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# Oscillation criteria for third-order nonlinear neutral differential equations with distributed deviating arguments

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**Abstract**

The aim of this paper is to investigate the oscillation and asymptotic behavior of a class of third-order nonlinear neutral differential equations with distributed deviating arguments. By means of Riccati transformation technique and some inequalities, we establish several sufficient conditions which ensure that every solution of the studied equation is either oscillatory or converges to zero. Two examples are provided to illustrate the main results. 

*Keywords:* Oscillation, asymptotic behavior, third-order neutral differential equation, distributed deviating argument.
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## 1. Introduction

During the past few decades, an increasing interest in obtaining sufficient conditions for oscillatory and nonoscillatory behavior of different classes of differential equations has been stimulated due to their applications in natural sciences and engineering (see Hale [9] and Wong [24]). This resulted in publication of several monographs [1, 11], numerous research papers [2–6, 8, 10, 12–23, 25, 26], and the references cited

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therein. Analysis of qualitative properties of neutral differential equations is important not only for the sake of further development of the oscillation theory, but for practical reasons too. In fact, neutral differential equations are used in modeling of networks containing lossless transmission lines (see, for instance, the paper by Driver [7]).

In what follows, let us present some background details which motivate our study. Assuming that

$$0 \le p(t) \le p_0 < \infty, \tag{1.1}$$

the oscillation of second-order neutral differential equation

$$[r(t)(x(t) + p(t)x[\tau(t)])']' + q(t)f(x(\sigma(t))) = 0,$$

and its particular cases were investigated by Baculíková and Džurina [4, 5], Fišnarová and Mařík [8], Li and Rogovchenko [14, 15], Li et al. [16], and Xing et al. [25]. For the oscillation of second-order neutral differential equations with distributed deviating arguments, Li et al. [12] and Li and Thandapani [17] established several oscillation criteria for

$$(r(t)|z'(t)|^{\alpha-1}z'(t))' + \int_a^b q(t,\xi)\,|x[g(t,\xi)]|^{\alpha-1}\,x[g(t,\xi)]\mathrm{d}\sigma(\xi) = 0,$$

where $z(t) := x(t) + p(t)x[\tau(t)]$. Compared with second-order neutral differential equations, there are few oscillation results for third-order neutral differential equations. Baculíková and Džurina [2, 3], Jiang and Li [10], and Li et al. [18] examined an equation of the form

$$(r(t)((x(t) + p(t)x(\tau(t)))'')^{\gamma})' + q(t)x^{\gamma}(\sigma(t)) = 0, \tag{1.2}$$

under the assumption that

$$0 \le p(t) \le p_0 < 1,$$

whereas Li and Rogovchenko [14], Thandapani and Li [20], and Xing et al. [25] deduced oscillation of (1.2) assuming that condition (1.1) is satisfied.

On the basis of the ideas exploited by Li et al. [12, 18], the objective of this paper is to establish several oscillation criteria for

$$(r(t)|z''(t)|^{\alpha-1}z''(t))' + \int_a^b q(t,\xi)\,|x[g(t,\xi)]|^{\alpha-1}\,x[g(t,\xi)]\mathrm{d}\sigma(\xi) = 0, \tag{1.3}$$

where $t \ge t_0 > 0$, $\alpha > 0$ is a constant, and $z(t) := x(t) + p(t)x[\tau(t)]$. As usual, a solution $x$ of (1.3) is called oscillatory, if the set of its zeros is unbounded from above, otherwise, it is said to be nonoscillatory.

In order to accomplish these tasks, it is necessary to make the following assumptions hold throughout this paper:

($A_1$) $r \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ and $p \in \mathrm{C}([t_0,\infty),[0,\infty))$;

($A_2$) $q \in \mathrm{C}([t_0,\infty)\times[a,b],[0,\infty))$ and $q(t,\xi)$ is not eventually zero on any $[t_\mu,\infty)\times[a,b]$, $t_\mu \in [t_0,\infty)$;

($A_3$) $g \in \mathrm{C}([t_0,\infty)\times[a,b],\mathbb{R})$ is a nondecreasing function for $\xi$ satisfying $\liminf_{t\to\infty} g(t,\xi) = \infty$ for $\xi \in [a,b]$;

($A_4$) $\tau \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$, $\tau'(t) \ge \tau_0 > 0$, $\lim_{t\to\infty}\tau(t) = \infty$, and $g(\tau(t),\xi) = \tau[g(t,\xi)]$;

($A_5$) $\sigma \in \mathrm{C}([a,b],\mathbb{R})$ is nondecreasing and the integral of (1.3) is taken in the sense of Riemann–Stieltjes.

Main results in this paper are organized into two parts in accordance with different assumptions on the coefficient $r$. In Section 2, oscillation results for (1.3) are established in the case where

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\mathrm{d}t = \infty. \tag{1.4}$$

By assuming that

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\mathrm{d}t < \infty, \tag{1.5}$$

oscillation criteria for (1.3) are obtained in Section 3. To illustrate the results reported in Sections 2 and 3, we give two examples in Section 4.

In the sequel, we use the following notations for a compact presentation of our results:

$$Q(t,\xi) := \min\{q(t,\xi), q(\tau(t),\xi)\}, \qquad R(t) := \max\{r(t), r[\tau(t)]\},$$

$$\rho'_+(t) := \max\{0, \rho'(t)\}, \qquad \vartheta(t) := \int_{\zeta(t)}^{\infty} r^{-1/\alpha}(s)\mathrm{d}s,$$

where $\rho$ and $\zeta$ will be explained later, and all functional inequalities are tacitly assumed to hold for all $t$ large enough, unless mentioned otherwise.

## 2. Oscillation criteria for the case (1.4)

In this section, we consider two cases $g(t,a) \le \tau(t)$ and $g(t,a) \ge \tau(t)$. Let us start with the first case.

