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A higher order frozen Jacobian iterative method for solving Hamilton-Jacobi equations

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Abstract

It is well-known that the solution of Hamilton-Jacobi equation may have singularity i.e., the solution is non-smooth or nearly non-smooth. We construct a frozen Jacobian multi-step iterative method for solving Hamilton-Jacobi equation under the assumption that the solution is nearly singular. The frozen Jacobian iterative methods are computationally very efficient because a single instance of the iterative method uses a single inversion (in the scene of LU factorization) of the frozen Jacobian. The multi-step part enhances the convergence order by solving lower and upper triangular systems. The convergence order of our proposed iterative method is $3(m-1)$ for $m \geq 3$. For attaining good numerical accuracy in the solution, we use Chebyshev pseudo-spectral collocation method. Some Hamilton-Jacobi equations are solved, and numerically obtained results show high accuracy.

Keywords: Hamilton-Jacobi equations, frozen Jacobian iterative methods, systems of nonlinear equations, Chebyshev pseudo-spectral collocation method.
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1. Introduction

The multi-dimensional Hamilton-Jacobi (HJ) equations can be written as

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$$\phi_t + H\left(\phi_{x_1}, \phi_{x_2}, \cdots, \phi_{x_n}\right) = 0, \quad \vec{\mathbf{x}} = (x_1, \cdots, x_n) \in \mathbb{R}^n, \tag{1.1}$$

where $\phi = \phi(\vec{x}, t)$ and H is the Hamiltonian. The solution of (1.1) may produce discontinuous spatial derivatives when the evolution of the solution passes a particular instant of time. The non-smoothness of the solution exhibits difficulties in the numerical approximation of HJ equations solutions. Many researchers [8, 9, 16, 17] have proposed method for solving HJ equations. Souganidis et al. [26] introduced first order converging methods for HJ equations. Osher et al. [21, 22] proposed high-order upwind schemes. The schemes proposed by Osher et al. were originated from essentially non-oscillatory (ENO) [13] and a monotone numerical flux. Jiang and his co-researchers constructed a more compact upwind scheme which is based on a weighted ENO (WENO)[14]. WENO scheme was reported in [15, 18]. The further extensions of these methods for unstructured grid can be found in [1]. Al-Aidarous et al. [4, 5] presented results concerning convergence result for the ergodic problem for Hamilton-Jacobi equations with Neumann-type boundary conditions and asymptotic analysis for the eikonal equation with the dynamical boundary conditions. The HJ equations contain partial derivatives in time and space. Many methods can be used to discretize the spatial domain, but we are interested in pseudospectral collocation methods both in space and time. The reason to choose spectral collocation methods is hidden in their numerical approximation accuracy. Interested readers can find the further information in [7, 10–12, 25, 27, 29]. When we use the pseudospectral collocation methods for time and space simultaneously to discretize some nonlinear partial or ordinary differential equations we obtain a system of nonlinear equations. It is of interest that we construct efficient iterative methods for solving system of nonlinear equations. Recently a large community of researchers [2, 3, 6, 23, 24, 30, 31] have contributed in the area of the iterative method for solving system of nonlinear equations associated with partial and ordinary equations. In the efficient class of iterative methods for solving system of nonlinear equations, the multi-step frozen Jacobian iterative methods are good candidates. After discretization, we can write (1.1) as

$$\mathbf{F}(\boldsymbol{\phi}) = D_t\, \boldsymbol{\phi} + H(D_{x_1}\, \boldsymbol{\phi}, D_{x_2}\, \boldsymbol{\phi}, \cdots, D_{x_n}\, \boldsymbol{\phi}) = \mathbf{0},$$

where D_t and D_{x_i} are spectral differentiation matrices for temporal derivative and spatial derivatives, respectively. The classical frozen Jacobian multi-step iterative method [20, 28] for solving system of nonlinear equations is the Newton frozen Jacobian multi-step iterative method

$$\text{MNR} = \begin{cases} \text{number of steps} = m, \\ \text{convergence order} = m+1, \\ \text{function evaluations} = m, \\ \text{Jacobian evaluations} = 1, \\ \text{number of LU-factorization} = 1, \\ \text{number of solutions of lower} \\ \ \text{and upper triangular systems } = m, \end{cases} \quad \begin{cases} \text{base method} \rightarrow & \begin{cases} \boldsymbol{\phi}_0 = \text{initial guess}, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\, \boldsymbol{\psi}_1 = \mathbf{F}(\boldsymbol{\phi}_0), \\ \boldsymbol{\phi}_1 = \boldsymbol{\phi}_0 - \boldsymbol{\psi}_1, \end{cases} \\ \text{multi-step part} \rightarrow & \begin{cases} \text{for } s = 1, m, \\ \qquad \mathbf{F}'(\boldsymbol{\phi}_0)\, \boldsymbol{\psi}_{s+1} = \mathbf{F}(\boldsymbol{\phi}_s), \\ \qquad \boldsymbol{\phi}_{s+1} = \boldsymbol{\phi}_s - \boldsymbol{\psi}_{s+1}, \\ \text{end}, \\ \boldsymbol{\phi}_0 = \boldsymbol{\phi}_{m+1}. \end{cases} \end{cases}$$

The frozen Jacobian multi-step iterative method MNR is an efficient iterative method for solving the system of nonlinear equations. But the low per step increment in the convergence makes this iterative method less attractive. We are interested in constructing frozen Jacobian multi-step iterative methods that offer high convergence order with reasonable computational cost. Recently three frozen Jacobian multi-step iterative methods HJ [19], FTUC [2], MSF [30] for the solving system of nonlinear equations are proposed by different authors. The convergence orders of HJ, FTUC, and MSF, are $2m$, $3m-4$ and $3m$, respectively. The applicability of MSF method is limited because it uses second order Fréchet derivative. There is one thing common in all iterative methods is that they utilize a single inversion of the Jacobian (in the scenes of LU factorization) and repeatably solve lower and upper triangular systems to attain the high order of convergence. The two operations are computationally very expensive namely the LU factorization and evaluation of Jacobians. Hence if a method uses least number of Jacobian evaluation and LU factorization and achieves a high convergence rate, then we way that particular method is efficient. The other expensive operations are the evaluation of the system of nonlinear equations, matrix-vector multiplications, and solution of lower and upper triangular systems. We are interested in constructing an efficient iterative method for solving system of nonlinear equations associated with partial and ordinary differential equations. When we talk about the high order of convergence, it means we have to guarantee the regularity of functional derivatives of the system of nonlinear equations.

$$\mathrm{HJ} = \begin{cases} \text{number of steps} = m \geq 2, \\ \text{convergence order} = 2m, \\ \text{function evaluations} = m-1, \\ \text{Jacobian evaluations} = 2, \\ \text{LU-factorization} = 1, \\ \text{matrix-vector multiplications} = m, \\ \text{vector-vector multiplications} = 2m, \\ \text{number of solutions of systems} \\ \text{of lower and upper triangular} \\ \text{systems of equations } = 2m-1, \end{cases} \quad \begin{cases} \text{base method} \longrightarrow \begin{cases} \mathbf{y}_0 = \text{initial guess}, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{y}_0)\,, \\ \mathbf{y}_1 = \mathbf{y}_0 - \frac{2}{3}\,\boldsymbol{\phi}_1, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_2 = \mathbf{F}'(\mathbf{y}_1)\,\boldsymbol{\phi}_1, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_3 = \mathbf{F}'(\mathbf{y}_1)\,\boldsymbol{\phi}_2, \\ \mathbf{y}_2 = \mathbf{y}_0 - \frac{23}{8}\,\boldsymbol{\phi}_1 + 3\,\boldsymbol{\phi}_2 - \frac{9}{8}\,\boldsymbol{\phi}_3, \end{cases} \\ \text{multi-step part} \rightarrow \begin{cases} \text{for } s = 3, m, \\ \quad \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_4 = \mathbf{F}(\mathbf{y}_{s+1})\,, \\ \quad \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_5 = \mathbf{F}'(\mathbf{y}_1)\,\boldsymbol{\phi}_4, \\ \quad \mathbf{y}_s = \mathbf{y}_{s-1} - \frac{5}{2}\,\boldsymbol{\phi}_4 + \frac{3}{2}\,\boldsymbol{\phi}_5, \\ \text{end}, \\ \mathbf{y}_0 = \mathbf{y}_m\,. \end{cases} \end{cases}$$

