Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 6161–6169
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Strong convergence of a hybrid algorithm in a Banach space

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Communicated by M. Eslamian

Abstract

In this paper, we study a hybrid algorithm for finding a common solution of a finite family of equilibrium problems which is also a common fixed point of a finite family of asymptotically quasi-ϕ-nonexpansive mappings in a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property.

Keywords: Mean valued algorithm, hybrid algorithm, convergence, variational inequality, hybrid algorithm.
2010 MSC: 65K15, 90C33.

1. Introduction and preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \|x\|^2 = \langle x, f^*\rangle = \|f^*\|^2\}.$$

Let B_E be the unit sphere of E. Recall that E is said to be a strictly convex space iff $\|x + y\| < 2$ for all $x, y \in B_E$ and $x \neq y$. Recall that E is said to have a Gâteaux differentiable norm iff $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$

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Received 2016-09-05

exists for each $x, y \in B_E$. In this case, we also say that E is smooth. E is said to have a uniformly Gâteaux differentiable norm, if for each $y \in B_E$, the limit is attained uniformly for all $x \in B_E$. E is also said to have a uniformly Fréchet differentiable norm iff the above limit is attained uniformly for $x, y \in B_E$. In this case, we say that E is uniformly smooth. It is known that if E is uniformly smooth, then duality mapping J is uniformly norm-to-norm continuous on every bounded subset of E. It is also known that E^* is uniformly convex, if and only if E is uniformly smooth.

Next, we assume that E is a smooth Banach space which means mapping J is single-valued. Consider the functional

$$\phi(x, y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x, y \in E.$$

In [2], Alber studied a new mapping Π_C in a Banach space E which is an analogue of P_C, the metric projection, in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x)$. It is obvious from the definition of function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y), \quad \forall x, y \in E.$$

Recall that E has the Kadec-Klee property, if $\lim_{n\to\infty} \|x_n - x\| = 0$, as $n \to \infty$, for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, as $n \to \infty$. It is known that every uniformly convex Banach space has the Kadec-Klee property.

Let $T : C \to C$ be a mapping. In this paper, we use $Fix(T)$ to denote the fixed point set of mapping T. T is said to be closed, if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x'$ and $\lim_{n\to\infty} Tx_n = y'$, we have $Tx' = y'$.

Recall that a point p is said to be an asymptotic fixed point of mapping T if and only if subset C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We use $\widetilde{Fix}(T)$ to denote the asymptotic fixed point set. Let K be a bounded subset of C. Recall that T is said to be uniformly asymptotically regular on C if and only if $\limsup_{n\to\infty} \sup_{x\in K}\{\|T^n x - T^{n+1}x\|\} = 0$.

Recall that T is said to be relatively nonexpansive [6] iff

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, \quad \forall p \in \widetilde{Fix}(T) = Fix(T) \ne \emptyset.$$

T is said to be relatively asymptotically nonexpansive [1] iff

$$\phi(p, T^n x) \le (\mu_n + 1)\phi(p, x), \quad \forall x \in C, \quad \forall p \in \widetilde{Fix}(T) = Fix(T) \ne \emptyset, \quad \forall n \ge 1,$$

where $\{\mu_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

T is said to be quasi-ϕ-nonexpansive [15] iff

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, \quad \forall p \in Fix(T) \ne \emptyset.$$

T is said to be asymptotically quasi-ϕ-nonexpansive [16] iff there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\phi(p, T^n x) \le (\mu_n + 1)\phi(p, x), \quad \forall x \in C, \quad \forall p \in Fix(T) \ne \emptyset, \quad \forall n \ge 1.$$

Remark 1.1. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require the strong restriction that the fixed point set equals the asymptotic fixed point set. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Hilbert spaces since $\sqrt{\phi(x, y)} = \|x - y\|$.

Mann iterative algorithm is efficient for studying fixed points of (asymptotically) nonexpansive operators. However, in the framework of infinite-dimensional Banach spaces, they are only weakly convergent (see [10] and the references therein). In many modern disciplines, problems arise in the framework of infinite dimension spaces. In such nonlinear problems, strong convergence is often much more desirable than the weak convergence. To guarantee the strong convergence of Mann iteration algorithms, many authors use different regularization methods in the framework of Banach spaces; see [4, 7, 13, 17] and the references therein.