**Theorem 2.1.** *Let conditions* $(\mathrm{A}_1)$-$(\mathrm{A}_5)$, (1.1), (1.4), *and* $\alpha \ge 1$ *be satisfied. Suppose that* $g(t,a) \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$, $g'(t,a) > 0$, $g(t,a) \le t$, *and* $g(t,a) \le \tau(t)$ *for* $t \ge t_0$. *If there exists a function* $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that, for all sufficiently large* $t_1 \ge t_0$ *and for some* $t_2 > t_1$,

$$\int^{\infty}\left[2^{1-\alpha}\rho(t)G_1(t)\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi) - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{r[g(t,a)](\rho'_+(t))^{\alpha+1}}{(\rho(t)g'(t,a))^{\alpha}}\right]\mathrm{d}t = \infty \tag{2.1}$$

*and*

$$\int^{\infty} u\left[\frac{1}{R(u)}\int_u^{\infty}\int_a^b Q(s,\xi)\mathrm{d}\sigma(\xi)\mathrm{d}s\right]^{1/\alpha}\mathrm{d}u = \infty, \tag{2.2}$$

*where*

$$G_1(t) := \left(\frac{\int_{t_2}^{g(t,a)}\int_{t_1}^{s} r^{-1/\alpha}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{g(t,a)} r^{-1/\alpha}(u)\mathrm{d}u}\right)^{\alpha}, \tag{2.3}$$

*then every solution* $x$ *of* (1.3) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

*Proof.* Assume that (1.3) has a nonoscillatory solution $x$. Without loss of generality, we may suppose that there exists a $t_1 \ge t_0$ such that $x(t) > 0$, $x[\tau(t)] > 0$ for $t \ge t_1$, and $x[g(t,\xi)] > 0$ for $(t,\xi) \in [t_1,\infty) \times [a,b]$. Then we have $z > 0$. It follows from (1.3) that

$$(r(t)|z''(t)|^{\alpha-1}z''(t))' \le 0, \tag{2.4}$$

and

$$(r(t)|z''(t)|^{\alpha-1}z''(t))' + \int_a^b q(t,\xi)x^{\alpha}[g(t,\xi)]\mathrm{d}\sigma(\xi) + \frac{p_0^{\alpha}}{\tau'(t)}(r[\tau(t)]|z''[\tau(t)]|^{\alpha-1}z''[\tau(t)])' + \int_a^b p_0^{\alpha}q(\tau(t),\xi)x^{\alpha}[g(\tau(t),\xi)]\mathrm{d}\sigma(\xi) = 0.$$

By virtue of (2.4) and $\tau'(t) \ge \tau_0 > 0$, we obtain

$$(r(t)|z''(t)|^{\alpha-1}z''(t))' + \int_a^b q(t,\xi)x^{\alpha}[g(t,\xi)]\mathrm{d}\sigma(\xi) + \int_a^b p_0^{\alpha}q(\tau(t),\xi)x^{\alpha}[g(\tau(t),\xi)]\mathrm{d}\sigma(\xi) + \frac{p_0^{\alpha}}{\tau_0}(r[\tau(t)]|z''[\tau(t)]|^{\alpha-1}z''[\tau(t)])' \le 0.$$

By using the latter inequality and condition $g(\tau(t),\xi) = \tau[g(t,\xi)]$, we have

$$\begin{aligned}(r(t)|z''(t)|^{\alpha-1}z''(t))' &+ \frac{p_0^\alpha}{\tau_0}(r[\tau(t)]|z''[\tau(t)]|^{\alpha-1}z''[\tau(t)])' \\ &\le -\int_a^b (q(t,\xi)x^\alpha[g(t,\xi)] + p_0^\alpha q(\tau(t),\xi)x^\alpha[g(\tau(t),\xi)])\,\mathrm{d}\sigma(\xi) \\ &\le -\int_a^b Q(t,\xi)\,(x^\alpha[g(t,\xi)] + p_0^\alpha x^\alpha[\tau(g(t,\xi))])\,\mathrm{d}\sigma(\xi). \qquad (2.5)\end{aligned}$$

In view of $0 \le p(t) \le p_0 < \infty$ and the inequality (see [5, Lemma 1])

$$A^\alpha + B^\alpha \ge \frac{1}{2^{\alpha-1}}(A+B)^\alpha \quad \text{for } A \ge 0,\ B \ge 0,\ \text{and } \alpha \ge 1,$$

we arrive at

$$x^\alpha[g(t,\xi)] + p_0^\alpha x^\alpha[\tau(g(t,\xi))] \ge \frac{(x[g(t,\xi)] + p_0 x[\tau(g(t,\xi))])^\alpha}{2^{\alpha-1}} \ge \frac{z^\alpha[g(t,\xi)]}{2^{\alpha-1}}. \qquad (2.6)$$

By combining (2.5) and (2.6), we conclude that

$$(r(t)|z''(t)|^{\alpha-1}z''(t))' + \frac{p_0^\alpha}{\tau_0}(r[\tau(t)]|z''[\tau(t)]|^{\alpha-1}z''[\tau(t)])' \le -\frac{1}{2^{\alpha-1}}\int_a^b Q(t,\xi)z^\alpha[g(t,\xi)]\mathrm{d}\sigma(\xi). \qquad (2.7)$$

Based on condition (1.4), $z$ satisfies two possible cases:

(I) $z > 0$, $z' > 0$, $z'' > 0$, and $(r|z''|^{\alpha-1}z'')' \le 0$;

(II) $z > 0$, $z' < 0$, $z'' > 0$, and $(r|z''|^{\alpha-1}z'')' \le 0$.