It is well-known that the HJ equations may produce discontinuous derivatives after a particular instant of time. It means that when the solution of HJ equations is not smooth, then the approximation method to approximate time and space derivatives may face difficulties. It should be noted that the regularity of functional derivatives of the system of nonlinear equations associated with HJ equations is different than the regularity of the time and space derivatives in HJ equations. In all developments, we assume regularity of functional derivatives of the system of nonlinear equations. Our iterative method works well and the solution of HJ equation is approximately regular.

2. Frozen Jacobian iterative method

We have constructed a new frozen Jacobian multi-step iterative method (EEAF). The convergence order of EEAF method is $3m-3$, and it uses two Jacobians and one LU-factorization that makes it computationally very efficient. A single evaluation of the system of nonlinear equations and three solutions of systems of lower and upper triangular systems make an increment of an additive factor of three in the convergence order of the iterative method per multi-step. The number of vector-vector and matrix-vector multiplications are $2m-1$ and $3m-1$, respectively. The computational cost of matrix-vector multiplications is considerable, but the achieved high order of convergence provides us a good efficiency index of the iterative method.

3. Convergence analysis

In this section, first, we will establish the proof of convergence order for the base method i.e., $m = 3$. At the second place, we will use the mathematical induction for the proof of convergence order of multi-step part i.e., $m \geq 3$. In our convergence analysis, the expansion system of nonlinear equations around the simple root is made by utilizing Taylor's series and hence, we deal with higher order Fréchet derivatives. The constraint of Fréchet differentiability on the system of nonlinear equations is essential because it is the Fréchet differentiability that is the responsible for linearization of the system of nonlinear equation. A function $\mathbf{F}(\cdot)$

is said to be Fréchet differentiable at a point $\boldsymbol{\phi}$ if there exists a linear operator $\mathbf{A} \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^q)$ such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{||\mathbf{F}(\boldsymbol{\phi}+\mathbf{h}) - \mathbf{F}(\boldsymbol{\phi}) - \mathbf{A}\mathbf{h}||}{||\mathbf{h}||} = 0\,.$$

The linear operator $\mathbf{A}$ is called the first order Fréchet derivative and we denote it by $\mathbf{F}'(\boldsymbol{\phi})$. The higher order Fréchet derivatives can be computed recursively as follows

$$\begin{aligned}\mathbf{F}'(\boldsymbol{\phi}) &= \text{Jacobian}\,(\mathbf{F}(\boldsymbol{\phi}))\,,\\ \mathbf{F}^j(\boldsymbol{\phi})\mathbf{v}^{j-1} &= \text{Jacobian}\left(\mathbf{F}^{j-1}(\boldsymbol{\phi})\mathbf{v}^{j-1}\right), \quad j \geq 2\,,\end{aligned}$$

where $\mathbf{v}$ is a vector independent from $\boldsymbol{\phi}$.

$$\text{FTUC} = \begin{cases} \text{number of steps} = m \geq 3, \\ \text{convergence-order} = 3m-4, \\ \text{function evaluations} = m-1, \\ \text{jacobian evaluations} = 2, \\ \text{LU-factorization } = 1, \\ \text{matrix-vector multiplications} = m-1, \\ \text{vector-vector multiplications} = m+1, \\ \text{number of solutions of systems} \\ \text{of lower and upper triangular} \\ \text{systems of equations} = 2m-2, \end{cases} \quad \begin{cases} \text{base method} \longrightarrow \begin{cases} \mathbf{y}_0 = \text{initial guess}, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{y}_0)\,, \\ \mathbf{y}_1 = \mathbf{y}_0 - \boldsymbol{\phi}_1, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_2 = \mathbf{F}(\mathbf{y}_1)\,, \\ \mathbf{y}_2 = \mathbf{y}_1 - 3\,\boldsymbol{\phi}_2, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_3 = \mathbf{F}'(\mathbf{y}_2)\,\boldsymbol{\phi}_2, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_4 = \mathbf{F}'(\mathbf{y}_2)\,\boldsymbol{\phi}_3, \\ \mathbf{y}_3 = \mathbf{y}_1 - \frac{7}{4}\,\boldsymbol{\phi}_2 + \frac{1}{2}\,\boldsymbol{\phi}_3 + \frac{1}{4}\,\boldsymbol{\phi}_4, \end{cases} \\ \text{multi-step part} \rightarrow \begin{cases} \text{for } s = 4, m, \\ \quad \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_5 = \mathbf{F}(\mathbf{y}_s)\,, \\ \quad \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_6 = \mathbf{F}'(\mathbf{y}_2)\,\boldsymbol{\phi}_5, \\ \quad \mathbf{y}_s = \mathbf{y}_{s-1} - 2\,\boldsymbol{\phi}_5 + \boldsymbol{\phi}_6, \\ \text{end}, \\ \mathbf{y}_0 = \mathbf{y}_m\,, \end{cases} \end{cases}$$

$$\text{MSF} = \begin{cases} \text{number of steps} = m, \\ \text{convergence-order} = 3m, \\ \text{function evaluations} = m, \\ \text{Jacobian evaluations} = 2, \\ \text{second-order Fréchet derivative} = 1, \\ \text{LU-factorization} = 1, \\ \text{matrix-vector multiplications} = 2m-2, \\ \text{vector-vector multiplications} = m+2, \\ \text{number of solutions of systems} \\ \text{of lower and upper triangular} \\ \text{systems of equations} = 3m-1, \end{cases} \quad \begin{cases} \text{base method} \longrightarrow \begin{cases} \mathbf{y}_0 = \text{initial guess}, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{y}_0)\,, \\ \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_2 = \mathbf{F}''(\mathbf{y}_0)\,\boldsymbol{\phi}_1^2, \\ \mathbf{y}_1 = \mathbf{y}_0 - \boldsymbol{\phi}_1 - \frac{1}{2}\,\boldsymbol{\phi}_2, \end{cases} \\ \text{multi-step part} \rightarrow \begin{cases} \text{for } s = 2, m, \\ \quad \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_3 = \mathbf{F}(\mathbf{y}_s)\,, \\ \quad \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_4 = \mathbf{F}'(\mathbf{y}_1)\,\boldsymbol{\phi}_3 \\ \quad \mathbf{F}'(\mathbf{y}_0)\,\boldsymbol{\phi}_5 = \mathbf{F}'(\mathbf{y}_1)\,\boldsymbol{\phi}_4, \\ \quad \mathbf{y}_s = \mathbf{y}_{s-1} - 3\,\boldsymbol{\phi}_3 \\ \qquad\quad +3\,\boldsymbol{\phi}_4 - \boldsymbol{\phi}_5, \\ \text{end}, \\ \mathbf{y}_0 = \mathbf{y}_m\,. \end{cases} \end{cases}$$