Let C be a nonempty closed and convex subset of E and let $B : C \times C \to \mathbb{R}$ be a function. Recall the following equilibrium problem [5]. Find $\bar{x} \in C$ such that $B(\bar{x}\, y) \geq 0$, for all $y \in C$. We use $Sol(B)$ to denote the solution set of the equilibrium problem. That is, $Sol(B) = \{x \in C : B(x, y) \geq 0, \forall y \in C\}$.

In order to study the equilibrium problem, we assume that B satisfies the following conditions:

(B1) $B(a, a) \equiv 0,\ \forall a \in C$;

(B2) $B(b, a) + B(a, b) \leq 0,\ \forall a, b \in C$;

(B3) $B(a, b) \geq \limsup_{t \downarrow 0} B(tc + (1-t)a, b),\ \ \forall a, b, c \in C$;

(B4) $b \mapsto B(a, b)$ is convex and weakly lower semi-continuous for all $a \in C$.

We remark here that B is said to be monotone iff $B(x, y) + B(y, x) \leq 0$, for all $x, y \in C$. $y \mapsto B(x, y)$ is convex iff $B(tx + (1-t)y, z) \leq tB(x, z) + (1-t)B(y, z)$ for all $x, y, z \in C$ and $t \in (0, 1)$. $y \mapsto B(x, y)$ is lower semi-continuous iff $B(x, y_n) \to B(x, y)$ whenever $y_n \to y$ as $n \to \infty$. It is known that the indicator function of an open set is lower semi-continuous. The equilibrium problem is dynamic and is experiencing an explosive growth in both theory and applications. It includes variational inequality problems, saddle problems, complementary problems, zero point problem as special cases, provides a unified framework for many problems in image recovery, traffic and network bandwidth allocation (see [3, 9, 11, 19, 24] and the references therein).

In this paper, we study a hybrid algorithm for a finite family of equilibrium problems and fixed point problems of asymptotically quasi-ϕ-nonexpansive mappings. Strong convergence of the algorithm is obtained in a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. From the framework of the space, the operator and the restrictions imposed on the control sequences, the results presented in this paper mainly improve the corresponding results in [8, 12, 14–16, 22, 23].

The following lemmas play an important role in this paper.

Lemma 1.2 ([20])**.** *Let r be a positive real number and let E be uniformly convex. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|(1-t)y + ta\|^2 + t(1-t)g(\|b - a\|) \leq t\|a\|^2 + (1-t)\|b\|^2$$

for all $a, b \in B^r := \{a \in E : \|a\| \leq r\}$ and $t \in [0, 1]$.

Lemma 1.3 ([2])**.** *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of E. Let $x \in E$. Then*

$$\phi(y, \Pi_C x) \leq \phi(y, x) - \phi(\Pi_C x, x), \quad \forall y \in C,$$

$$\langle y - x_0, Jx - Jx_0\rangle \leq 0, \quad \forall y \in C,$$

if and only if $x_0 = \Pi_C x$.

Lemma 1.4 ([16])**.** *Let E be a strictly convex, smooth, and reflexive Banach space and let C be a closed convex subset of E. Let $T : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping. Then $Fix(T)$ is convex and closed.*

Lemma 1.5 ([15, 21])**.** *Let E be a strictly convex, smooth, and reflexive Banach space and let C be a closed convex subset of E. Let B be a function, which satisfies* (B1)-(B4)*, from $C \times C$ to $\mathbb{R}$. Let $x \in E$ and let $r > 0$. Define a mapping $W_{G,r} : E \to C$ by*

$$R^{B,r}x = \{z \in C : rB(z, y) + \langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C\}.$$

Then, there exists $z \in C$ *such that* $rB(z,y)+\langle z-y, Jz-Jx\rangle \leq 0$, *for all* $y \in C$ *and the following conclusions hold:*

(1) $R^{B,r}$ *is single-valued quasi-*ϕ*-nonexpansive and*

$$\langle R^{B,r}x - R^{B,r}y, JR^{B,r}x - JR^{B,r}y\rangle \leq \langle R^{B,r}x - R^{B,r}y, Jx - Jy\rangle$$

for all $x, y \in E$.

(2) $\phi(q, R^{B,r}x) + \phi(R^{B,r}x, x) \leq \phi(q, x)$, $\forall q \in Fix(R^{B,r})$.