Assume first that case (I) holds. By using $z' > 0$, $z'' > 0$, and the fact that $g(t,\xi)$ is a nondecreasing function for $\xi \in [a,b]$, we have by (2.7) that

$$(r(t)(z''(t))^\alpha)' + \frac{p_0^\alpha}{\tau_0}(r[\tau(t)](z''[\tau(t)])^\alpha)' \le -\frac{z^\alpha[g(t,a)]}{2^{\alpha-1}}\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi). \qquad (2.8)$$

Define a Riccati transformation $\omega$ by

$$\omega(t) := \rho(t)\frac{r(t)(z''(t))^\alpha}{(z'[g(t,a)])^\alpha}, \quad t \ge t_1. \qquad (2.9)$$

Clearly, $\omega > 0$ and

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'[g(t,a)])^\alpha} - \alpha r(t)\rho(t)g'(t,a)\frac{(z''(t))^\alpha z''[g(t,a)]}{(z'[g(t,a)])^{\alpha+1}}.$$

Applying the monotonicity of $r|z''|^{\alpha-1}z''$ and $g(t,a) \le t$ implies that

$$z''[g(t,a)] \ge \left(\frac{r(t)}{r[g(t,a)]}\right)^{1/\alpha} z''(t).$$

Then, combining the latter inequality and (2.9), we conclude that

$$\omega'(t) \le \frac{\rho'_+(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'[g(t,a)])^\alpha} - \frac{\alpha g'(t,a)}{(\rho(t)r[g(t,a)])^{1/\alpha}}\omega^{(\alpha+1)/\alpha}(t). \qquad (2.10)$$

Furthermore, we define another function $\nu$ by

$$\nu(t) := \rho(t)\frac{r[\tau(t)](z''[\tau(t)])^\alpha}{(z'[g(t,a)])^\alpha}, \quad t \ge t_1. \qquad (2.11)$$

Thus, we have $\nu > 0$ and

$$\nu'(t) = \frac{\rho'(t)}{\rho(t)}\nu(t) + \rho(t)\frac{(r[\tau(t)](z''[\tau(t)])^\alpha)'}{(z'[g(t,a)])^\alpha} - \alpha\rho(t)g'(t,a)r[\tau(t)]\frac{(z''[\tau(t)])^\alpha z''[g(t,a)]}{(z'[g(t,a)])^{\alpha+1}}.$$

We derive from the monotonicity of $r|z''|^{\alpha-1}z''$ and $g(t,a) \le \tau(t)$ that

$$z''[g(t,a)] \ge \left(\frac{r[\tau(t)]}{r[g(t,a)]}\right)^{1/\alpha} z''[\tau(t)].$$

By using the latter inequality and (2.11), we deduce that

$$\nu'(t) \le \frac{\rho'_+(t)}{\rho(t)}\nu(t) + \rho(t)\frac{(r[\tau(t)](z''[\tau(t)])^\alpha)'}{(z'[g(t,a)])^\alpha} - \frac{\alpha g'(t,a)}{(\rho(t)r[g(t,a)])^{1/\alpha}}\nu^{(\alpha+1)/\alpha}(t). \tag{2.12}$$

It follows from (2.10) and (2.12) that

$$\begin{aligned}\omega'(t) + \frac{p_0^\alpha}{\tau_0}\nu'(t) \le \rho(t)&\left[\frac{(r(t)(z''(t))^\alpha)'}{(z'[g(t,a)])^\alpha} + \frac{p_0^\alpha}{\tau_0}\frac{(r[\tau(t)](z''[\tau(t)])^\alpha)'}{(z'[g(t,a)])^\alpha}\right]\\ &+ \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \frac{\alpha g'(t,a)}{(\rho(t)r[g(t,a)])^{1/\alpha}}\omega^{(\alpha+1)/\alpha}(t)\\ &+ \frac{p_0^\alpha}{\tau_0}\left[\frac{\rho'_+(t)}{\rho(t)}\nu(t) - \frac{\alpha g'(t,a)}{(\rho(t)r[g(t,a)])^{1/\alpha}}\nu^{(\alpha+1)/\alpha}(t)\right].\end{aligned}$$

Let

$$C := \frac{\rho'_+(t)}{\rho(t)} \quad \text{and} \quad D := \frac{\alpha g'(t,a)}{(\rho(t)r[g(t,a)])^{1/\alpha}}.$$

By using the inequality (see [13])

$$Cu - Du^{(\alpha+1)/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{C^{\alpha+1}}{D^\alpha}, \quad D > 0, \tag{2.13}$$

we conclude that

$$\frac{\rho'_+(t)}{\rho(t)}\omega(t) - \frac{\alpha g'(t,a)}{(\rho(t)r[g(t,a)])^{1/\alpha}}\omega^{(\alpha+1)/\alpha}(t) \le \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[g(t,a)](\rho'_+(t))^{\alpha+1}}{(\rho(t)g'(t,a))^\alpha}$$

and

$$\frac{\rho'_+(t)}{\rho(t)}\nu(t) - \frac{\alpha g'(t,a)}{(\rho(t)r[g(t,a)])^{1/\alpha}}\nu^{(\alpha+1)/\alpha}(t) \le \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[g(t,a)](\rho'_+(t))^{\alpha+1}}{(\rho(t)g'(t,a))^\alpha}.$$

By combining the latter inequalities and (2.8), we obtain

$$\begin{aligned}\omega'(t) + \frac{p_0^\alpha}{\tau_0}\nu'(t) \le -&\frac{\rho(t)}{2^{\alpha-1}}\left(\frac{z[g(t,a)]}{z'[g(t,a)]}\right)^\alpha\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi)\\ &+ \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^\alpha}{\tau_0}\right)\frac{r[g(t,a)](\rho'_+(t))^{\alpha+1}}{(\rho(t)g'(t,a))^\alpha}.\end{aligned} \tag{2.14}$$

By virtue of $(r(z'')^\alpha)' \le 0$,

$$z'(t) = z'(t_1) + \int_{t_1}^t z''(s)\mathrm{d}s = z'(t_1) + \int_{t_1}^t \frac{(r(s)(z''(s))^\alpha)^{1/\alpha}}{r^{1/\alpha}(s)}\mathrm{d}s \ge r^{1/\alpha}(t)z''(t)\int_{t_1}^t r^{-1/\alpha}(s)\mathrm{d}s.$$