$$
\text{EEAF} = \begin{cases} \text{number of steps} = m \geq 3, \\ \text{convergence order} = 3m-3, \\ \text{function evaluations} = m-1, \\ \text{Jacobian evaluations} = 2, \\ \text{LU-factorization} = 1, \\ \text{matrix-vector} \\ \quad \text{multiplications} = 2m-3, \\ \text{vector-vector} \\ \quad \text{multiplications} = 3m-4, \\ \text{number of lower and} \\ \text{upper triangular systems } = 3m-5, \end{cases} \quad \begin{cases} \text{base method} \longrightarrow & \begin{cases} \boldsymbol{\phi}_0 = \text{initial guess}, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_1 = \mathbf{F}(\boldsymbol{\phi}_0), \\ \boldsymbol{\phi}_1 = \boldsymbol{\phi}_0 - \boldsymbol{\psi}_1, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_2 = \mathbf{F}(\boldsymbol{\phi}_1), \\ \boldsymbol{\phi}_2 = \boldsymbol{\phi}_1 - 1/2\,\boldsymbol{\psi}_2, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_3 = \mathbf{F}'(\boldsymbol{\phi}_2)\,\boldsymbol{\psi}_2, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_4 = \mathbf{F}'(\boldsymbol{\phi}_2)\,\boldsymbol{\psi}_3, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_5 = \mathbf{F}'(\boldsymbol{\phi}_2)\,\boldsymbol{\psi}_4, \\ \boldsymbol{\phi}_3 = \boldsymbol{\phi}_1 - 17/4\,\boldsymbol{\psi}_2 + 27/4\,\boldsymbol{\psi}_3 \\ \qquad -19/4\,\boldsymbol{\psi}_4 + 5/4\,\boldsymbol{\psi}_5, \end{cases} \\ \text{multi-step part} \rightarrow & \begin{cases} \text{for} \quad s = 4, m, \\ \qquad \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_6 = \mathbf{F}(\boldsymbol{\phi}_{s-1}), \\ \qquad \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_7 = \mathbf{F}'(\boldsymbol{\phi}_2)\,\boldsymbol{\psi}_6, \\ \qquad \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_8 = \mathbf{F}'(\boldsymbol{\phi}_2)\,\boldsymbol{\psi}_7, \\ \qquad \boldsymbol{\phi}_s = \boldsymbol{\phi}_{s-1} - 13/4\,\boldsymbol{\psi}_6, \\ \qquad\qquad +7/2\,\boldsymbol{\psi}_7 - 5/4\,\boldsymbol{\psi}_8, \\ \text{end}, \\ \boldsymbol{\phi}_0 = \boldsymbol{\phi}_m\,. \end{cases} \end{cases}
$$

Theorem 3.1. *Let $\mathbf{F} : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently Fréchet differentiable function on an open convex neighborhood Γ of $\boldsymbol{\phi}^* \in \mathbb{R}^n$ with $\mathbf{F}(\boldsymbol{\phi}^*) = 0$ and $\det(\mathbf{F}'(\boldsymbol{\phi}^*)) \neq 0$, where $\mathbf{F}'(\boldsymbol{\phi})$ denotes the Fréchet derivative of $\mathbf{F}(\boldsymbol{\phi})$. Let $\mathbf{A}_1 = \mathbf{F}'(\boldsymbol{\phi}^*)$ and $\mathbf{A}_j = \frac{1}{j!}\,\mathbf{F}'(\boldsymbol{\phi}^*)^{-1}\,\mathbf{F}^{(j)}(\boldsymbol{\phi}^*)$ for $j \geq 2$, where $\mathbf{F}^{(j)}(\boldsymbol{\phi})$ denotes j-order Fréchet derivative of $\mathbf{F}(\boldsymbol{\phi})$. Then, for $m = 3$, with an initial guess in the neighborhood of $\boldsymbol{\phi}^*$, the sequence $\{\boldsymbol{\phi}_m\}$ generated by EEAF converges to $\boldsymbol{\phi}^*$ with local order of convergence at least six and error*

$$
\mathbf{e}_3 = \mathbf{L}{\mathbf{e}_0}^6 + O\left({\mathbf{e}_0}^7\right)\,,
$$

where $\mathbf{e}_0 = \boldsymbol{\phi}_0 - \boldsymbol{\phi}^*$, ${\mathbf{e}_0}^p = \overbrace{(\mathbf{e}_0, \mathbf{e}_0, \ldots, \mathbf{e}_0)}^{p\ times}$, and $\mathbf{L} = 26\mathbf{A}_2^5 + \mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 13/4\mathbf{A}_3\mathbf{A}_2^3$ is a 6-linear function, i.e., $\mathbf{L} \in \mathbb{L}\,\overbrace{(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^n, \cdots, \mathbb{R}^n)}^{6\text{-}times}$ with $\mathbf{L}{\mathbf{e}_0}^6 \in \mathbb{R}^n$.

Proof. We define the error at the nth step $\mathbf{e}_n = \boldsymbol{\phi}_n - \boldsymbol{\phi}^*$. To complete the convergence proof, we performed the detailed computations by using Maple and details are provided below in sequence.