(3) $Fix(R^{B,r}) = Sol(B)$ *is closed and convex.*

2. Convergence theorems

Theorem 2.1. *Let* E *be a uniformly smooth and strictly convex Banach space. Let* C *be a convex and closed subset of* E. *Let* N *be some positive integer. Let* B_i *be a bifunction with restrictions* (B1), (B2), (B3), (B4) *and let* $T_i : C \to C$ *be an asymptotically quasi-*ϕ*-nonexpansive mapping such that* T_i *is uniformly asymptotically regular and closed on* C *for each* $1 \leq i \leq N$. *Assume* $\cap_{i=1}^N\big(Sol(B_i) \cap Fix(T_i)\big)$ *is nonempty and bounded. Let* $\{x_n\}$ *be a sequence generated in the following process.*

$$\begin{cases} C_{(1,i)} = C, \quad \forall 1 \leq i \leq N, \\ C_1 = \cap_{i=1}^N C_{(1,i)}, \\ x_1 = \Pi_{C_1}x_0, \\ y_{(n,i)} = J^{-1}\big((1-\alpha_{(n,i)})JT_i^n x_n + \alpha_{(n,i)}Jx_n\big), \\ r_{(n,i)}B_i(u_{(n,i)}, y) + \langle u_{(n,i)} - y, Ju_{(n,i)} - Jy_{(n,i)}\rangle \leq 0, \quad \forall y \in C_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \mu_{(n,i)}M_{(n,i)} + \phi(z, x_n) \geq \phi(z, u_{(n,i)})\}, \\ C_{n+1} = \cap_{i=1}^N C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}}x_1, \end{cases}$$

where $M_{(n,i)} = \sup\{\phi(p, x_n) : p \in \cap_{i=1}^N\big(Sol(B_i) \cap Fix(T_i)\big)\}$, $\{r_{(n,i)}\}$ *is a real sequence in* $[r, \infty)$, *where* r *is some positive real number and* $\{\alpha_{(n,i)}\}$ *is a real sequence in* $[a, b]$, *where* $0 < a < b < 1$. *If* E *has the Kadec-Klee property, then* $\{x_n\}$ *converges strongly to* $\Pi_{\cap_{i=1}^N\left(Sol(B_i)\cap Fix(T_i)\right)}x_1$.

Proof. From Lemmas 1.4 and 1.5, we find that $\cap_{i=1}^N\big(Sol(B_i) \cap Fix(T_i)\big)$ is convex and closed. Hence, $\Pi_{\cap_{i\in\Lambda}S(G_i)\bigcap\cap_{i\in\Lambda}F(T_i)}x$ is well-defined for any element in E.

Next, we prove that C_n is convex and closed. We show this by the induction. It is obvious that $C_{(1,i)} = C$ is convex and closed. Assume that $C_{(m,i)}$ is convex and closed for some $m \geq 1$. Let $z_1, z_2 \in C_{(m+1,i)}$. Hence $z_1, z_2 \in C_{(m,i)}$. Therefore, $z = tz_1+(1-t)z_2 \in C_{(m,i)}$, where $t \in (0, 1)$. Notice that $\phi(z_1, u_{(m,i)})-\phi(z_1, x_m) \leq \mu_{(m,i)}M_{(m,i)}$, and $\phi(z_2, u_{(m,i)}) - \phi(z_2, x_m) \leq \mu_{(m,i)}M_{(m,i)}$. Hence, one has

$$2\langle z_1, Jx_m - Ju_{(m,i)}\rangle - \|x_m\|^2 + \|u_{(m,i)}\|^2 \leq \mu_{(m,i)}M_{(m,i)},$$

and

$$2\langle z_2, Jx_k - Ju_{(m,i)}\rangle - \|x_m\|^2 + \|u_{(m,i)}\|^2 \leq \mu_{(m,i)}M_{(m,i)}.$$