Hence, we get

$$\left(\frac{z'(t)}{\int_{t_1}^{t} r^{-1/\alpha}(s)\mathrm{d}s}\right)' \leq 0,$$

which implies that, for $t \geq t_2 > t_1$,

$$\begin{aligned} z(t) &= z(t_2) + \int_{t_2}^{t} z'(s)\mathrm{d}s = z(t_2) + \int_{t_2}^{t} \frac{z'(s)}{\int_{t_1}^{s} r^{-1/\alpha}(u)\mathrm{d}u} \int_{t_1}^{s} r^{-1/\alpha}(u)\mathrm{d}u\mathrm{d}s \\ &\geq \frac{z'(t)}{\int_{t_1}^{t} r^{-1/\alpha}(u)\mathrm{d}u} \int_{t_2}^{t}\int_{t_1}^{s} r^{-1/\alpha}(u)\mathrm{d}u\mathrm{d}s, \end{aligned}$$

and so

$$\frac{z(t)}{z'(t)} \geq \frac{\int_{t_2}^{t}\int_{t_1}^{s} r^{-1/\alpha}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t} r^{-1/\alpha}(u)\mathrm{d}u}. \tag{2.15}$$

Then, we have

$$\left(\frac{z[g(t,a)]}{z'[g(t,a)]}\right)^{\alpha} \geq G_1(t),$$

where $G_1$ is defined by (2.3). Substitution of this inequality into (2.14) yields

$$\begin{aligned} \omega'(t) + \frac{p_0^{\alpha}}{\tau_0}\nu'(t) \leq& -2^{1-\alpha}\rho(t)G_1(t)\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi) \\ &+ \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{r[g(t,a)](\rho'_+(t))^{\alpha+1}}{(\rho(t)g'(t,a))^{\alpha}}. \end{aligned}$$

Integrating the latter inequality from $t_3$ ($t_3 > t_2$) to $t$, we conclude that

$$\int_{t_3}^{t}\left[\frac{\rho(s)G_1(s)}{2^{\alpha-1}}\int_a^b Q(s,\xi)\mathrm{d}\sigma(\xi) - \frac{1+p_0^{\alpha}/\tau_0}{(\alpha+1)^{\alpha+1}}\frac{r[g(s,a)](\rho'_+(s))^{\alpha+1}}{(\rho(s)g'(s,a))^{\alpha}}\right]\mathrm{d}s \leq \omega(t_3) + \frac{p_0^{\alpha}}{\tau_0}\nu(t_3),$$

which contradicts (2.1).

Assume now that case (II) holds. On the basis of the monotonicities of $z$ and $g(t,\xi)$, we have $z[g(t,\xi)] \geq z[g(t,b)]$. By taking into account that $z'' > 0$, inequality (2.7) becomes

$$(r(t)(z''(t))^{\alpha})' + \frac{p_0^{\alpha}}{\tau_0}(r[\tau(t)](z''[\tau(t)])^{\alpha})' \leq -\frac{z^{\alpha}[g(t,b)]}{2^{\alpha-1}}\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi).$$

By using a similar proof of [14, Theorem 15], we can obtain $\lim_{t\to\infty} x(t) = 0$ when using (2.2). This completes the proof. □

Now, we turn our attention to the case when $g(t,a) \geq \tau(t)$.

**Theorem 2.2.** *Let conditions* ($A_1$)-($A_5$), (1.1), (1.4), (2.2), *and* $\alpha \geq 1$ *be satisfied. Suppose that* $\tau(t) \leq t$ *and* $g(t,a) \geq \tau(t)$ *for* $t \geq t_0$. *If there exists a function* $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that for all sufficiently large* $t_1 \geq t_0$ *and for some* $t_2 > t_1$,

$$\int^{\infty}\left[2^{1-\alpha}\rho(t)G_2(t)\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi) - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{r[\tau(t)](\rho'_+(t))^{\alpha+1}}{(\tau_0\rho(t))^{\alpha}}\right]\mathrm{d}t = \infty, \tag{2.16}$$

*where*

$$G_2(t) := \left(\frac{\int_{t_2}^{\tau(t)}\int_{t_1}^{s} r^{-1/\alpha}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{\tau(t)} r^{-1/\alpha}(u)\mathrm{d}u}\right)^{\alpha}, \tag{2.17}$$

*then the conclusion of Theorem* 2.1 *remains intact.*

*Proof.* Assume that (1.3) has a nonoscillatory solution $x$. Without loss of generality, we may suppose that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x[\tau(t)] > 0$ for $t \geq t_1$, and $x[g(t,\xi)] > 0$ for $(t,\xi) \in [t_1,\infty) \times [a,b]$. As in the proof of Theorem 2.1, we have (2.4), (2.7), and two possible cases (I) and (II) (as those in the proof of Theorem 2.1) for $z$.

Assume first that case (I) holds. It follows from $g(t,\xi) \geq g(t,a) \geq \tau(t)$, $z' > 0$, and $z'' > 0$ that

$$(r(t)(z''(t))^\alpha)' + \frac{p_0^\alpha}{\tau_0}(r[\tau(t)](z''[\tau(t)])^\alpha)' \leq -\frac{z^\alpha[\tau(t)]}{2^{\alpha-1}}\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi). \tag{2.18}$$

Define a Riccati transformation $\omega$ by

$$\omega(t) := \rho(t)\frac{r(t)(z''(t))^\alpha}{(z'[\tau(t)])^\alpha}, \quad t \geq t_1. \tag{2.19}$$

Then $\omega > 0$. Applying (2.4) and $\tau(t) \leq t$ yields

$$z''[\tau(t)] \geq \left(\frac{r(t)}{r[\tau(t)]}\right)^{1/\alpha} z''(t).$$

By differentiating (2.19), we conclude that

$$\begin{aligned}
\omega'(t) &= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'[\tau(t)])^\alpha} - \alpha r(t)\rho(t)\tau'(t)\frac{(z''(t))^\alpha z''[\tau(t)]}{(z'[\tau(t)])^{\alpha+1}} \\
&\leq \frac{\rho'_+(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'[\tau(t)])^\alpha} - \alpha\rho(t)\tau'(t)\frac{r^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}[\tau(t)]}\left(\frac{z''(t)}{z'[\tau(t)]}\right)^{\alpha+1} \\
&= \frac{\rho'_+(t)}{\rho(t)}\omega(t) + \rho(t)\frac{(r(t)(z''(t))^\alpha)'}{(z'[\tau(t)])^\alpha} - \frac{\alpha\tau'(t)}{(\rho(t)r[\tau(t)])^{1/\alpha}}\omega^{(\alpha+1)/\alpha}(t).
\end{aligned} \tag{2.20}$$

