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Huge analysis of Hepatitis C model within the scope of fractional calculus

Badr Saad T. Alkahtani[a], Abdon Atangana[b], Ilknur Koca[c,*]

[a]Department of mathematics, college of science, King Saud University, P. O. Box 1142, Riyadh, 11989, Saudi Arabia.

[b]Institute for groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, 9300 Bloemfontein South Africa.

[c]Department of Mathematics, Faculty of Sciences, Mehmet Akif Ersoy University, 15100, Burdur, Turkey.

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Abstract

A model of Hepatitis C is considered using the concept of derivative with fractional order. Using the benefits associated to Caputo derivative with fractional order, we study the existence and uniqueness of the system solutions with the help of fixed-point theorem. We derive special solutions using an iterative method. To see the efficiency of the used method, we present in detail the stability analysis of this method together with the uniqueness of the special solutions.

Keywords: Hepatitis C model, special solution, fixed point theorem, iterative method.
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1. Introduction

In many discipline and manufacturing fields, the concept of memory effect or properties have been extensively found in complicated classifications [5, 6]. To mention, the memory effect has been found in uncharacteristic transmission, in viscoelastic deformation medium for instance an aquifer, stock market, bacterial chemo-taxis and many other complex networks [3, 8, 10–12]. Nonetheless, how to portray truthfully the memory property of systems is still remain a thought-provoking subject in objective modeling and phenomenological portrayal. From widespread hypothetical and investigational analysis, the concept of fractional operators has been contemplated as one of the superlative mathematical tools to exemplify the

*Corresponding authors
Email addresses: balqahtani1@ksu.edu.sa (Badr Saad T. Alkahtani), AtanganaA@ufs.ac.za (Abdon Atangana), ikoca@mehmetakif.edu.tr (Ilknur Koca)

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memory property of complicated systems and certain materials [4, 7, 13]. But these fractional operators with constant order have failed to accurately characterize more complex system as for instance the movement of pollution within a deformable aquifer [2]. But the variable-order fractional derivative, which are extension of constant-order fractional derivative have been suggested and have been considered as the best mathematical operators to depict the memory property which changes with time or spatial location [1, 9]. However, there are couples of problems associate to these variable order derivatives. For instance differential equations with these derivatives cannot be solved analytically, and we cannot find a clear relation between these derivatives with the well-known integral transform like Mellin transform, Laplace transform, Fourier transform and Sumudu transform. With these derivatives, one cannot calculate the derivative of a simple function for instance sinus or exponential function. Therefore a new fractional variable order is needed which can be handled analytically and also have a relationship with the well-known integral transform. In the work, we will propose a fractional variable order derivative, that can be handled analytically and also have relationship with some other integral transform. The rest of the paper is structured as follows: in Section 2, we present some definitions of existing fractional variable order operators.

2. Model derivation with Caputo-Fabrizio derivative with fractional order

We first give the definitions of Caputo-Fabrizio derivative with fractional order. Caputo-Fabrizio derivative with fractional order has been considered with no singular kernel [5, 10].

Definition 2.1. Let $f \in H^1(a,b)$, $b > a, \alpha \in [0,1]$ then the new Caputo derivative of fractional order is defined as:

$$D_t^{\alpha}(f(t)) = \frac{M(\alpha)}{1-\alpha}\int_a^t f^{'}(x)\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx,$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$, [5, 10]. But, if the function does not belong to $H^1(a,b)$, then the derivative can be reformulated as

$$D_t^{\alpha}(f(t)) = \frac{\alpha M(\alpha)}{1-\alpha}\int_a^t (f(t)-f(x))\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx.$$

Remark 2.2. The investigators observed that, if $\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty)$, $\alpha = \frac{1}{1+\alpha} \in [0,1]$, then new Caputo derivative of fractional order assumes the form

$$D_t^{\alpha}(f(t)) = \frac{N(\sigma)}{\sigma}\int_a^t f^{'}(x)\exp\left[-\frac{t-x}{\sigma}\right]dx,\ N(0) = N(\infty) = 1.$$

In addition,

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t).$$

At this instant subsequent to the preface of the novel derivative, the connected anti-derivative turns out to be imperative; the connected integral of the derivative was proposed by Nieto and Losada [10].