This finds $\phi(z, u_{(m,i)}) \leq \phi(z, x_m) + \mu_{(m,i)}M_{(m,i)}$, where $z \in C_{(m,i)}$. This shows that $C_{(m+1,i)}$ is closed and convex. Hence, $C_n = \cap_{i=1}^N C_{(n,i)}$ is a convex and closed set. This proves that $\Pi_{C_{n+1}}x_1$ is well-defined. On the other hand, we have $\cap_{i=1}^N\big(Sol(B_i) \cap Fix(T_i)\big)$ is in C_n, for each $n \geq 1$. And $\cap_{i=1}^N\big(Sol(B_i) \cap Fix(T_i)\big) \subset$

$C_1 = C$ is obvious. Suppose that $\cap_{i=1}^N \big(Sol(B_i) \cap Fix(T_i)\big) \subset C_{(m,i)}$ for some positive integer m. For any $w \in \cap_{i=1}^N \big(Sol(B_i) \cap Fix(T_i)\big) \subset C_{(m,i)}$, we see that

$$\begin{aligned}\phi(w, u_{(m,i)}) &\leq \phi(w, y_{(m,i)})\\ &= \|w\|^2 + \|\alpha_{(m,i)} Jx_m + (1-\alpha_{(m,i)})JT_i^m x_m\|^2\\ &\quad - 2\langle w, \alpha_{(m,i)}Jx_m + (1-\alpha_{(m,i)})JT_i^m x_m\rangle\\ &\leq \|w\|^2 - 2(1-\alpha_{(m,i)})\langle w, JT_i^m x_m\rangle - 2\alpha_{(m,i)}\langle w, Jx_m\rangle\\ &\quad + (1-\alpha_{(m,i)})\|T_i^m x_m\|^2 + \alpha_{(m,i)}\|x_m\|^2\\ &\leq (1-\alpha_{(m,i)})\phi(w, x_m) + \alpha_{(m,i)}\phi(w, x_m)\\ &\quad + (1-\alpha_{(m,i)})\mu_{(m,i)}\phi(w, x_m)\\ &\leq \phi(w, x_m) + \mu_{(m,i)}\phi(w, x_m),\end{aligned}$$

which shows that $w \in C_{(m+1,i)}$. This completes the proof that $\cap_{i=1}^N \big(Sol(B_i) \cap Fix(T_i)\big) \subset C_{(n,i)}$. Hence, we find that $\cap_{i=1}^N \big(Sol(B_i) \cap Fix(T_i)\big) \subset \cap_{i=1}^N C_{(n,i)}$. From Lemma 1.3, one has $\langle z - x_n, Jx_1 - Jx_n\rangle \leq 0$, for any $z \in C_n$. Hence, we have

$$\langle w - x_n, Jx_1 - Jx_n\rangle \leq 0, \quad \forall w \in \cap_{i=1}^N \big(Sol(B_i) \cap Fix(T_i)\big). \tag{2.1}$$

By using Lemma 1.3, we have $\phi(x_n, x_1) \leq \phi(\Pi_{\cap_{i=1}^N \left(Sol(B_i)\cap Fix(T_i)\right)} x_1, x_1)$, which shows that $\{\phi(x_n, x_1)\}$ is a bounded sequence. Hence, $\{x_n\}$ is also a bounded sequence. Since the framework of E is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$. It follows that $\bar{x} \in C_n$. Therefore, $\phi(x_n, x_1) \leq \phi(\bar{x}, x_1)$. Since the norm function is a weakly lower semicontinuous function, we have

$$\phi(\bar{x}, x_1) \leq \liminf_{n\to\infty}(\|x_n\|^2 + \|x_1\|^2 - 2\langle x_n, Jx_1\rangle) = \liminf_{n\to\infty} \phi(x_n, x_1) \leq \phi(\bar{x}, x_1).$$

It follows that $\lim_{n\to\infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1)$. Hence, we have $\lim_{n\to\infty} \|x_n\| = \|\bar{x}\|$. By using the Kadec-Klee property of the spaces, one obtains that x_n converges strongly to $\bar{x}$ as $n \to \infty$. On the other hand, we find that $\phi(x_{n+1}, x_1) \geq \phi(x_n, x_1)$, which shows that $\{\phi(x_n, x_1)\}$ is a nondecreasing sequence. Therefore, one has $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. It follows that $\phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x_1) - \phi(x_n, x_1)$. Therefore, we have $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. On the other hand, since $x_{n+1} \in C_{n+1}$, one sees that

$$\phi(x_{n+1}, x_n) + \mu_{(n,i)}M_{(n,i)} \geq \phi(x_{n+1}, u_{(n,i)}) \geq 0.$$