Similarly, define another Riccati transformation $\nu$ by

$$\nu(t) := \rho(t)\frac{r[\tau(t)](z''[\tau(t)])^\alpha}{(z'[\tau(t)])^\alpha}, \quad t \geq t_1.$$

Clearly, $\nu > 0$ and

$$\begin{aligned}
\nu'(t) &= \frac{\rho'(t)}{\rho(t)}\nu(t) + \rho(t)\frac{(r[\tau(t)](z''[\tau(t)])^\alpha)'}{(z'[\tau(t)])^\alpha} - \alpha\rho(t)\tau'(t)r[\tau(t)]\left(\frac{z''[\tau(t)]}{z'[\tau(t)]}\right)^{\alpha+1} \\
&\leq \frac{\rho'_+(t)}{\rho(t)}\nu(t) + \rho(t)\frac{(r[\tau(t)](z''[\tau(t)])^\alpha)'}{(z'[\tau(t)])^\alpha} - \frac{\alpha\tau'(t)}{(\rho(t)r[\tau(t)])^{1/\alpha}}\nu^{(\alpha+1)/\alpha}(t).
\end{aligned} \tag{2.21}$$

In view of (2.20) and (2.21), we get

$$\begin{aligned}
\omega'(t) + \frac{p_0^\alpha}{\tau_0}\nu'(t) \leq \rho(t)&\left[\frac{(r(t)(z''(t))^\alpha)'}{(z'[\tau(t)])^\alpha} + \frac{p_0^\alpha}{\tau_0}\frac{(r[\tau(t)](z''[\tau(t)])^\alpha)'}{(z'[\tau(t)])^\alpha}\right] \\
&+ \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \frac{\alpha\tau'(t)}{(\rho(t)r[\tau(t)])^{1/\alpha}}\omega^{(\alpha+1)/\alpha}(t) \\
&+ \frac{p_0^\alpha}{\tau_0}\left[\frac{\rho'_+(t)}{\rho(t)}\nu(t) - \frac{\alpha\tau'(t)}{(\rho(t)r[\tau(t)])^{1/\alpha}}\nu^{(\alpha+1)/\alpha}(t)\right].
\end{aligned}$$

Set

$$C := \frac{\rho'_+(t)}{\rho(t)} \quad \text{and} \quad D := \frac{\alpha\tau'(t)}{(\rho(t)r[\tau(t)])^{1/\alpha}}.$$

By virtue of (2.13) and (2.18), we have

$$\omega'(t)+\frac{p_0^\alpha}{\tau_0}\nu'(t)\le -\frac{\rho(t)}{2^{\alpha-1}}\left(\frac{z[\tau(t)]}{z'[\tau(t)]}\right)^\alpha\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi)+\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^\alpha}{\tau_0}\right)\frac{r[\tau(t)](\rho'_+(t))^{\alpha+1}}{(\rho(t)\tau'(t))^\alpha}.$$

Similarly, as in the proof of Theorem 2.1, we obtain (2.15), and hence

$$\left(\frac{z[\tau(t)]}{z'[\tau(t)]}\right)^\alpha\ge G_2(t),$$

where $G_2$ is defined as in (2.17). Therefore,

$$\omega'(t)+\frac{p_0^\alpha}{\tau_0}\nu'(t)\le -\frac{\rho(t)G_2(t)}{2^{\alpha-1}}\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi)+\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^\alpha}{\tau_0}\right)\frac{r[\tau(t)](\rho'_+(t))^{\alpha+1}}{(\tau_0\rho(t))^\alpha}.$$

By integrating the latter inequality from $t_3$ ($t_3>t_2$) to $t$, we have

$$\int_{t_3}^t\left[\frac{\rho(s)G_2(s)}{2^{\alpha-1}}\int_a^b Q(s,\xi)\mathrm{d}\sigma(\xi)-\frac{1+p_0^\alpha/\tau_0}{(\alpha+1)^{\alpha+1}}\frac{r[\tau(s)](\rho'_+(s))^{\alpha+1}}{(\tau_0\rho(s))^\alpha}\right]\mathrm{d}s\le\omega(t_3)+\frac{p_0^\alpha}{\tau_0}\nu(t_3),$$

which contradicts (2.16).

Assume now that case (II) holds. As in the proof of Case (II) in Theorem 2.1, we arrive at the desired conclusion. The proof is complete. □

## 3. Oscillation criteria for the case (1.5)

In this section, we establish some oscillation criteria for (1.3) under the assumption that (1.5) holds. Similarly, as in Section 2, we begin with the case when $g(t,a)\le\tau(t)$ holds.

**Theorem 3.1.** *Let conditions* $(A_1)$-$(A_5)$, (1.1), (1.5), (2.2), *and* $\alpha\ge1$ *be satisfied. Suppose that* $g(t,a)\in C^1([t_0,\infty),\mathbb{R})$, $g'(t,a)>0$, *and* $g(t,a)\le\tau(t)\le t$ *for* $t\ge t_0$. *Assume further that there exists a function* $\rho\in C^1([t_0,\infty),(0,\infty))$ *such that* (2.1) *holds for all sufficiently large* $t_1\ge t_0$ *and for some* $t_2>t_1$. *If there exists a function* $\zeta\in C^1([t_0,\infty),\mathbb{R})$ *such that,* $\zeta(t)\ge t$, $\zeta(t)\ge g(t,a)$, $\zeta'(t)>0$ *for* $t\ge t_0$, *and for all sufficiently large* $t_1\ge t_0$,

$$\int^\infty\left[2^{1-\alpha}\vartheta^\alpha(t)G_3(t)\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(1+\frac{p_0^\alpha}{\tau_0}\right)\frac{\zeta'(t)}{\vartheta(t)r^{1/\alpha}[\zeta(t)]}\right]\mathrm{d}t=\infty,\tag{3.1}$$

*where*

$$G_3(t):=(g(t,a)-t_1)^\alpha,\tag{3.2}$$

*then every solution* $x$ *of* (1.3) *is either oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

*Proof.* Assume that (1.3) has a nonoscillatory solution $x$. Without loss of generality, we may suppose that there exists a $t_1\ge t_0$ such that $x(t)>0$, $x[\tau(t)]>0$ for $t\ge t_1$, and $x[g(t,\xi)]>0$ for $(t,\xi)\in[t_1,\infty)\times[a,b]$. Then we have $z>0$. Based on condition (1.5), there exist three possible cases (I), (II) (as those in the proof of Theorem 2.1), and

(III) $z>0$, $z'>0$, $z''<0$, and $(r|z''|^{\alpha-1}z'')'\le0$.