Definition 2.3. Let $0 < \alpha < 1$. The fractional integral of order α of a function f is defined by

$$I_{\alpha}^t(f(t)) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(s)ds,\ t \geq 0.$$

Remark 2.4. Note that, according to above definition, the fractional integral of Caputo type of function of order $0 < \alpha < 1$ is an average between function f and its integral of order one. This therefore imposes

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1.$$

The above expression yields an explicit formula for

$$M(\alpha) = \frac{2}{(2-\alpha)},\ 0 \leq \alpha \leq 1.$$

Because of the above, Nieto and Losada proposed that the new Caputo derivative of order $0 < \alpha < 1$ can be reformulated as

$$D_t^\alpha(f(t)) = \frac{1}{1-\alpha}\int_a^t f^{'}(x)\exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx.$$

2.1. Hepatitis C model with Caputo-Fabrizio fractional derivative

The model that we study in this paper is a Caputo-Fabrizio derivative and α satisfying $0 < \alpha \leq 1$ with fractional order is presented via following system:

$$\begin{aligned} {}_0^{CF}D_t^\alpha T(t) &= s - dT(t) - (1-\eta)\beta V(t)T(t), \\ {}_0^{CF}D_t^\alpha I(t) &= (1-\eta)\beta V(t)T(t) - \delta I(t), \\ {}_0^{CF}D_t^\alpha V(t) &= (1-\varepsilon)pI(t) - cV(t), \end{aligned} \tag{2.1}$$

where T represents uninfected hepatocytes, I represents infected hepatocytes and V represents virus. The model assumes that uninfected hepatocytes are produced at a constant rate s, die at rate d, per cell and are infected at constant rate β. Infected hepatocytes are lost at a rate δ per cell. Viral particles (virions) are produced at rate p per infected hepatocyte and cleared at rate c per virion.

Let $\alpha \in (0, 1]$ and consider the system

$$\begin{cases} {}_0^{CF}D_t^\alpha T(t) = f_1(T, I, V), \\ {}_0^{CF}D_t^\alpha I(t) = f_2(T, I, V), \\ {}_0^{CF}D_t^\alpha V(t) = f_3(T, I, V) \end{cases}$$

with the initial conditions $T_1(0) = T_{01}$, $I_1(0) = I_{01}$ and $V_1(0) = V_{01}$. Also the efficacy of treatment in blocking virion production and reducing new infections are described by two parameters, $0 < \varepsilon < 1$ and $0 < \eta < 1$, respectively. Here

$$\begin{aligned} f_1(T, I, V) &= s - dT(t) - (1-\eta)\beta V(t)T(t), \\ f_2(T, I, V) &= (1-\eta)\beta V(t)T(t) - \delta I(t), \\ f_3(T, I, V) &= (1-\varepsilon)pI(t) - cV(t). \end{aligned}$$

3. Derivation of the special solution

The aim of this section is to provide a special solution of the above equation (2.1) applying the Sumudu transform on both sides of equation (2.1) together with an iterative method. We shall give the Sumudu transform in the following theorem.

Theorem 3.1. *Let $f(t)$ be a function for which the Caputo-Fabrizio exists, then the Sumudu transform of the Caputo-Fabrizio fractional derivative of $f(t)$ is given as:*

$$ST\left({}_0^{CF}D_t^\alpha\right)((f(t)) = M(\alpha)\frac{SF(f(t)) - f(0)}{1-\alpha+\alpha u}.$$