This yields that $\lim_{n\to\infty} \phi(x_{n+1}, u_{(n,i)}) = 0$. Hence, one has $\lim_{n\to\infty}(\|u_{(n,i)}\| - \|x_{n+1}\|) = 0$. This implies that $\lim_{n\to\infty} \|u_{(n,i)}\| = \|\bar{x}\|$. That is,

$$\lim_{n\to\infty} \|Ju_{(n,i)}\| = \lim_{n\to\infty} \|u_{(n,i)}\| = \|\bar{x}\| = \|J\bar{x}\|.$$

This implies that $\{Ju_{(n,i)}\}$ is bounded. Assume that $Ju_{(n,i)}$ converges weakly to $u^{(*,i)} \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $u^i \in E$ such that $Ju^i = u^{(*,i)}$. It follows that

$$\phi(x_{n+1}, u_{(n,i)}) + 2\langle x_{n+1}, Ju_{(n,i)}\rangle = \|x_{n+1}\|^2 + \|Ju_{(n,i)}\|^2.$$

By taking $\liminf_{n\to\infty}$, one has

$$\phi(\bar{x}, u^i) = \|\bar{x}\|^2 + \|Ju^i\|^2 - 2\langle \bar{x}, Ju^i\rangle = \|\bar{x}\|^2 - 2\langle \bar{x}, u^{(*,i)}\rangle + \|u^{(*,i)}\|^2 \leq 0.$$

That is, $\bar{x} = u^i$, which in turn implies that $J\bar{x} = u^{(*,i)}$. Hence, $Ju_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* is uniformly convex. Hence, it has the Kadec-Klee property, we obtain that $J\bar{x} = \lim_{n\to\infty} Ju_{(n,i)}$. Since $J^{-1} : E^* \to E$

is demi-continuous and E has the Kadec-Klee property, one gets that $u_{(n,i)} \to \bar{x}$, as $n \to \infty$. This implies $\lim_{n\to\infty} \|x_n - u_{(n,i)}\| = 0$. Hence, we have $\lim_{n\to\infty}(\phi(w, u_{(n,i)}) - \phi(w, x_n)) = 0$. Since E^* is uniformly convex, we find from Lemma 1.2 that

$$\begin{aligned}
\phi(w, u_{(n,i)}) &\leq \|w\|^2 + \|\alpha_{(n,i)} Jx_n + (1-\alpha_{(n,i)})JT_i^n x_n\|^2 \\
&\quad - 2\langle w, (1-\alpha_{(n,i)})JT_i^n x_n + \alpha_{(n,i)} Jx_n\rangle \\
&\leq \|w\|^2 - 2(1-\alpha_{(n,i)})\langle w, JT_i^n x_n\rangle - 2\alpha_{(n,i)}\langle w, Jx_n\rangle \\
&\quad - \alpha_{(n,i)}(1-\alpha_{(n,i)})g(\|Jx_n - JT_i^n x_n\|) \\
&\quad + \alpha_{(n,i)}\|x_n\|^2 + (1-\alpha_{(n,i)})\|T_i^n x_n\|^2 \\
&\leq \phi(w, x_n) + (1-\alpha_{(n,i)})\mu_{(n,i)}\phi(w, x_n) - \alpha_{(n,i)}(1-\alpha_{(n,i)})g(\|Jx_n - JT_i^n x_n\|) \\
&\leq \phi(w, x_n) + \mu_{(n,i)}M_{(n,i)} - \alpha_{(n,i)}(1-\alpha_{(n,i)})g(\|Jx_n - JT_i^n x_n\|).
\end{aligned}$$

It follows that

$$\alpha_{(n,i)}(1-\alpha_{(n,i)})g(\|Jx_n - JT_i^n x_n\|) \leq \phi(w, x_n) + \mu_{(n,i)}M_{(n,i)} - \phi(w, u_{(n,i)}).$$