$$
\begin{aligned}
\mathbf{F}(\boldsymbol{\phi}_0) =& \mathbf{A}_1\Big(\mathbf{e}_0 + \mathbf{A}_2\mathbf{e}_0^2 + \mathbf{A}_3\mathbf{e}_0^3 + \mathbf{A}_4\mathbf{e}_0^4 + \mathbf{A}_5\mathbf{e}_0^5 + \mathbf{A}_6\mathbf{e}_0^6 + \mathbf{A}_7\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right)\Big), \\
\mathbf{F}^{-1}(\boldsymbol{\phi}_0) =& \Big(\mathbf{I} - 2\mathbf{A}_2\mathbf{e}_0 + \Big(-3\mathbf{A}_3 + 4\mathbf{A}_2^2\Big)\mathbf{e}_0^2 + \Big(-4\mathbf{A}_4 + 6\mathbf{A}_3\mathbf{A}_2 + 6\mathbf{A}_2\mathbf{A}_3 - 8\mathbf{A}_2^3\Big)\mathbf{e}_0^3 + \Big(-5\mathbf{A}_5 + 8\mathbf{A}_4\mathbf{A}_2 \\
& + 9\mathbf{A}_3^2 + 8\mathbf{A}_2\mathbf{A}_4 - 12\mathbf{A}_3\mathbf{A}_2^2 - 12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 12\mathbf{A}_2^2\mathbf{A}_3 + 16\mathbf{A}_2^4\Big)\mathbf{e}_0^4 + \Big(-6\mathbf{A}_6 + 10\mathbf{A}_5\mathbf{A}_2 \\
& + 12\mathbf{A}_4\mathbf{A}_3 + 12\mathbf{A}_3\mathbf{A}_4 + 10\mathbf{A}_2\mathbf{A}_5 - 16\mathbf{A}_4\mathbf{A}_2^2 - 18\mathbf{A}_3^2\mathbf{A}_2 - 16\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 18\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 18\mathbf{A}_2\mathbf{A}_3^2 \\
& - 16\mathbf{A}_2^2\mathbf{A}_4 + 24\mathbf{A}_3\mathbf{A}_2^3 + 24\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 24\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 + 24\mathbf{A}_2^3\mathbf{A}_3 - 32\mathbf{A}_2^5\Big)\mathbf{e}_0^5 + \cdots + O\Big(\mathbf{e}_0^8\Big)\Big)\mathbf{A}_1^{-1}, \\
\mathbf{e}_1 =& \mathbf{A}_2\mathbf{e}_0^2 + \Big(2\mathbf{A}_3 - 2\mathbf{A}_2^2\Big)\mathbf{e}_0^3 + \Big(4\mathbf{A}_2^3 + 3\mathbf{A}_4 - 4\mathbf{A}_2\mathbf{A}_3 - 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(6\mathbf{A}_3\mathbf{A}_2^2 + 6\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 8\mathbf{A}_2^4 \\
& + 8\mathbf{A}_2^2\mathbf{A}_3 + 4\mathbf{A}_5 - 6\mathbf{A}_2\mathbf{A}_4 - 6\mathbf{A}_3^2 - 4\mathbf{A}_4\mathbf{A}_2\Big)\mathbf{e}_0^5 + \cdots + O\Big(\mathbf{e}_0^8\Big),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}(\boldsymbol{\phi}_1) =& \mathbf{A}_1\Big(\mathbf{A}_2\mathbf{e}_0^2 + \Big(2\mathbf{A}_3 - 2\mathbf{A}_2^2\Big)\mathbf{e}_0^3 + \Big(5\mathbf{A}_2^3 + 3\mathbf{A}_4 - 4\mathbf{A}_2\mathbf{A}_3 - 3\mathbf{A}_3\mathbf{A}_2\Big)\mathbf{e}_0^4 + \Big(6\mathbf{A}_3\mathbf{A}_2^2 + 8\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 \\
& - 12\mathbf{A}_2^4 + 10\mathbf{A}_2^2\mathbf{A}_3 + 4\mathbf{A}_5 - 6\mathbf{A}_2\mathbf{A}_4 - 6\mathbf{A}_3^2 - 4\mathbf{A}_4\mathbf{A}_2\Big)\mathbf{e}_0^5 + \Big(- 19\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 19\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 \\
& - 11\mathbf{A}_3\mathbf{A}_2^3 + 11\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 9\mathbf{A}_3^2\mathbf{A}_2 + 8\mathbf{A}_4\mathbf{A}_2^2 + 16\mathbf{A}_2\mathbf{A}_3^2 + 12\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 15\mathbf{A}_2^2\mathbf{A}_4 + 28\mathbf{A}_2^5 \\
& + 5\mathbf{A}_6 - 24\mathbf{A}_2^3\mathbf{A}_3 - 8\mathbf{A}_2\mathbf{A}_5 - 9\mathbf{A}_3\mathbf{A}_4 - 8\mathbf{A}_4\mathbf{A}_3 - 5\mathbf{A}_5\mathbf{A}_2\Big)\mathbf{e}_0^6 + \cdots + O\Big(\mathbf{e}_0^8\Big)\Big), \\
\boldsymbol{\psi}_2 =& \mathbf{A}_2\mathbf{e}_0^2 + \Big(- 4\mathbf{A}_2^2 + 2\mathbf{A}_3\Big)\mathbf{e}_0^3 + \Big(- 6\mathbf{A}_3\mathbf{A}_2 + 13\mathbf{A}_2^3 - 8\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_4\Big)\mathbf{e}_0^4 + \Big(- 38\mathbf{A}_2^4 - 8\mathbf{A}_4\mathbf{A}_2 \\
& + 20\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 18\mathbf{A}_3\mathbf{A}_2^2 - 12\mathbf{A}_3^2 + 26\mathbf{A}_2^2\mathbf{A}_3 - 12\mathbf{A}_2\mathbf{A}_4 + 4\mathbf{A}_5\Big)\mathbf{e}_0^5 + \Big(- 59\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
& - 55\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 50\mathbf{A}_3\mathbf{A}_2^3 + 27\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 27\mathbf{A}_3^2\mathbf{A}_2 + 24\mathbf{A}_4\mathbf{A}_2^2 + 40\mathbf{A}_2\mathbf{A}_3^2 + 36\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 \\
& + 39\mathbf{A}_2^2\mathbf{A}_4 + 104\mathbf{A}_2^5 + 5\mathbf{A}_6 - 76\mathbf{A}_2^3\mathbf{A}_3 - 16\mathbf{A}_2\mathbf{A}_5 - 18\mathbf{A}_3\mathbf{A}_4 - 16\mathbf{A}_4\mathbf{A}_3 - 10\mathbf{A}_5\mathbf{A}_2\Big)\mathbf{e}_0^6 \\
& + \cdots + O\Big(\mathbf{e}_0^8\Big), \\
\mathbf{e}_2 =& 1/2\mathbf{A}_2\mathbf{e}_0^2 + 1/2\mathbf{A}_3\mathbf{e}_0^3 + (3\mathbf{A}_4 - 5/2\mathbf{A}_2^3)\mathbf{e}_0^4 + (2\mathbf{A}_5 - 5\mathbf{A}_2^2\mathbf{A}_3 + 11\mathbf{A}_2^4 - 4\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 3\mathbf{A}_3\mathbf{A}_2^2)\mathbf{e}_0^5 \\
& + (-36\mathbf{A}_2^5 - 15/2\mathbf{A}_2^2\mathbf{A}_4 + 22\mathbf{A}_2^3\mathbf{A}_3 + 35/2\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 + 31/2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 13\mathbf{A}_3\mathbf{A}_2^3 \\
& - 11/2\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 - 9/2\mathbf{A}_3^2\mathbf{A}_2 - 4\mathbf{A}_4\mathbf{A}_2^2 + 5/2\mathbf{A}_6 - 8\mathbf{A}_2\mathbf{A}_3^2 - 6\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3)\mathbf{e}_0^6 + (24\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 \\
& + 24\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 + 20\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 + 21\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 + 17\mathbf{A}_3^2\mathbf{A}_2^2 + 18\mathbf{A}_4\mathbf{A}_2^3 - 10\mathbf{A}_2^2\mathbf{A}_5 - 7\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 \\
& - 6\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 - 6\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 - 5\mathbf{A}_5\mathbf{A}_2^2 + 35\mathbf{A}_2^2\mathbf{A}_3^2 + 26\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 + 31\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 - 57\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 \\
& - 72\mathbf{A}_2^4\mathbf{A}_3 - 53\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 + 104\mathbf{A}_2^6 - 11\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 + 33\mathbf{A}_2^3\mathbf{A}_4 - 9\mathbf{A}_3^3 - 8\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 - 48\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 \\
& - 42\mathbf{A}_3\mathbf{A}_2^4 - 12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4 - 9\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 + 3\mathbf{A}_7)\mathbf{e}_0^7 + O\left(\mathbf{e}_0^8\right), \\
\mathbf{F}'(\boldsymbol{\phi}_2) =& \mathbf{A}_1\Big(\mathbf{I} + \mathbf{A}_2^2\mathbf{e}_0^2 + 2\mathbf{A}_2\mathbf{A}_3\mathbf{e}_0^3 + (3/4\mathbf{A}_3\mathbf{A}_2^2 + 3\mathbf{A}_2\mathbf{A}_4 - 5\mathbf{A}_2^4)\mathbf{e}_0^4 + (3/2\mathbf{A}_3^2\mathbf{A}_2 + 3/2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 \\
& - 6\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 8\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 10\mathbf{A}_2^3\mathbf{A}_3 + 4\mathbf{A}_2\mathbf{A}_5 + 22\mathbf{A}_2^5)\mathbf{e}_0^5 + (-11\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 - 9\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 \\
& - 8\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 + 1/2\mathbf{A}_4\mathbf{A}_2^3 + 9/4\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 - 16\mathbf{A}_2^2\mathbf{A}_3^2 - 12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 35\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 \\
& + 31\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 + 3\mathbf{A}_3^3 + 26\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3 - 15/2\mathbf{A}_3\mathbf{A}_2^4 + 9/4\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 - 15\mathbf{A}_2^3\mathbf{A}_4 - 72\mathbf{A}_2^6 + 5\mathbf{A}_2\mathbf{A}_6 \\
& + 44\mathbf{A}_2^4\mathbf{A}_3)\mathbf{e}_0^6 + (-114\mathbf{A}_2^4\mathbf{A}_3\mathbf{A}_2 - 106\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2^2 - 96\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^3 - 84\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^4 + 33\mathbf{A}_3\mathbf{A}_2^5 \\
& - 22\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_3 - 18\mathbf{A}_2\mathbf{A}_3^3 - 16\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_4\mathbf{A}_2^2\mathbf{A}_3 + 3\mathbf{A}_3\mathbf{A}_5\mathbf{A}_2 + 52\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 + 66\mathbf{A}_2^4\mathbf{A}_4 \\
& + 9/2\mathbf{A}_3\mathbf{A}_4\mathbf{A}_3 - 15\mathbf{A}_3\mathbf{A}_2^3\mathbf{A}_3 + 6\mathbf{A}_2\mathbf{A}_7 - 24\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_4 - 18\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4 - 20\mathbf{A}_2^3\mathbf{A}_5 + 9/2\mathbf{A}_3^2\mathbf{A}_4 \\
& + 62\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 48\mathbf{A}_2^3\mathbf{A}_4\mathbf{A}_2 + 48\mathbf{A}_2^2\mathbf{A}_3^2\mathbf{A}_2 + 40\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 27/2\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 \\
& + 42\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2^2 + 34\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2^2 - 21/2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 + 36\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^3 - 12\mathbf{A}_3^2\mathbf{A}_2^3 + 3\mathbf{A}_3\mathbf{A}_2\mathbf{A}_5 \\
& - 144\mathbf{A}_2^5\mathbf{A}_3 - 14\mathbf{A}_2^2\mathbf{A}_5\mathbf{A}_2 - 12\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 - 12\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 - 10\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2^2 + \mathbf{A}_4\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 \\
& + 70\mathbf{A}_2^3\mathbf{A}_3^2 + \mathbf{A}_4\mathbf{A}_3\mathbf{A}_2^2 + 208\mathbf{A}_2^7)\mathbf{e}_0^7\Big) + O\left(\mathbf{e}_0^8\right), \\
\boldsymbol{\psi}_3 =& \mathbf{A}_2\mathbf{e}_0^2 + (-6\mathbf{A}_2^2 + 2\mathbf{A}_3)\mathbf{e}_0^3 + (26\mathbf{A}_2^3 - 9\mathbf{A}_3\mathbf{A}_2 - 12\mathbf{A}_2\mathbf{A}_3 + 3\mathbf{A}_4)\mathbf{e}_0^4 + (-94\mathbf{A}_2^4 + 40\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 \\
& + 36\mathbf{A}_3\mathbf{A}_2^2 - 12\mathbf{A}_4\mathbf{A}_2 + 52\mathbf{A}_2^2\mathbf{A}_3 - 18\mathbf{A}_3^2 - 18\mathbf{A}_2\mathbf{A}_4 + 4\mathbf{A}_5)\mathbf{e}_0^5 + (300\mathbf{A}_2^5 + 78\mathbf{A}_2^2\mathbf{A}_4 - 188\mathbf{A}_2^3\mathbf{A}_3 \\
& - 24\mathbf{A}_2\mathbf{A}_5 - 27\mathbf{A}_3\mathbf{A}_4 - 24\mathbf{A}_4\mathbf{A}_3 - 15\mathbf{A}_5\mathbf{A}_2 - 145\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2 - 135\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2 - 509/4\mathbf{A}_3\mathbf{A}_2^3 \\
& + 54\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2 + 54\mathbf{A}_3^2\mathbf{A}_2 + 48\mathbf{A}_4\mathbf{A}_2^2 + 5\mathbf{A}_6 + 80\mathbf{A}_2\mathbf{A}_3^2 + 72\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3)\mathbf{e}_0^6 + (-196\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2 \\
& - 204\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2 - 389/2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2 - 182\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2 - 353/2\mathbf{A}_3^2\mathbf{A}_2^2 - 172\mathbf{A}_4\mathbf{A}_2^3 + 104\mathbf{A}_2^2\mathbf{A}_5 \\
& + 68\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2 + 72\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2 + 72\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2 + 60\mathbf{A}_5\mathbf{A}_2^2 - 30\mathbf{A}_2\mathbf{A}_6 - 36\mathbf{A}_3\mathbf{A}_5 - 36\mathbf{A}_4^2 - 30\mathbf{A}_5\mathbf{A}_3 \\
& - 18\mathbf{A}_6\mathbf{A}_2 - 290\mathbf{A}_2^2\mathbf{A}_3^2 - 509/2\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3 - 270\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3 + 462\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2 + 600\mathbf{A}_2^4\mathbf{A}_3 \\
& + 434\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2 - 868\mathbf{A}_2^6 + 108\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3 - 282\mathbf{A}_2^3\mathbf{A}_4 + 108\mathbf{A}_3^3 + 96\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3 + 837/2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3
\end{aligned}
$$