Proof. The proof of the Theorem 3.1 can be found in [2]. □

To solve the above equation (2.1), we apply the Sumudu transform on both sides of equation (2.1), we obtain

$$\begin{aligned}
M(\alpha)\frac{SF(T(t))-T(0)}{1-\alpha+\alpha s} &= SL\left\{s-dT(t)-(1-\eta)\beta V(t)T(t)\right\},\\
M(\alpha)\frac{SF(I(t))-I(0)}{1-\alpha+\alpha s} &= SL\left\{(1-\eta)\beta V(t)T(t)-\delta I(t)\right\},\\
M(\alpha)\frac{SF(V(t))-V(0)}{1-\alpha+\alpha s} &= SL\left\{(1-\varepsilon)pI(t)-cV(t)\right\}.
\end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}
SF(T(t)) &= T(0)+\frac{(1-\alpha+\alpha s)}{M(\alpha)}SL\left\{s-dT(t)-(1-\eta)\beta V(t)T(t)\right\},\\
SF(I(t)) &= I(0)+\frac{(1-\alpha+\alpha s)}{M(\alpha)}SL\left\{(1-\eta)\beta V(t)T(t)-\delta I(t)\right\},\\
SF(V(t)) &= V(0)+\frac{(1-\alpha+\alpha s)}{M(\alpha)}SL\left\{(1-\varepsilon)pI(t)-cV(t)\right\}.
\end{aligned}\tag{3.1}$$

Now applying the inverse Sumudu transform on both sides of equation (3.1), we obtain

$$\begin{aligned}
T(t) &= T(0)+SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)}SL\left\{\begin{array}{c} s\\ -dT(t)\\ -(1-\eta)\beta V(t)T(t)\end{array}\right\}\right\},\\
I(t) &= I(0)+SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)}SL\left\{(1-\eta)\beta V(t)T(t)-\delta I(t)\right\}\right\},\\
V(t) &= V(0)+SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)}SL\left\{(1-\varepsilon)pI(t)-cV(t)\right\}\right\}.
\end{aligned}$$

We next obtain the following recursive formula

$$\begin{aligned}
T_{(n+1)}(t) &= T_{(n)}(0)+SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)}SL\left\{\begin{array}{c} s\\ -dT_{(n)}(t)\\ -(1-\eta)\beta V_{(n)}(t)T_{(n)}(t)\end{array}\right\}\right\},\\
I_{(n+1)}(t) &= I_{(n)}(0)+SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)}SL\left\{\begin{array}{c} (1-\eta)\beta V_{(n)}(t)T_{(n)}(t)\\ -\delta I_{(n)}(t)\end{array}\right\}\right\},\\
V_{(n+1)}(t) &= V_{(n)}(0)+SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)}SL\left\{(1-\varepsilon)pI_{(n)}(t)-cV_{(n)}(t)\right\}\right\}.
\end{aligned}\tag{3.2}$$

And the solution of (3.2) is provided by

$$\begin{aligned}
T(t) &= \lim_{n\to\infty} T_n(t),\\
I(t) &= \lim_{n\to\infty} I_n(t),\\
V(t) &= \lim_{n\to\infty} V_n(t).
\end{aligned}$$

3.1. Application of fixed-point theorem for stability analysis of iteration method

Let $(X,\|.\|)$ be a Banach space and H a self-map of X. Let $y_{n+1}=g(H,y_n)$ be particular recurcive procedure. Suppose that, $F(H)$ the fixed-point set of H has at least one element and that y_n converges to a point $p\in F(H)$. Let $\{x_n\}\subseteq X$ and define $e_n=\|x_{n+1}-g(H,x_n)\|$. If $\lim_{n\to\infty} e^n=0$ implies that $\lim_{n\to\infty} x^n=p$, then the iteration method $y_{n+1}=g(H,y_n)$ is said to be H-stable. Without any loss of generality, we must assume that, our sequence $\{x_n\}$ has an upper boundary; otherwise we cannot expect the possibility of convergence. If all these conditions are satisfied for $y_{n+1}=Hy_n$ which is known as Picard's iteration, consequently the iteration will be H-stable. We shall then state the following theorem.

Theorem 3.2. *Let $(X, \|.\|)$ be a Banach space and H a self-map of X satisfying*

$$\|H_x - H_y\| \leq C \|x - H_x\| + c \|x - y\|$$

for all x, y in X where $0 \leq C$, $0 \leq c < 1$. Suppose that H is Picard H-stable [12].