This yields from the restriction imposed on $\{\alpha_{(n,i)}\}$ that $\lim_{n\to\infty} \|Jx_n - JT_i^n x_n\| = 0$. Therefore, we have

$$\lim_{n\to\infty} \|J\bar{x} - JT_i^n x_n\| = 0.$$

Since $J^{-1} : E^* \to E$ is demi-continuous, one has $T_i^n x_n \rightharpoonup \bar{x}$. Hence, one has $\lim_{n\to\infty} \|T_i^n x_n\| = \|\bar{x}\|$. Since E has the Kadec-Klee property, we obtain

$$\lim_{n\to\infty} \|\bar{x} - T_i^n x_n\| = 0.$$

Since each T_i is uniformly asymptotically regular, one has $\lim_{n\to\infty} \|T_i^{n+1} x_n - \bar{x}\| = 0$. That is, $T_i(T_i^n x_n) \to \bar{x}$. Since T_i is a closed mapping, we find $\bar{x} = T_i\bar{x}$ for each $1 \leq i \leq N$. This proves $\bar{x} \in \cap_{i=1}^N Fix(T_i)$.

On the other hand, we have $\lim_{n\to\infty}(\|y_{(n,i)}\| - \|u_{(n,i)}\|) = 0$. Since $u_{(n,i)} \to \bar{x}$ as $n \to \infty$, we find that $\lim_{n\to\infty} \|Jy_{(n,i)}\| = \|J\bar{x}\|$. This shows that $\{Jy_{(n,i)}\}$ is bounded. Since E is uniformly smooth, one sees that E^* is reflexive. We may assume that $Jy_{(n,i)} \rightharpoonup y^{(*,i)} \in E^*$. There exists $y^i \in E$ such that $Jy^i = y^{(*,i)}$. It follows that

$$\|u_{(n,i)}\|^2 + \|Jy_{(n,i)}\|^2 = \phi(u_{(n,i)}, y_{(n,i)}) + 2\langle u_{(n,i)}, Jy_{(n,i)}\rangle.$$

Hence, we have

$$0 \leq \phi(\bar{x}, y^i) = \|\bar{x}\|^2 + \|y^i\|^2 - 2\langle \bar{x}, Jy^i\rangle = \|\bar{x}\|^2 + \|y^{(*,i)}\|^2 - 2\langle \bar{x}, y^{(*,i)}\rangle \leq 0.$$

That is, $\bar{x} = y^i$. Hence, we have $y^{(*,i)} = J\bar{x}$. It follows that $Jy_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* is uniformly convex, it has the Kadec-Klee property, we obtain that $Jy_{(n,i)} - J\bar{x} \to 0$ as $n \to \infty$. Since $J^{-1} : E^* \to E$ is demi-continuous, we see that $y_{(n,i)} \rightharpoonup \bar{x}$. By using the Kadec-Klee property, we obtain that $y_{(n,i)} \to \bar{x}$ as $n \to \infty$. Since E is uniformly smooth, $\lim_{n\to\infty} \|Jy_{(n,i)} - Ju_{(n,i)}\| = 0$. Since B_i is monotone, we find that

$$r_{(n,i)}B_i(y, u_{(n,i)}) \leq \|y - u_{(n,i)}\|\|Ju_{(n,i)} - Jy_{(n,i)}\|, \quad \forall y \in C_n.$$

Therefore, one sees $B_i(y, \bar{x}) \leq 0$ for all $y \in C$. For $y \in C$ and $0 < t_i < 1$, define $y_{(t,i)} = (1-t_i)\bar{x} + t_i y$. This implies that $0 \geq B_i(y_{(t,i)}, \bar{x})$. Hence, we have

$$0 = B_i(y_{(t,i)}, y_{(t,i)}) \leq t_i B_i(y_{(t,i)}, y).$$

It follows that $B_i(\bar{x}, y) \geq 0$ for all $y \in C$. This implies that $\bar{x} \in Sol(B_i)$ for every $1 \leq i \leq N$.

Finally, we prove $\bar{x} = \Pi_{\cap_{i=1}^N \left(Sol(B_i)\cap Fix(T_i)\right)} x_1$. By letting $n \to \infty$ in (2.1), we arrive at $\langle \bar{x} - w, Jx_1 - J\bar{x}\rangle \geq 0$ $w \in \cap_{i=1}^N \left(Sol(B_i) \cap Fix(T_i)\right)$. From Lemma 1.3, we find that $\bar{x} = \Pi_{\cap_{i=1}^N \left(Sol(B_i)\cap Fix(T_i)\right)} x_1$. This completes the proof. □

For the class of quasi-ϕ-nonexpansive mappings, we find the following result immediately.

