (-1)^{n+1}\frac{2^\alpha}{n^{3\alpha}}E_\alpha(n^{2\alpha}t^\alpha)]\sin_\alpha(nx)^\alpha.$$

Example 4.3. The local fractional modified diffusion equation is written in the following form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} - \frac{1}{(2h)^\alpha}\int_{x-h}^{x+h} u(\mu,t)(d\mu)^\alpha = 0 \tag{4.14}$$

subject to the boundary and initial conditions described by

$$\begin{cases} \dfrac{\partial^\alpha u(x,0)}{\partial t^\alpha} = \dfrac{(\pi-x)^\alpha}{2^\alpha}, \\ u(0,t) = u(\pi,t) = 0. \end{cases}$$

According to the expression of the Eq. (3.1), we let

$$R[u(x,t)] = -\frac{1}{(2h)^\alpha}\int_{x-h}^{x+h} u(\mu,t)(d\mu)^\alpha.$$

In view of the expression of Eqs. (3.2), (3.3), and (4.14), we let

$$\begin{cases} u(x,t) = \displaystyle\sum_{n=1}^{\infty} u_n(t)\sin_\alpha(nx)^\alpha, \\ \dfrac{1}{(2h)^\alpha}\displaystyle\int_{x-h}^{x+h} u(\mu,t)(d\mu)^\alpha = \sum_{n=1}^{\infty}\frac{\sin_\alpha(nh)^\alpha}{(nh)^\alpha}u_n(t)\sin_\alpha(nx)^\alpha. \end{cases} \tag{4.15}$$

Substituting Eq. (4.15) into Eq. (4.14) and then comparing the coefficient of like $\sin_\alpha(nx)^\alpha$ of the transformed equation, we can deduce

$$u_n^{(\alpha)}(t) - n^{2\alpha}u_n(t) = \frac{\sin_\alpha(nh)^\alpha}{(nh)^\alpha}u_n(t). \tag{4.16}$$

Solving Eq. (4.16), we can derive

$$u_n^{(\alpha)}(t) = [c_n + \frac{1}{n^{2\alpha} + \frac{\sin_\alpha(nh)^\alpha}{(nh)^\alpha}})]E_\alpha[n^{2\alpha}t^\alpha + \frac{\sin_\alpha(nh)^\alpha}{(nh)^\alpha}t^\alpha]. \tag{4.17}$$

By means of the following condition

$$\frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = \frac{(\pi-x)^\alpha}{2^\alpha} = \sum_{n=1}^{\infty}\frac{\sin_\alpha(nx)^\alpha}{n^\alpha}, \tag{4.18}$$

it is readily derived from Eqs. (4.17) and (4.18) that

$$c_n = \frac{1 - n^\alpha}{n^\alpha[n^{2\alpha} + \frac{\sin_\alpha((nh)^\alpha)}{(nh)^\alpha}]}. \tag{4.19}$$

Substituting Eq. (4.19) into Eq. (4.17), we yield

$$u_n(t) = \frac{1}{n^{3\alpha} + \frac{\sin_\alpha(nh)^\alpha}{(nh)^\alpha}} E_\alpha[n^{2\alpha}t^\alpha + \frac{\sin_\alpha((nh)^\alpha)}{(nh)^\alpha} t^\alpha].$$

Thus, the exact solution of Eq. (4.14) is formulated as follows:

$$u_n(x,t) = \sum_{n=1}^{\infty}[\frac{1}{n^{3\alpha} + \frac{\sin_\alpha(nh)^\alpha}{(nh)^\alpha}} E_\alpha(n^{2\alpha}t^\alpha + \frac{\sin_\alpha(nh)^\alpha}{(nh)^\alpha} t^\alpha)] \sin_\alpha(nx)^\alpha.$$

5. Conclusions

In this article, the local fractional Fourier series method is introduced for solving modified diffusion equations with local fractional operators in details. The test examples show that the suggested method can be regarded as a simple and efficient tool for computing local fractional modified diffusion equations.

Acknowledgment

This work is supported by the foundation and advanced technology research program of Henan province (142300410385).

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