Let us take into account the following recursive formula equation (3.2) connected to equation (2.1).

$$\begin{aligned}
T_{(n+1)}(t) &= T_{(n)}(0) + SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)} SL \left\{\begin{array}{c} s - dT_{(n)}(t) \\ -(1-\eta)\beta V_{(n)}(t) T_{(n)}(t) \end{array}\right\}\right\}, \\
I_{(n+1)}(t) &= I_{(n)}(0) + SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)} SL \left\{\begin{array}{c} (1-\eta)\beta V_{(n)}(t) T_{(n)}(t) \\ -\delta I_{(n)}(t) \end{array}\right\}\right\}, \\
V_{(n+1)}(t) &= V_{(n)}(0) + SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)} SL \left\{(1-\varepsilon)pI_{(n)}(t) - cV_{(n)}(t)\right\}\right\},
\end{aligned}$$

where $\frac{1+(s-1)\alpha}{M(\alpha)}$ is the fractional Lagrange multiplier.

Theorem 3.3. *Let P be a self-map defined as*

$$\begin{cases}
P(T_{(n)}(t)) = T_{(n+1)}(t) = T_{(n)}(t) + SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)} SL \left\{\begin{array}{c} s - dT_{(n)}(t) \\ -(1-\eta)\beta V_{(n)}(t) T_{(n)}(t) \end{array}\right\}\right\}, \\
P(I_{(n)}(t)) = I_{(n+1)}(t) = I_{(n)}(t) + SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)} SL \left\{\begin{array}{c} (1-\eta)\beta V_{(n)}(t) T_{(n)}(t) \\ -\delta I_{(n)}(t) \end{array}\right\}\right\}, \\
P(V_{(n)}(t)) = V_{(n+1)}(t) = V_{(n)}(t) + SL^{-1}\left\{\frac{(1-\alpha+\alpha s)}{M(\alpha)} SL \left\{(1-\varepsilon)pI_{(n)}(t) - cV_{(n)}(t)\right\}\right\}
\end{cases}$$

is $P-$stable in $L^1(a, b)$ if

$$\begin{cases}
\{1 - df(\gamma) + (\eta - 1)\beta(K + S)g(\gamma)\} < 1, \\
\{1 + (1-\eta)\beta(K + S)h(\gamma) - sQ(\gamma)\} < 1, \\
\{1 + (1-\varepsilon)pL(\gamma) - cN(\gamma)\} < 1.
\end{cases}$$

Proof. The fist step of the proof will consist on showing that P has a fixed point. To achieve this, we evaluate the followings for all $(n, m) \in \mathbb{N} \times \mathbb{N}$.

$$\begin{aligned}
P(T_{(n)}(t)) - P(T_{(m)}(t)) = T_{(n)}(t) - T_{(m)}(t) \\
+ SL^{-1}\left\{\frac{1+(s-1)\alpha}{M(\alpha)} SL \left\{s - dT_{(n)}(t) - (1-\eta)\beta V_{(n)}(t) T_{(n)}(t)\right\}\right\} \\
- SL^{-1}\left\{\frac{1+(s-1)\alpha}{M(\alpha)} SL \left\{s - dT_{(m)}(t) - (1-\eta)\beta V_{(m)}(t) T_{(m)}(t)\right\}\right\},
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
P(I_{(n)}(t)) - P(I_{(m)}(t)) = I_{(n)}(t) - I_{(m)}(t) \\
+ SL^{-1}\left\{\frac{1+(s-1)\alpha}{M(\alpha)} SL \left\{(1-\eta)\beta V_{(n)}(t) T_{(n)}(t) - \delta I_{(n)}(t)\right\}\right\} \\
- SL^{-1}\left\{\frac{1+(s-1)\alpha}{M(\alpha)} SL \left\{(1-\eta)\beta V_{(m)}(t) T_{(m)}(t) - \delta I_{(m)}(t)\right\}\right\},
\end{aligned}$$

and

$$\begin{aligned}
P(V_{(n)}(t)) - P(V_{(m)}(t)) = V_{(n)}(t) - V_{(m)}(t) \\
+ SL^{-1}\left\{\frac{1+(s-1)\alpha}{M(\alpha)} SL \left\{(1-\varepsilon)pI_{(n)}(t) - cV_{(n)}(t)\right\}\right\} \\
- SL^{-1}\left\{\frac{1+(s-1)\alpha}{M(\alpha)} SL \left\{(1-\varepsilon)pI_{(m)}(t) - cV_{(m)}(t)\right\}\right\}.
\end{aligned}$$