$$
\begin{aligned}
&+411\mathbf{A}_3\mathbf{A}_2^4+120\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4+108\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4+6\mathbf{A}_7)\mathbf{e}_0^7+\left(\mathbf{e}_0^8\right),\\
\boldsymbol{\psi}_4 =&\mathbf{A}_2\mathbf{e}_0^2+(-8\mathbf{A}_2^2+2\mathbf{A}_3)\mathbf{e}_0^3+(3\mathbf{A}_4+43\mathbf{A}_2^3-16\mathbf{A}_2\mathbf{A}_3-12\mathbf{A}_3\mathbf{A}_2)\mathbf{e}_0^4+(4\mathbf{A}_5-24\mathbf{A}_3^2-16\mathbf{A}_4\mathbf{A}_2\\
&-186\mathbf{A}_2^4+66\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2+60\mathbf{A}_3\mathbf{A}_2^2+86\mathbf{A}_2^2\mathbf{A}_3-24\mathbf{A}_2\mathbf{A}_4)\mathbf{e}_0^5+(5\mathbf{A}_6+693\mathbf{A}_2^5+129\mathbf{A}_2^2\mathbf{A}_4\\
&-372\mathbf{A}_2^3\mathbf{A}_3-32\mathbf{A}_2\mathbf{A}_5-36\mathbf{A}_3\mathbf{A}_4-32\mathbf{A}_4\mathbf{A}_3-20\mathbf{A}_5\mathbf{A}_2-286\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2\\
&-267\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2-511/2\mathbf{A}_3\mathbf{A}_2^3+89\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2+90\mathbf{A}_3^2\mathbf{A}_2+80\mathbf{A}_4\mathbf{A}_2^2+132\mathbf{A}_2\mathbf{A}_3^2+120\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3)\mathbf{e}_0^6\\
&+(-386\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2-402\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2-391\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2-360\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2-355\mathbf{A}_3^2\mathbf{A}_2^2-344\mathbf{A}_4\mathbf{A}_2^3\\
&+172\mathbf{A}_2^2\mathbf{A}_5+112\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2+120\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2+120\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2+100\mathbf{A}_5\mathbf{A}_2^2-40\mathbf{A}_2\mathbf{A}_6-48\mathbf{A}_3\mathbf{A}_5\\
&-48\mathbf{A}_4^2-40\mathbf{A}_5\mathbf{A}_3-24\mathbf{A}_6\mathbf{A}_2-572\mathbf{A}_2^2\mathbf{A}_3^2-511\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3-534\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3+1064\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2\\
&+1386\mathbf{A}_2^4\mathbf{A}_3+996\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2-2296\mathbf{A}_2^6+178\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3-558\mathbf{A}_2^3\mathbf{A}_4+180\mathbf{A}_3^3+160\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3\\
&+1951/2\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3+1929/2\mathbf{A}_3\mathbf{A}_2^4+198\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4+180\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4+6\mathbf{A}_7)\mathbf{e}_0^7+O\left(\mathbf{e}_0^8\right),\\
\boldsymbol{\psi}_5 =&\mathbf{A}_2\mathbf{e}_0^2+(-10\mathbf{A}_2^2+2\mathbf{A}_3)\mathbf{e}_0^3+(64\mathbf{A}_2^3-15\mathbf{A}_3\mathbf{A}_2-20\mathbf{A}_2\mathbf{A}_3+3\mathbf{A}_4)\mathbf{e}_0^4+(-322\mathbf{A}_2^4+98\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\\
&+90\mathbf{A}_3\mathbf{A}_2^2-20\mathbf{A}_4\mathbf{A}_2+128\mathbf{A}_2^2\mathbf{A}_3-30\mathbf{A}_3^2-30\mathbf{A}_2\mathbf{A}_4+4\mathbf{A}_5)\mathbf{e}_0^5+(1375\mathbf{A}_2^5+192\mathbf{A}_2^2\mathbf{A}_4\\
&-644\mathbf{A}_2^3\mathbf{A}_3-40\mathbf{A}_2\mathbf{A}_5-45\mathbf{A}_3\mathbf{A}_4-40\mathbf{A}_4\mathbf{A}_3-25\mathbf{A}_5\mathbf{A}_2-494\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2-463\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^2\\
&-1787/4\mathbf{A}_3\mathbf{A}_2^3+132\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2+135\mathbf{A}_3^2\mathbf{A}_2+120\mathbf{A}_4\mathbf{A}_2^2+5\mathbf{A}_6+196\mathbf{A}_2\mathbf{A}_3^2+180\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3)\mathbf{e}_0^6\\
&+(-666\mathbf{A}_2^2\mathbf{A}_4\mathbf{A}_2-696\mathbf{A}_2\mathbf{A}_3^2\mathbf{A}_2-1367/2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2-624\mathbf{A}_2\mathbf{A}_4\mathbf{A}_2^2-1247/2\mathbf{A}_3^2\mathbf{A}_2^2\\
&-600\mathbf{A}_4\mathbf{A}_2^3+256\mathbf{A}_2^2\mathbf{A}_5+166\mathbf{A}_2\mathbf{A}_5\mathbf{A}_2+180\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2+180\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2+150\mathbf{A}_5\mathbf{A}_2^2-50\mathbf{A}_2\mathbf{A}_6\\
&-60\mathbf{A}_3\mathbf{A}_5-60\mathbf{A}_4^2-50\mathbf{A}_5\mathbf{A}_3-30\mathbf{A}_6\mathbf{A}_2-988\mathbf{A}_2^2\mathbf{A}_3^2-1787/2\mathbf{A}_3\mathbf{A}_2^2\mathbf{A}_3-926\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3\\
&+2108\mathbf{A}_2^3\mathbf{A}_3\mathbf{A}_2+2750\mathbf{A}_2^4\mathbf{A}_3+1974\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2^2-5170\mathbf{A}_2^6+264\mathbf{A}_2\mathbf{A}_4\mathbf{A}_3-966\mathbf{A}_2^3\mathbf{A}_4+270\mathbf{A}_3^3\\
&+240\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3+1949\mathbf{A}_2\mathbf{A}_3\mathbf{A}_2^3+3849/2\mathbf{A}_3\mathbf{A}_2^4+294\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4+270\mathbf{A}_3\mathbf{A}_2\mathbf{A}_4+6\mathbf{A}_7)\mathbf{e}_0^7+O\left(\mathbf{e}_0^8\right),\\
\mathbf{e}_3 =&\left(26\mathbf{A}_2^5+\mathbf{A}_2^2\mathbf{A}_3\mathbf{A}_2-13/4\mathbf{A}_3\mathbf{A}_2^3\right)\mathbf{e}_0^6+O\left(\mathbf{e}_0^7\right).
\end{aligned}
$$