Corollary 2.2. *Let E be a uniformly smooth and strictly convex Banach space. Let C be a convex and closed subset of E. Let N be some positive integer. Let B_i be a bifunction with restrictions* (B1), (B2), (B3), (B4) *and let $T_i : C \to C$ be a quasi-ϕ-nonexpansive mapping such that T_i is uniformly asymptotically regular and closed on C for each $1 \le i \le N$. Assume $\cap_{i=1}^N \big(Sol(B_i) \cap Fix(T_i)\big)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following process.*

$$\begin{cases} C_{(1,i)} = C, \forall 1 \le i \le N, \\ C_1 = \cap_{i=1}^N C_{(1,i)}, \\ x_1 = \Pi_{C_1} x_0, \\ y_{(n,i)} = J^{-1}\big((1-\alpha_{(n,i)})JT_i x_n + \alpha_{(n,i)} J x_n\big), \\ r_{(n,i)} B_i(u_{(n,i)}, y) + \langle u_{(n,i)} - y, Ju_{(n,i)} - Jy_{(n,i)} \rangle \le 0, \forall y \in C_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(z, u_{(n,i)}) \le \phi(z, x_n)\}, \\ C_{n+1} = \cap_{i=1}^N C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{r_{(n,i)}\}$ is a real sequence in $[r, \infty)$, where r is some positive real number and $\{\alpha_{(n,i)}\}$ is a real sequence in $[a, b]$, where $0 < a < b < 1$. If E has the Kadec-Klee property, then $\{x_n\}$ converges strongly to $\Pi_{\cap_{i=1}^N \left(Sol(B_i)\cap Fix(T_i)\right)} x_1$.

In the framework of Hilbert spaces, the generalized projection Π is reduced to the metric projection $Proj$, $\phi(x, y) = \|x - y\|^2$ and the class of asymptotically quasi-ϕ-nonexpansive mappings is reduced to the class of asymptotically quasi-nonexpansive mappings. From Theorem 2.1, we find the following results immediately.

Corollary 2.3. *Let E be a Hilbert space and let C be a convex and closed subset of E. Let N be some positive integer. Let B_i be a bifunction with restrictions* (B1), (B2), (B3), (B4) *and let $T_i : C \to C$ be an asymptotically quasi-nonexpansive mapping such that T_i is uniformly asymptotically regular and closed on C for each $1 \le i \le N$. Assume $\cap_{i=1}^N \big(Sol(B_i) \cap Fix(T_i)\big)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following process.*

$$\begin{cases} C_{(1,i)} = C, \quad \forall 1 \le i \le N, \\ C_1 = \cap_{i=1}^N C_{(1,i)}, \\ x_1 = Proj_{C_1} x_0, \\ y_{(n,i)} = (1-\alpha_{(n,i)}) T_i^n x_n + \alpha_{(n,i)} x_n, \\ r_{(n,i)} B_i(u_{(n,i)}, y) + \langle u_{(n,i)} - y, u_{(n,i)} - y_{(n,i)} \rangle \le 0, \quad \forall y \in C_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|u_{(n,i)} - z\|^2 \le \mu_{(n,i)} M_{(n,i)} + \|x_n - z\|^2\}, \\ C_{n+1} = \cap_{i=1}^N C_{(n+1,i)}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $M_{(n,i)} = \sup\{\|x_n - p\|^2 : p \in \cap_{i=1}^N \big(Sol(B_i) \cap Fix(T_i)\big)\}$, $\{r_{(n,i)}\}$ is a real sequence in $[r, \infty)$, where r is some positive real number and $\{\alpha_{(n,i)}\}$ is a real sequence in $[a, b]$, where $0 < a < b < 1$. Then $\{x_n\}$ converges strongly to $Proj_{\cap_{i=1}^N \left(Sol(B_i)\cap Fix(T_i)\right)} x_1$.

Finally, we give an application to variational inequalities. Let $A : C \to E^*$ be a single-valued monotone operator which is continuous along each line segment in C with respect to the weak* topology of E^*. Recall the following variational inequality. Find a point $x \in C$ such that $\langle x - y, Ax \rangle \le 0$ for all $y \in C$. The symbol $N_C(x)$ stands for the normal cone for C at a point $x \in C$; that is, $N_C(x) = \{x^* \in E^* : \langle x - y, x^* \rangle \ge 0, \ \forall y \in C\}$. Next, we use $VI(C, A)$ to denote the solution set of the variational inequality.