Let consider the equality (3.3) and applying norm on both sides and without loss of generality

$$\begin{aligned}&\left\|P(T_{(n)}(t))-P(T_{(m)}(t))\right\|\\&\quad=\left\|T_{(n)}(t)-T_{(m)}(t)\right.\\&\quad\quad+SL^{-1}\left\{\frac{1+(s-1)\,\alpha}{M(\alpha)}SL\left\{\begin{array}{c}-d\left\{T_{(n)}(t)-T_{(m)}(t)\right\}\\-(1-\eta)\beta\left\{V_{(n)}(t)T_{(n)}(t)-V_{(m)}(t)T_{(m)}(t)\right\}\end{array}\right\}\right\}\left.\right\|.\end{aligned}\tag{3.4}$$

Using the properties of the norm in particular the triangular inequality, the right hand side of equation (3.4) is converted to

$$\begin{aligned}\left\|P(T_{(n)}(t))-P(T_{(m)}(t))\right\|&\leq\left\|T_{(n)}(t)-T_{(m)}(t)\right\|\\&+\left\|SL^{-1}\left\{\begin{array}{c}\frac{1+(s-1)\alpha}{M(\alpha)}\\SL\left\{\begin{array}{c}-d\left\{T_{(n)}(t)-T_{(m)}(t)\right\}\\-(1-\eta)\beta\left\{\begin{array}{c}V_{(n)}(t)\left\{T_{(n)}(t)-T_{(m)}(t)\right\}\\+T_{(m)}(t)\left\{V_{(n)}(t)-V_{(m)}(t)\right\}\end{array}\right\}\end{array}\right\}\end{array}\right\}\right\|.\end{aligned}$$

The above can further be transformed using the property of norm and integral as follows

$$\begin{aligned}\left\|P(T_{(n)}(t))-P(T_{(m)}(t))\right\|&\leq\left\|T_{(n)}(t)-T_{(m)}(t)\right\|\\&+SL^{-1}\left\{\frac{1+(s-1)\,\alpha}{M(\alpha)}SL\left\{\left\|-d\left\{T_{(n)}(t)-T_{(m)}(t)\right\}\right\|\right\}\right\}\\&+SL^{-1}\left\{\begin{array}{c}\frac{1+(s-1)\alpha}{M(\alpha)}\\SL\left\{\left\|(\eta-1)\beta\left\{\begin{array}{c}V_{(n)}(t)\left\{T_{(n)}(t)-T_{(m)}(t)\right\}\\+T_{(m)}(t)\left\{V_{(n)}(t)-V_{(m)}(t)\right\}\end{array}\right\}\right\|\right\}\end{array}\right\}.\end{aligned}\tag{3.5}$$

Since both the solutions play the same role, we shall assume in this case that

$$\begin{aligned}\left\|T_{(n)}(t)-T_{(m)}(t)\right\|&\cong\left\|V_{(n)}(t)-V_{(m)}(t)\right\|,\\\left\|T_{(n)}(t)-T_{(m)}(t)\right\|&\cong\left\|I_{(n)}(t)-I_{(m)}(t)\right\|.\end{aligned}$$

Replacing this in equation (3.5), we obtain the following relation

$$\begin{aligned}\left\|P(T_{(n)}(t))-P(T_{(m)}(t))\right\|&\leq\left\|T_{(n)}(t)-T_{(m)}(t)\right\|\\&+SL^{-1}\left\{\frac{1+(s-1)\,\alpha}{M(\alpha)}SL\left\{\left\|-d\left\{T_{(n)}(t)-T_{(m)}(t)\right\}\right\|\right\}\right\}\\&+SL^{-1}\left\{\begin{array}{c}\frac{1+(s-1)\alpha}{M(\alpha)}\\SL\left\{\left\|(\eta-1)\beta\left\{\begin{array}{c}V_{(n)}(t)\left\{T_{(n)}(t)-T_{(m)}(t)\right\}\\+T_{(m)}(t)\left\{T_{(n)}(t)-T_{(m)}(t)\right\}\end{array}\right\}\right\|\right\}\end{array}\right\}.\end{aligned}\tag{3.6}$$