Assume that case (I) and case (II) hold. By using the proof of Theorem 2.1, we get the conclusion of Theorem 3.1.

Assume now that case (III) holds. In view of $g(t,\xi)\ge g(t,a)$, $z'>0$, and $z''<0$, inequality (2.7) reduces to

$$(-r(t)(-z''(t))^\alpha)'+\frac{p_0^\alpha}{\tau_0}(-r[\tau(t)](-z''[\tau(t)])^\alpha)'\le-\frac{z^\alpha[g(t,a)]}{2^{\alpha-1}}\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi).\tag{3.3}$$

From $(r|z''|^{\alpha-1}z'')' \leq 0$, we have $(r(-z'')^{\alpha})' \geq 0$, which shows that $r(-z'')^{\alpha}$ is nondecreasing. Thus, we obtain

$$z''(s) \leq \frac{r^{1/\alpha}(t)}{r^{1/\alpha}(s)} z''(t), \quad s \geq t \geq t_1.$$

An integration from $\zeta(t)$ to $l$ yields

$$z'(l) \leq z'[\zeta(t)] + r^{1/\alpha}(t) z''(t) \int_{\zeta(t)}^{l} r^{-1/\alpha}(s) \mathrm{d}s.$$

By passing to the limit as $l \to \infty$, we get

$$0 \leq z'[\zeta(t)] + r^{1/\alpha}(t) z''(t) \vartheta(t),$$

that is,

$$-\vartheta(t) \frac{r^{1/\alpha}(t) z''(t)}{z'[\zeta(t)]} \leq 1.$$

Define a function $\varphi$ by

$$\varphi(t) := -\frac{r(t)(-z''(t))^{\alpha}}{(z'[\zeta(t)])^{\alpha}}, \quad t \geq t_1. \tag{3.4}$$

Clearly, $\varphi < 0$ and

$$-\vartheta^{\alpha}(t)\varphi(t) \leq 1. \tag{3.5}$$

Similarly, we define another function $\phi$ by

$$\phi(t) := -\frac{r[\tau(t)](-z''[\tau(t)])^{\alpha}}{(z'[\zeta(t)])^{\alpha}}, \quad t \geq t_1. \tag{3.6}$$

Then $\phi < 0$. From the monotonicity of $r(-z'')^{\alpha}$ and $\tau(t) \leq t$, we obtain

$$r[\tau(t)](-z''[\tau(t)])^{\alpha} \leq r(t)(-z''(t))^{\alpha}.$$

Hence, $0 < -\phi(t) < -\varphi(t)$. By virtue of (3.5), we have

$$-\vartheta^{\alpha}(t)\phi(t) \leq 1. \tag{3.7}$$

Now, by differentiating (3.4), we arrive at

$$\varphi'(t) = \frac{(-r(t)(-z''(t))^{\alpha})'}{(z'[\zeta(t)])^{\alpha}} + \frac{\alpha r(t)\zeta'(t)(-z''(t))^{\alpha} z''[\zeta(t)]}{(z'[\zeta(t)])^{\alpha+1}}.$$

By virtue of $\zeta(t) \geq t$ and the fact that $r(-z'')^{\alpha}$ is nondecreasing, we get

$$z''[\zeta(t)] \leq \frac{r^{1/\alpha}(t)}{r^{1/\alpha}[\zeta(t)]} z''(t),$$

and so

$$\varphi'(t) \leq \frac{(-r(t)(-z''(t))^{\alpha})'}{(z'[\zeta(t)])^{\alpha}} - \frac{\alpha\zeta'(t)}{r^{1/\alpha}[\zeta(t)]}(-\varphi(t))^{(\alpha+1)/\alpha}. \tag{3.8}$$

Similarly, by differentiating (3.6), we have

$$\phi'(t) = \frac{(-r[\tau(t)](-z''[\tau(t)])^{\alpha})'}{(z'[\zeta(t)])^{\alpha}} + \frac{\alpha\zeta'(t) r[\tau(t)](-z''[\tau(t)])^{\alpha} z''[\zeta(t)]}{(z'[\zeta(t)])^{\alpha+1}}.$$

By taking into account that $r(-z'')^\alpha$ is nondecreasing and $\zeta(t) \geq \tau(t)$, we conclude that

$$z''[\zeta(t)] \leq \frac{r^{1/\alpha}[\tau(t)]}{r^{1/\alpha}[\zeta(t)]} z''[\tau(t)],$$

and hence

$$\phi'(t) \leq \frac{(-r[\tau(t)](-z''[\tau(t)])^\alpha)'}{(z'[\zeta(t)])^\alpha} - \frac{\alpha\zeta'(t)}{r^{1/\alpha}[\zeta(t)]}(-\phi(t))^{(\alpha+1)/\alpha}. \tag{3.9}$$

It follows from (3.3), (3.8), and (3.9) that

$$\begin{aligned}\varphi'(t) + \frac{p_0^\alpha}{\tau_0}\phi'(t) &\leq \frac{(-r(t)(-z''(t))^\alpha)'}{(z'[\zeta(t)])^\alpha} + \frac{p_0^\alpha}{\tau_0}\frac{(-r[\tau(t)](-z''[\tau(t)])^\alpha)'}{(z'[\zeta(t)])^\alpha} \\ &\quad - \frac{\alpha\zeta'(t)}{r^{1/\alpha}[\zeta(t)]}(-\varphi(t))^{(\alpha+1)/\alpha} - \frac{p_0^\alpha}{\tau_0}\frac{\alpha\zeta'(t)}{r^{1/\alpha}[\zeta(t)]}(-\phi(t))^{(\alpha+1)/\alpha} \\ &\leq -2^{1-\alpha}\left(\frac{z[g(t,a)]}{z'[\zeta(t)]}\right)^\alpha \int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi) \\ &\quad - \frac{\alpha\zeta'(t)}{r^{1/\alpha}[\zeta(t)]}(-\varphi(t))^{(\alpha+1)/\alpha} - \frac{p_0^\alpha}{\tau_0}\frac{\alpha\zeta'(t)}{r^{1/\alpha}[\zeta(t)]}(-\phi(t))^{(\alpha+1)/\alpha}.\end{aligned}$$