□

Now we present the proof of convergence of EEAF via the mathematical induction.

Theorem 3.2. *The convergence order of EEAF method is $3m-3$ for $m \geq 3$.*

Proof. All the computation are made under the assumption of Theorem 3.1. We know from Theorem 3.1 that the convergence order of EEAF method is six for $m = 3$. Now we assume that the convergence order of EEAF is $3s-3$ for $s \geq 3$, and we will prove that the convergence order of EEAF is $3s$ for $(s+1)$-th step. If the convergence order of EEAF is $3s-3$ then

$$\mathbf{e}_s = \mathbf{y}_s - \mathbf{y}^* \sim d_1\mathbf{e}_0^{3s-3}, \tag{3.1}$$

where d_1 is the asymptotic constant and symbol $\sim$ means the approximation. By using (3.1), we perform the following steps to complete the proof.

$$
\begin{aligned}
\mathbf{F}(\mathbf{y}_0)^{-1} &\sim (\mathbf{I}-2\mathbf{A}_2\mathbf{e}_0)\,\mathbf{A}_1^{-1},\\
\mathbf{F}(\mathbf{y}_s) &\sim \mathbf{A}_1 d_1\mathbf{e}_0^{3s-3},\\
\mathbf{F}'(\mathbf{y}_2) &\sim \mathbf{A}_1\left(\mathbf{I}+\mathbf{A}_2^2\mathbf{e}_0^2\right),\\
\boldsymbol{\phi}_6 &\sim (\mathbf{I}-2\mathbf{e}_0\mathbf{A}_2)d_1\mathbf{e}_0^{3s-3},\\
\boldsymbol{\phi}_7 &\sim \left(d_1-4\mathbf{A}_2d_1\mathbf{e}_0+5\mathbf{A}_2^2d_1\mathbf{e}_0^2-4\mathbf{A}_2^3d_1\mathbf{e}_0^3+4\mathbf{A}_2^4d_1\mathbf{e}_0^4\right)\mathbf{e}_0^{3s-3},\\
\boldsymbol{\phi}_8 &\sim \left(d_1-6\mathbf{A}_2d_1\mathbf{e}_0+14\mathbf{A}_2^2d_1\mathbf{e}_0^2-20\mathbf{A}_2^3d_1\mathbf{e}_0^3+25\mathbf{A}_2^4d_1\mathbf{e}_0^4-22\mathbf{A}_2^5d_1\mathbf{e}_0^5+12\mathbf{A}_2^6d_1\mathbf{e}_0^6-8\mathbf{A}_2^7d_1\mathbf{e}_0^7\right)\mathbf{e}_0^{3s-3},\\
\mathbf{e}_{s+1} &\sim \left(11\mathbf{A}_2^3d_1\mathbf{e}_0^3-(69\mathbf{A}_2^4d_1\mathbf{e}_0^4)/4+(55\mathbf{A}_2^5d_1\mathbf{e}_0^5)/2-15\mathbf{A}_2^6d_1\mathbf{e}_0^6+10\mathbf{A}_2^7d_1\mathbf{e}_0^7\right)\mathbf{e}_0^{3s-3},\\
\mathbf{e}_{s+1} &\sim 11\mathbf{A}_2^3d_1\mathbf{e}_0^{3s}.
\end{aligned}
$$