Since $T_{(n)}(t)$, $T_{(m)}(t)$ and $V_{(n)}(t)$ are bounded, we can find three different positive constants, R, S, K such that for all t.

$$\begin{aligned}&\left\|T_{(n)}(t)\right\|<R,\ \left\|T_{(m)}(t)\right\|<S,\\&\left\|V_{(n)}(t)\right\|<K,\ (n,m)\in\mathbb{N}\times\mathbb{N}.\end{aligned}\tag{3.7}$$

Now considering equation (3.6) with (3.7), we obtain the following

$$\left\|P(T_{(n)}(t))-P(T_{(m)}(t))\right\|\leq\left\{\begin{array}{c}1-df(\gamma)\\+(\eta-1)\beta(K+S)g(\gamma)\end{array}\right\}\left\|T_{(n)}(t)-T_{(m)}(t)\right\|,\tag{3.8}$$

where f,g are functions from $SL^{-1}\left\{\frac{1+(s-1)\alpha}{M(\alpha)}SL\right\}$.

In the same way, we get

$$\left\|P(I_{(n)}(t))-P(I_{(m)}(t))\right\|\leq\left\{\begin{array}{c}1+(1-\eta)\beta(K+S)h(\gamma)\\-\delta Q(\gamma)\end{array}\right\}\left\|I_{(n)}(t)-I_{(m)}(t)\right\| \tag{3.9}$$

and

$$\left\|P(V_{(n)}(t))-P(V_{(m)}(t))\right\|\leq\{1+(1-\varepsilon)pL(\gamma)-cN(\gamma)\}\left\|V_{(n)}(t)-V_{(m)}(t)\right\|. \tag{3.10}$$

For

$$\begin{aligned}\{1-df(\gamma)+(\eta-1)\beta(K+S)g(\gamma)\}&<1,\\ \{1+(1-\eta)\beta(K+S)h(\gamma)-\delta Q(\gamma)\}&<1,\\ \{1+(1-\varepsilon)pL(\gamma)-cN(\gamma)\}&<1.\end{aligned}$$

Then the nonlinear P-self mapping has a fixed point. We next show that, P satisfies the conditions in Theorem 3.2. Let (3.8), (3.9), and (3.10) hold, thus putting

$$c=(0,0,0),\ C=\left\{\begin{array}{l}1-df(\gamma)+(\eta-1)\beta(K+S)g(\gamma),\\ 1+(1-\eta)\beta(K+S)h(\gamma)-\delta Q(\gamma),\\ 1+(1-\varepsilon)pL(\gamma)-cN(\gamma),\end{array}\right.$$

then the above shows that condition of Theorem (3.2) hold for the nonlinear mapping P. Therefore since all conditions in Theorem 3.2 hold for the defined non-linear mapping P, then P is Picards $P-$stable. This completes the proof of Theorem 3.3. □

4. Uniqueness of the special solution

In this section, we show that the special solution of equation (2.1) using the iteration method is unique. We shall first assume that, equation (2.1) has an exact solution via which, the special solution converges for a large number m. We consider the following Hilbert space $H=L^2\left((a,b)\times(0,T)\right)$ that can be defined as the set of those functions.

$$v:(a,b)\times[0,T]\rightarrow\mathbb{R},\quad\int\int uvdudv<\infty.$$

We now, consider the following operator

$$P(T,I,V)=\left\{\begin{array}{l}s-dT(t)-(1-\eta)\beta V(t)T(t),\\ (1-\eta)\beta V(t)T(t)-\delta I(t),\\ (1-\varepsilon)pI(t)-cV(t).\end{array}\right.$$