Applying $z > 0$ and $z'' < 0$ implies that

$$z(t) = z(t_1) + \int_{t_1}^t z'(s)\mathrm{d}s \geq (t - t_1)z'(t).$$

Hence, we have

$$\left(\frac{z[g(t,a)]}{z'[\zeta(t)]}\right)^\alpha = \left(\frac{z[g(t,a)]}{z'[g(t,a)]}\frac{z'[g(t,a)]}{z'[\zeta(t)]}\right)^\alpha \geq G_3(t),$$

where $G_3$ is as in (3.2). Then, we have

$$\begin{aligned}\varphi'(t) + \frac{p_0^\alpha}{\tau_0}\phi'(t) &\leq -2^{1-\alpha}G_3(t)\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi) - \frac{\alpha\zeta'(t)}{r^{1/\alpha}[\zeta(t)]}(-\varphi(t))^{(\alpha+1)/\alpha} \\ &\quad - \frac{\alpha p_0^\alpha}{\tau_0}\frac{\zeta'(t)}{r^{1/\alpha}[\zeta(t)]}(-\phi(t))^{(\alpha+1)/\alpha}.\end{aligned}$$

By multiplying the latter inequality by $\vartheta^\alpha(t)$ and integrating the resulting inequality from $t_2$ ($t_2 > t_1$) to $t$, we get

$$\begin{aligned}\vartheta^\alpha(t)\varphi(t) - \vartheta^\alpha(t_2)\varphi(t_2) &+ \alpha\int_{t_2}^t\left[\frac{\vartheta^{\alpha-1}(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}\varphi(s) + \frac{\vartheta^\alpha(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}(-\varphi(s))^{(\alpha+1)/\alpha}\right]\mathrm{d}s \\ &+ \frac{p_0^\alpha}{\tau_0}(\vartheta^\alpha(t)\phi(t) - \vartheta^\alpha(t_2)\phi(t_2)) + \frac{\alpha p_0^\alpha}{\tau_0}\int_{t_2}^t\left[\frac{\vartheta^{\alpha-1}(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}\phi(s) + \frac{\vartheta^\alpha(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}(-\phi(s))^{(\alpha+1)/\alpha}\right]\mathrm{d}s \\ &+ \int_{t_2}^t 2^{1-\alpha}\vartheta^\alpha(s)G_3(s)\int_a^b Q(s,\xi)\mathrm{d}\sigma(\xi)\mathrm{d}s \leq 0.\end{aligned}$$

Set

$$A := -\left[\frac{\vartheta^\alpha(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}\right]^{\alpha/(\alpha+1)}\varphi(s) \quad \text{and} \quad B := \left[\frac{\alpha}{\alpha+1}\frac{\vartheta^{\alpha-1}(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}\left[\frac{\vartheta^\alpha(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}\right]^{-\alpha/(\alpha+1)}\right]^\alpha.$$

By using the inequality (see [12])

$$\frac{\alpha+1}{\alpha}AB^{1/\alpha} - A^{(\alpha+1)/\alpha} \leq \frac{1}{\alpha}B^{(\alpha+1)/\alpha}, \quad \text{for } A \geq 0 \text{ and } B \geq 0, \tag{3.10}$$

we obtain

$$\frac{\vartheta^{\alpha-1}(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}\varphi(s)+\frac{\vartheta^{\alpha}(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}(-\varphi(s))^{(\alpha+1)/\alpha}\geq-\frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\zeta'(s)}{\vartheta(s)r^{1/\alpha}[\zeta(s)]}.$$

On the other hand, define

$$A:=-\left[\frac{\vartheta^{\alpha}(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}\right]^{\alpha/(\alpha+1)}\phi(s)\quad\text{and}\quad B:=\left[\frac{\alpha}{\alpha+1}\frac{\vartheta^{\alpha-1}(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}\left[\frac{\vartheta^{\alpha}(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}.$$

By virtue of (3.10), we have

$$\frac{\vartheta^{\alpha-1}(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}\phi(s)+\frac{\vartheta^{\alpha}(s)\zeta'(s)}{r^{1/\alpha}[\zeta(s)]}(-\phi(s))^{(\alpha+1)/\alpha}\geq-\frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\zeta'(s)}{\vartheta(s)r^{1/\alpha}[\zeta(s)]}.$$

By using (3.5) and (3.7), we conclude that

$$\begin{aligned}\int_{t_2}^{t}\left[2^{1-\alpha}\vartheta^{\alpha}(s)G_3(s)\int_a^b Q(s,\xi)\mathrm{d}\sigma(\xi)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{\zeta'(s)}{\vartheta(s)r^{1/\alpha}[\zeta(s)]}\right]\mathrm{d}s\\ \leq\vartheta^{\alpha}(t_2)\varphi(t_2)+\frac{p^{\alpha}}{\tau_0}\vartheta^{\alpha}(t_2)\phi(t_2)+1+\frac{p_0^{\alpha}}{\tau_0},\end{aligned}$$

which contradicts (3.1). This completes the proof. □

With a proof similar to the proof of Theorems 2.2 and 3.1, we can obtain the following criterion for (1.3) assuming that $g(t,a)\geq\tau(t)$.