This completes the proof. □

4. Comparison of computational cost

The Table 1 shows that the iterative methods FTUC and EEAF are the competitors. If we look at the last column of the Table 1, the order of convergence of the iterative method EEAF is one more than that of FTUC. But the increment in the convergence order of EEAF compare to FUTC is at the cost of additional $m-2$ Matrix-vector multiplications, $2m-5$ vector-vector multiplications and $m-3$ solutions of lower and upper triangular systems. As the number of steps will increase, the efficiency of FTUC is higher than that of EEAF but for lower number of steps the efficiency of EEAF is comparable. For instance $m = 4$, we have the following version of EEAF

$$\text{EEAF}(m=4) = \begin{cases} \boldsymbol{\phi}_0 = \text{initial guess}, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_1 = \mathbf{F}(\boldsymbol{\phi}_0), \\ \boldsymbol{\phi}_1 = \boldsymbol{\phi}_0 - \boldsymbol{\psi}_1, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_2 = \mathbf{F}(\boldsymbol{\phi}_1), \\ \boldsymbol{\phi}_2 = \boldsymbol{\phi}_1 - 1/2\,\boldsymbol{\psi}_2, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_3 = \mathbf{F}'(\boldsymbol{\phi}_2)\,\boldsymbol{\psi}_2, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_4 = \mathbf{F}'(\boldsymbol{\phi}_2)\,\boldsymbol{\psi}_3, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_5 = \mathbf{F}'(\boldsymbol{\phi}_2)\,\boldsymbol{\psi}_4, \\ \boldsymbol{\phi}_3 = \boldsymbol{\phi}_1 - 17/4\,\boldsymbol{\psi}_2 + 27/4\,\boldsymbol{\psi}_3 - 19/4\,\boldsymbol{\psi}_4 + 5/4\,\boldsymbol{\psi}_5, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_6 = \mathbf{F}(\boldsymbol{\phi}_3), \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_7 = \mathbf{F}'(\boldsymbol{\phi}_2)\,\boldsymbol{\psi}_6, \\ \mathbf{F}'(\boldsymbol{\phi}_0)\,\boldsymbol{\psi}_8 = \mathbf{F}'(\boldsymbol{\phi}_2)\,\boldsymbol{\psi}_7, \\ \boldsymbol{\phi}_4 = \boldsymbol{\phi}_3 - 13/4\,\boldsymbol{\psi}_6 + 7/2\,\boldsymbol{\psi}_7 - 5/4\,\boldsymbol{\psi}_8, \\ \boldsymbol{\phi}_0 = \boldsymbol{\phi}_4\,. \end{cases}$$

The efficiency of iterative method EEAF($m = 4$) is higher than that of FTUC($m = 4$).

Table 1: Comparison of computational cost of different iterative methods

	HJ	MSF	FTUC	EEAF	EEAF-FTUC
Convergence order	$2m$	$3m$	$3m-4$	$3m-3$	1
Function evaluations	$m-1$	$m+1$	$m-1$	$m-1$	0
Jacobian evaluations	2	2	2	2	0
LU-factorization	1	1	1	1	0
Matrix-vector Multiplications	m	$2m-2$	$m-1$	$2m-3$	$m-2$
Vector-vector Multiplications	$2m$	$m+2$	$m+1$	$3m-4$	$2m-5$
Number of lower and upper triangular systems	$2m-1$	$3m-1$	$2m-2$	$3m-5$	$m-3$

5. Numerical testing

In this section, we will show the efficiency of our proposed iterative method for solving system of nonlinear equations associated with HJ equations. We use Chebyshev pseudospectral collocation method to discretize HJ equations for space and time. The temporal discretization makes the problem implicit in time and nonlinearity of Hamiltonian gives us the system of nonlinear equations.

The verification of CO is important and we adopt the following definition of computational CO (COC)

$$\text{COC} = \frac{\log\big(||\mathbf{F}(\mathbf{x}_{k+1})||_\infty / ||\mathbf{F}(\mathbf{x}_k)||_\infty\big)}{\log\big(||\mathbf{F}(\mathbf{x}_k)||_\infty / ||\mathbf{F}(\mathbf{x}_{k-1})||_\infty\big)}.$$

Consider the following system of nonlinear equations $\mathbf{F}(\mathbf{x}) = [F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x}), F_4(\mathbf{x})]^T = \mathbf{0}$,

$$\begin{aligned} F_1(\mathbf{x}) &= x_3\,x_2 + (x_2 + x_3)\,x_4 = 0,\\ F_2(\mathbf{x}) &= x_3\,x_1 + (x_1 + x_3)\,x_4 = 0,\\ F_3(\mathbf{x}) &= x_2\,x_1 + (x_1 + x_2)\,x_4 = 0,\\ F_4(\mathbf{x}) &= x_2\,x_1 + (x_1 + x_2)\,x_3 = 1. \end{aligned} \tag{5.1}$$

Let $\mathbf{d} = [d_1, d_2, d_3, d_4]^T$ be a constant vector, and $\mathbf{F}'(\mathbf{x})$ and $\mathbf{F}''(\mathbf{x})\mathbf{d} = (\mathbf{F}'(\mathbf{x})\mathbf{d})'$ can be written as

$$\mathbf{F}'(\mathbf{x}) = \begin{bmatrix} 0 & x_3 + x_4 & x_2 + x_4 & x_2 + x_3 \\ x_3 + x_4 & 0 & x_1 + x_4 & x_1 + x_3 \\ x_2 + x_4 & x_1 + x_4 & 0 & x_1 + x_2 \\ x_2 + x_3 & x_1 + x_3 & x_1 + x_2 & 0 \end{bmatrix}.$$

Table 2 shows that the computational COs are in agreement with the theoretical CO of the iterative scheme EEAF.

Table 2: EEAF : verification of CO for the problem (5.1).

Iter \	Steps	$m = 4$	$m = 5$	$m = 6$	$m = 7$
1	$\|\|\mathbf{F}(\mathbf{x}_k)\|\|_\infty$	1.48e-7	1.05e-9	7.43e-12	5.27e-14
2	-	2.00e-64	2.45e-111	3.81e-171	7.54e-244
3	-	3.06e-576	6.45e-1331	1.68e-2560	4.71e-4381
COC		9	12	15	18
Theoretical CO ($3m - 3$)		9	12	15	18

We also select a list of 1-D and 2-D HJ equations (5.2), (5.3) and (5.4), (5.5) (5.6), respectively.

$$\begin{cases} \boldsymbol{\phi}_t + \frac{1}{2}\left(\phi_x + 1\right)^2 = 0\,, & x \in (0, 2),\\ \text{initial condition:} & \phi(x, 0) = -\cos\left(\pi x\right),\\ \text{boundary condition:} & \phi(0, t) = \phi(2, t),\\ \qquad\text{functional derivative} & \\ F(\phi) = \phi_t + \frac{1}{2}\left(\phi_x + 1\right)^2, & \\ F'(\phi) = \partial_t + \left(\phi_x + 1\right)\partial_x, & \\ \mathbf{F}'(\boldsymbol{\phi}) = D_t + \operatorname{diag}\left(D_x\boldsymbol{\phi} + \mathbf{1}\right)D_x, & \end{cases} \tag{5.2}$$

$$\begin{cases} \boldsymbol{\phi}_t - \cos(\phi_x + 1) = 0\,, & x \in (0, 2),\\ \text{initial condition:} & \phi(x, 0) = -\cos\left(\pi x\right),\\ \text{boundary condition:} & \phi(0, t) = \phi(2, t),\\ \qquad\text{functional derivative} & \\ F(\phi) = \phi_t - \cos\left(\phi_x + 1\right), & \\ F'(\phi) = \partial_t + \sin\left(\phi_x + 1\right)\partial_x, & \\ \mathbf{F}'(\boldsymbol{\phi}) = D_t + \operatorname{diag}\left(\sin\left(D_x\boldsymbol{\phi} + \mathbf{1}\right)\right)D_x, & \end{cases} \tag{5.3}$$