The aim of this part is to prove that the inner product of

$$\left(P(X_{11}-X_{12},X_{21}-X_{22},X_{31}-X_{32}),(w_1,w_2,w_3)\right),$$

where $(X_{11}-X_{12})$, $(X_{21}-X_{22})$ and $(X_{31}-X_{32})$ are special solution of system. However,

$$\begin{aligned}&\left(P(X_{11}-X_{12},X_{21}-X_{22},X_{31}-X_{32}),(w_1,w_2,w_3)\right)\\ &\quad=\left\{\begin{array}{l}\left(-d\left(X_{11}-X_{12}\right)-(1-\eta)\beta\left(X_{31}-X_{32}\right)\left(X_{11}-X_{12}\right),w_1\right),\\ \left((1-\eta)\beta\left(X_{31}-X_{32}\right)\left(X_{11}-X_{12}\right)-\delta\left(X_{21}-X_{22}\right),w_2\right),\\ \left((1-\varepsilon)p\left(X_{21}-X_{22}\right)-c\left(X_{31}-X_{32}\right),w_3\right).\end{array}\right.\end{aligned} \tag{4.1}$$

We shall evaluate the first equation in the system without loss of generality

$$\begin{aligned}&\left(-d\left(X_{11}-X_{12}\right)-(1-\eta)\beta\left(X_{31}-X_{32}\right)\left(X_{11}-X_{12}\right),w_1\right)\\ &\quad\cong\left(-d\left(X_{11}-X_{12}\right),w_1\right)+\left(-(1-\eta)\beta\left(X_{31}-X_{32}\right)\left(X_{11}-X_{12}\right),w_1\right)\end{aligned} \tag{4.2}$$

Since both solutions play almost the same role, we can assume that,

$$X_{11} - X_{12} \cong X_{21} - X_{22} \cong X_{31} - X_{32},$$

then the equation (4.2) becomes

$$\begin{aligned}&(-d\left(X_{11} - X_{12}\right) - (1-\eta)\beta\left(X_{31} - X_{32}\right)\left(X_{11} - X_{12}\right), w_1)\\ &\quad\cong (-d\left(X_{11} - X_{12}\right), w_1) + \Big((1-\eta)\beta\left(X_{11} - X_{12}\right)^2, w_1\Big).\end{aligned}$$

Nevertheless, using the relationship between the norm and the inner product, we obtain the following inequality

$$\begin{aligned}&(-d\left(X_{11} - X_{12}\right) - (1-\eta)\beta\left(X_{31} - X_{32}\right)\left(X_{11} - X_{12}\right), w_1)\\ &\quad\le (d\left(X_{12} - X_{11}\right), w_1) + \Big((1-\eta)\beta\left(X_{12} - X_{11}\right)^2, w_1\Big)\\ &\quad\le d\left\|X_{12} - X_{11}\right\|\left\|w_1\right\| + (1-\eta)\beta\left\|X_{12} - X_{11}\right\|^2\left\|w_1\right\|\\ &\quad= (d + (1-\eta)\beta\overline{w})\left\|X_{12} - X_{11}\right\|\left\|w_1\right\|.\end{aligned} \tag{4.3}$$

Using the same routine, the second equation of the system can be evaluated as follows

$$\begin{aligned}&((1-\eta)\beta\left(X_{31} - X_{32}\right)\left(X_{11} - X_{12}\right) - \delta\left(X_{21} - X_{22}\right), w_2)\\ &\quad\le (1-\eta)\beta\left\|X_{22} - X_{21}\right\|^2\left\|w_2\right\| + \delta\left\|X_{22} - X_{21}\right\|\left\|w_2\right\|\\ &\quad= (1-\eta)\beta\overline{f} + \delta\left\|X_{22} - X_{21}\right\|\left\|w_2\right\|.\end{aligned} \tag{4.4}$$

Finally, third equation of the system is considered as below

$$\begin{aligned}&((1-\varepsilon)p\left(X_{21} - X_{22}\right) - c\left(X_{31} - X_{32}\right), w_3)\\ &\quad\le (\varepsilon-1)p\left\|X_{32} - X_{31}\right\|\left\|w_3\right\| + c\left\|X_{32} - X_{31}\right\|\left\|w_3\right\|\\ &\quad\quad((\varepsilon-1)p + c)\left\|X_{32} - X_{31}\right\|\left\|w_3\right\|.\end{aligned} \tag{4.5}$$