**Theorem 3.2.** *Let conditions* ($A_1$)-($A_5$), (1.1), (1.5), (2.2), *and* $\alpha\geq 1$ *be satisfied. Suppose that* $\tau(t)\leq t$ *and* $g(t,a)\geq\tau(t)$ *for* $t\geq t_0$. *Assume also that there exists a function* $\rho\in C^1([t_0,\infty),(0,\infty))$ *such that* (2.16) *holds for all sufficiently large* $t_1\geq t_0$ *and for some* $t_2>t_1$. *If there exists a function* $\zeta\in C^1([t_0,\infty),\mathbb{R})$ *such that* $\zeta(t)\geq t$, $\zeta(t)\geq g(t,a)$, $\zeta'(t)>0$ *for* $t\geq t_0$, *and* (3.1) *holds for all sufficiently large* $t_1\geq t_0$, *then the conclusion of Theorem* 3.1 *remains intact.*

## 4. Examples

Similar results can be obtained under the assumption that $0<\alpha\leq 1$. In this case, utilizing [5, Lemma 2], one has to replace $Q(t,\xi):=\min\{q(t,\xi),q(\tau(t),\xi)\}$ with $Q(t,\xi):=2^{\alpha-1}\min\{q(t,\xi),q(\tau(t),\xi)\}$ and proceed as above. In this section, we illustrate possible applications with two examples.

**Example 4.1.** For $t\geq 1$, consider a third-order neutral differential equation

$$[x(t)+x(t-2\pi)]'''+\int_{-4\pi}^{\pi}x[t+\xi]d\xi=0. \tag{4.1}$$

Let $\alpha=1$, $a=-4\pi$, $b=\pi$, $r(t)=1$, $p(t)=p_0=1$, $\tau(t)=t-2\pi$, $q(t,\xi)=1$, $g(t,\xi)=t+\xi$, and $\sigma(\xi)=\xi$. Note that $Q(t,\xi)=\min\{q(t,\xi),q(\tau(t),\xi)\}=1$, $g'(t,a)=1>0$, $g(t,a)=t-4\pi<t$, and $g(t,a)<\tau(t)$. Moreover, let $\tau_0=1$ and $\rho(t)=1$, then

$$G_1(t)=\frac{\int_{t_2}^{t-4\pi}\int_{t_1}^{s}r^{-1/\alpha}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t-4\pi}r^{-1/\alpha}(u)\mathrm{d}u}=\frac{t^2/2-(4\pi+t_1)t+\beta}{t-(4\pi+t_1)},\quad\beta=8\pi^2-\frac{t_2^2}{2}+4\pi t_1+t_1t_2,$$

and

$$\int^{\infty}\left[2^{1-\alpha}\rho(t)G_1(t)\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi)-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{r[g(t,a)](\rho'_+(t))^{\alpha+1}}{(\rho(t)g'(t,a))^{\alpha}}\right]\mathrm{d}t$$

$$= 5\pi \int^{\infty} G_1(t)\mathrm{d}t = \frac{5\pi}{2}\int^{\infty} \frac{t^2 - 2(4\pi + t_1)t + 2\beta}{t-(4\pi+t_1)}\mathrm{d}t = \infty.$$

Hence, by Theorem 2.1, every solution $x$ of (4.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. As a matter of fact, $x(t) = \sin t$ is an oscillatory solution to (4.1).

**Example 4.2.** For $t \geq 1$, consider a third-order neutral differential equation

$$\left[t^2(x(t) + p(t)x(t-\gamma))''\right]' + \int_0^1 (\xi+1)x[t+\xi]d\xi = 0, \tag{4.2}$$

where $0 < p(t) \leq p_0$, $p_0$ and $\gamma$ are positive constants. Let $\alpha = 1$, $a = 0$, $b = 1$, $r(t) = t^2$, $\tau(t) = t-\gamma$, $q(t,\xi) = \xi+1$, $g(t,\xi) = t+\xi$, and $\sigma(\xi) = \xi$. Note that $Q(t,\xi) = \min\{q(t,\xi), q(\tau(t),\xi)\} = \xi+1$, $\tau(t) = t-\gamma \leq t$, and $g(t,a) = t \geq \tau(t)$. Moreover, let $\tau_0 = 1$, $\rho(t) = 1$, and $\zeta(t) = t+1$, then we have $\vartheta(t) = 1/(t+1)$,

$$G_2(t) = \frac{\int_{t_2}^{t-\gamma}\int_{t_1}^{s} r^{-1/\alpha}(u)\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t-\gamma} r^{-1/\alpha}(u)\mathrm{d}u} = \frac{\int_{t_2}^{t-\gamma}\int_{t_1}^{s} u^{-2}\mathrm{d}u\mathrm{d}s}{\int_{t_1}^{t-\gamma} u^{-2}\mathrm{d}u} = \frac{(t-\gamma)^2 + (t_1\ln t_2 - t_2)(t-\gamma) - t_1(t-\gamma)\ln(t-\gamma)}{t-\gamma-t_1},$$

and

$$G_3(t) = (g(t,a) - t_1)^{\alpha} = t - t_1,$$

and thus

$$\int^{\infty}\left[2^{1-\alpha}\rho(t)G_2(t)\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi) - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{r[\tau(t)](\rho'_+(t))^{\alpha+1}}{(\tau_0\rho(t))^{\alpha}}\right]\mathrm{d}t$$
$$= \frac{3}{2}\int^{\infty} G_2(t)\mathrm{d}t = \frac{3}{2}\int^{\infty}\frac{(t-\gamma)^2 + (t_1\ln t_2 - t_2)(t-\gamma) - t_1(t-\gamma)\ln(t-\gamma)}{t-\gamma-t_1}\mathrm{d}t = \infty,$$

and

$$\int^{\infty}\left[2^{1-\alpha}\vartheta^{\alpha}(t)G_3(t)\int_a^b Q(t,\xi)\mathrm{d}\sigma(\xi) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{\zeta'(t)}{\vartheta(t)r^{1/\alpha}[\zeta(t)]}\right]\mathrm{d}t$$
$$= \int^{\infty}\left[\frac{3(t-t_1)}{2(t+1)} - \frac{1+p_0}{4(t+1)}\right]\mathrm{d}t = \infty.$$

Therefore, by Theorem 3.2, every solution $x$ of (4.2) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

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