Corollary 2.4. *Let E be a uniformly smooth and strictly convex Banach space. Let C be a convex and closed subset of E. Let N be some positive integer. Let $A_i : C \to E^*$ be a single-valued, monotone and hemicontinuous operator and let B_i be a bifunction satisfying* (B1), (B2), (B3) *and* (B4) *for each $1 \le i \le N$. Assume that $\cap_{i=1}^N \big(Sol(B_i) \cap VI(C, A_i)\big)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following process. $x_0 \in E$ chosen arbitrarily and*

$$\begin{cases} C_{1,i} = C, \quad \forall 1 \le i \le N, \\ C_1 = \cap_{i=1}^N C_{(1,i)}, \\ x_1 = \Pi_{C_1} x_0, \\ z_{(n,i)} = VI(C, A_i + \frac{1}{r_i}(J - Jx_n)), \\ Jy_{(n,i)} = (1 - \alpha_{(n,i)}) Jz_{n,i} + \alpha_{(n,i)} Jx_n, \quad n \ge 1, \\ r_{(n,i)} B_i(u_{(n,i)}, y) + \langle y - u_{(n,i)}, Ju_{(n,i)} - Jy_{(n,i)} \rangle \ge 0, \quad \forall y \in C_n, \\ C_{(n+1,i)} = \{w \in C_{(n,i)} : \phi(w, u_{n,i}) \le \phi(w, x_n)\}, \\ C_{n+1} = \cap_{i \in \Delta} C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{cases}$$

where $\{r_{(n,i)}\}$ is a real sequence in $[r, \infty)$, where r is some positive real number and $\{\alpha_{(n,i)}\}$ is a real sequence in $[a, b]$, where $0 < a < b < 1$. Then $\{x_n\}$ converges strongly to $\Pi_{\cap_{i=1}^N \left(Sol(B_i) \cap VI(C,A_i)\right)} x_1$.

Proof. For each $1 \le i \le N$, define a mapping W_i by

$$W_i x = \begin{cases} \emptyset, & x \notin C, \\ A_i x + N_C x, & x \in C. \end{cases}$$

From Rockafellar [18], we see that W_i is a maximal monotone operator with $VI(C, A_i) = W_i^{-1}(0)$. For each $r_i > 0$, and $x \in E$, we see that there exists a unique x_{r_i} in the domain of W_i such that $Jx \in Jx_{r_i} + r_i T_i(x_{r_i})$, where $x_{r_i} = (J + r_i W_i)^{-1} Jx$. For each $1 \le i \le N$,

$$z_{n,i} = VI(C, \frac{1}{r_i}(J - Jx_n) + A_i),$$

which is equivalent to

$$\langle z_{n,i} - y, A_i z_{n,i} + \frac{1}{r_i}(Jz_{n,i} - Jx_n) \rangle \le 0, \quad \forall y \in C,$$

that is,

$$\frac{1}{r_i}\big(Jx_n - Jz_{n,i}\big) \in N_C(z_{n,i}) + A_i z_{n,i}.$$

This implies that $z_{n,i} = (J + r_i T_i)^{-1} Jx_n$. Since $(J + r_i T_i)^{-1} J$ is closed quasi-ϕ-nonexpansive with $Fix((J + r_i T_i)^{-1} J) = T_i^{-1}(0)$, by using Theorem 2.1, we find the desired conclusion immediately. □

Remark 2.5. To construct a mathematical model which is as close as possible to a real world problem, we often have to use more than one constraint. Solving such real world problems, we have to obtain some solution which is simultaneously the solution of two or more subproblems. In this paper, we study a hybrid algorithm for finding a common solution of a finite family of equilibrium problems which is also a common fixed point of a finite family of asymptotically quasi-ϕ-nonexpansive mappings. Strong convergence theorems are established without any compact assumption in a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property.

Acknowledgment

The authors are grateful to the reviewers for useful suggestions which improved the contents of this article. The first author was supported by the Youth Foundation of Linyi University (LYDX2016BS023) and the AMEP of Linyi University, China.

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