Replacing equations (4.3), (4.4), and (4.5) into equation (4.1), we obtain

$$\begin{aligned}&(P(X_{11} - X_{12}, X_{21} - X_{22}, X_{31} - X_{32}), (w_1, w_2, w_3))\\ &\quad\le \begin{cases} (d + (1-\eta)\beta\overline{w})\left\|X_{12} - X_{11}\right\|\left\|w_1\right\|,\\ (1-\eta)\beta\overline{f} + \delta\left\|X_{22} - X_{21}\right\|\left\|w_2\right\|,\\ ((\varepsilon-1)p + c)\left\|X_{32} - X_{31}\right\|\left\|w_3\right\|.\end{cases}\end{aligned} \tag{4.6}$$

But, for large numbers m, n and k, both solutions converge to the exact solution, using the topology concept; we can find three very small positive parameters l_n, l_m, l_k such that

$$\begin{aligned}\left\|T - X_{11}\right\|, \left\|T - X_{12}\right\| &< \frac{l_n}{3\left(d + (1-\eta)\beta\overline{w}\right)\left\|w_1\right\|},\\ \left\|I - X_{21}\right\|, \left\|I - X_{22}\right\| &< \frac{l_m}{3(1-\eta)\beta\overline{f} + \delta\left\|w_2\right\|},\\ \left\|V - X_{31}\right\|, \left\|V - X_{32}\right\| &< \frac{l_k}{3\left((\varepsilon-1)p + c\right)\left\|w_3\right\|}.\end{aligned}$$

Thus introducing the exact solution in the right hand side of equation (4.6), using the triangular inequality and finally taking $M = max(n, m, k), l = max(l_n, l_m, l_k)$.

$$\begin{cases} (d + (1-\eta)\beta\overline{w})\left\|X_{12} - X_{11}\right\|\left\|w_1\right\|,\\ (1-\eta)\beta\overline{f} + \delta\left\|X_{22} - X_{21}\right\|\left\|w_2\right\|,\\ ((\varepsilon-1)p + c)\left\|X_{32} - X_{31}\right\|\left\|w_3\right\|,\end{cases} < \begin{cases} l,\\ l,\\ l,\end{cases}$$

Since l is a very small positive parameter, we conclude based on the topology idea that

$$\begin{cases} (d+(1-\eta)\beta\overline{w})\left\|X_{12}-X_{11}\right\|\left\|w_1\right\|, \\ (1-\eta)\beta\overline{f}+\delta\left\|X_{22}-X_{21}\right\|\left\|w_2\right\|, \\ ((\varepsilon-1)p+c)\left\|X_{32}-X_{31}\right\|\left\|w_3\right\|, \end{cases} < \begin{cases} 0, \\ 0, \\ 0, \end{cases}$$

but

$$\begin{aligned} &(d+(1-\eta)\beta\overline{w}), (1-\eta)\beta\overline{f}+\delta, ((\varepsilon-1)p+c) \neq 0, \left\|w_1\right\|, \left\|w_2\right\|, \left\|w_3\right\| \neq 0 \\ &\qquad \Rightarrow \left\|X_{12}-X_{11}\right\| = \left\|X_{22}-X_{21}\right\| = \left\|X_{31}-X_{32}\right\| = 0 \\ &\qquad \Rightarrow X_{11}=X_{12}, X_{21}=X_{22} \text{ and } X_{31}=X_{32}. \end{aligned}$$

This completes the proof of uniqueness.

5. Conclusion

A model of Hepatitis C was considered using the concept of derivative with fractional order. We used the newly proposed derivative with fractional order. The design of the definition allows the description of the memory without any singularity like in the case of Caputo and Riemann-Liouville derivative. With the benefits of the new derivative, we presented the stability of the equilibrium points. We derived the solution of the new equation using an iterative scheme. In order to show the efficiency of the used method, we employed the fixed-point theorem to study the stability analysis of the method for solving the new equation. We presented in detail the uniqueness of the special derived solution.

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