$$\begin{cases} \boldsymbol{\phi}_t + \sqrt{\phi_x + \phi_y + 1} = 0\,, & (x,y) \in (0,1)^2, \\ \text{initial condition:} & \phi(x,y,0) = 0.25\,(\cos(2\pi x) - 1) \\ & \qquad (\cos(2\pi y) - 1) - 1, \\ \text{boundary condition:} & \text{periodic boundary condition,} \\ \qquad \text{functional derivative} & \\ F(\phi) = \phi_t + \sqrt{\phi_x + \phi_y + 1}, & \\ F'(\phi) = \partial_t + \frac{\partial_x + \partial_y}{\sqrt{\phi_x + \phi_y + 1}}, & \\ \mathbf{F}'(\boldsymbol{\phi}) = D_t + \operatorname{diag}\left(\sqrt{D_x\phi + D_y\phi + 1}\right)^{-1} & \\ \qquad (D_x + D_y)\,, & \end{cases} \tag{5.4}$$

$$\begin{cases} \boldsymbol{\phi}_t + \frac{1}{2}\,(\phi_x + \phi_y + 1)^2 = 0\,, & (x,y) \in (-2,2)^2, \\ \text{initial condition:} & \phi(x,y,0) = -\cos\left(\frac{\pi}{2}(x+y)\right), \\ \text{boundary condition:} & \text{periodic boundary condition,} \\ \qquad \text{Functional derivative} & \\ F(\phi) = \phi_t + \frac{1}{2}\,(\phi_x + \phi_y + 1)^2\,, & \\ F'(\phi) = \partial_t + (\phi_x + \phi_y + 1)\,(\partial_x + \partial_y), & \\ \mathbf{F}'(\boldsymbol{\phi}) = D_t + \operatorname{diag}\,(D_x\boldsymbol{\phi} + D_y\boldsymbol{\phi} + \mathbf{1})\,(D_x + D_y), & \end{cases} \tag{5.5}$$

$$\begin{cases} \boldsymbol{\phi}_t - cos\,(\phi_x + \phi_y + 1) = 0\,, & (x,y) \in (-2,2)^2, \\ \text{initial condition:} & \phi(x,y,0) = -\cos\left(\frac{\pi}{2}(x+y)\right), \\ \text{boundary condition:} & \text{periodic boundary condition,} \\ \qquad \text{Functional derivative} & \\ F(\phi) = \phi_t - \cos\,(\phi_x + \phi_y + 1)\,, & \\ F'(\phi) = \partial_t + \sin\,(\phi_x + \phi_y + 1)\,(\partial_x + \partial_y), & \\ \mathbf{F}'(\boldsymbol{\phi}) = D_t + \operatorname{diag}\,(\sin\,(D_x\boldsymbol{\phi} + D_y\boldsymbol{\phi} + \mathbf{1}))\,(D_x + D_y)\,, & \end{cases} \tag{5.6}$$

In all problems, we begin with an initial guess $\boldsymbol{\phi} = \mathbf{0}$ but this initial guess does not satisfy the initial condition. If we impose the initial condition on $\boldsymbol{\phi} = \mathbf{0}$ then the initial guess becomes non-smooth and it may happen that the algorithm faces divergences. The question is what is the remedy. We can write the each system of nonlinear equations in the form

$$\mathbf{F}(\boldsymbol{\phi}) = A\boldsymbol{\phi} + N(\boldsymbol{\phi}) - \mathbf{p}\,,$$

where the A is the linear operator and $N(\cdot)$ is nonlinear operator and the vector $\mathbf{p}$ contains the initial condition information. We modify the linear and nonlinear parts to adjust the initial and boundary conditions. In all our simulations, we perform a smoothing step

$$\boldsymbol{\phi} = A^{-1}\,(\mathbf{p} - N(\boldsymbol{\phi}))\,. \tag{5.7}$$

We have observed in all simulations that without performing smoothing step we face divergence in most of the simulations. The numerical solutions of (5.2), (5.3), (5.4), (5.5), and (5.6) are shown in Figures {1, 2}, 3, 4, 5, 6, respectively. In most of the cases, we require only one iteration(iter) of our proposed iterative method to get convergence for a given final time and spatial domain. When the problem becomes stiff, we are forced to use more than one iterations. For instance in the case of problem (5.2) for $t_f = 1.0/\pi^2$, we use two iterations to get convergence of the iterative method EEAF. The infinity norm of residue of system of nonlinear equations associated with HJ equations are depicted in Tables 3, 4, 5, 6, 7, and 8. In each table, we also showed the computational cost of the iterative method EEAF for given number of steps. When we increase the time length, the initial guess becomes more important for the convergence of our proposed iterative method EEAF and the convergence also requires more than one iteration with a sufficient number of multi-steps. A large of some multi-steps are required for the convergence iterative method when a single iteration is selected. Concerning how many steps should be chosen? It depends on the reduction of the norm of the residue of the system of nonlinear equations. Usually after a reasonable number of steps we observe no decrease of the norm of the residue of the system of nonlinear equations and we stop the simulation. The other possible difficulty is the computation of derivative of the non-smooth solution of HJ equations. When the solution becomes non-smooth, we face divergence. It is recommended to overcome the divergence of the iterative method, which one could get convergence for a small interval of time and uses the computed solution as an initial guess for a slightly bigger time interval. The aforementioned strategy works very well to get convergence for time intervals when the solution of HJ equations becomes non-smooth.

Table 3: 1-D nonlinear HJ equation (5.2): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 0.8/\pi^2$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 30$, $n_x = 100$.

Iterative methods		EEAF
Problem size		3000
Number of iterations		1
Number of steps		13
Theoretical convergence-order		36
Number of function evaluations per iteration		13
Solutions of system of linear equations per iteration		38
Number of Jacobian evaluations per iteration		2
Number of Jacobian LU-factorizations per iteration		1
Number of matrix-vector multiplications per iteration		25
Number of vector-vector multiplications per iteration		38
	Steps (iter=1)	
$\|\|\mathbf{F}(\boldsymbol{\phi})\|\|_\infty$	1	$3.18e+00$
	5	$2.55e-02$
	9	$1.34e-06$
	11	$1.97e-09$
	13	$5.81e-11$
Simulation Time (sec)		12.686

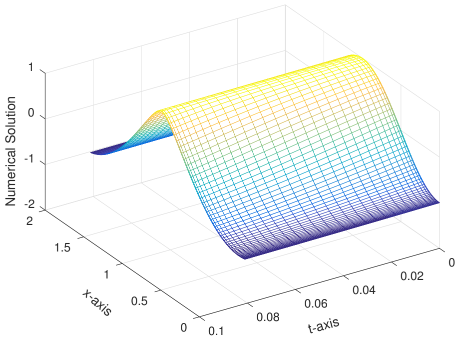

Figure 1: 1-D nonlinear HJ equation (5.2): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 0.8/\pi^2$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 30$, $n_x = 100$.

Table 4: 1-D nonlinear HJ equation (5.2): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 1.0/\pi^2$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 30$, $n_x = 100$.

Iterative methods		EEAF
Problem size		3000
Number of iterations		2
		[iter=1, iter=2]
Number of steps		[4, 7]
Theoretical convergence-order		[9, 18]
Number of function evaluations per iteration		[4, 7]
Solutions of system of linear equations per iteration		[11, 20]
Number of Jacobian evaluations per iteration		[2, 2]
Number of Jacobian LU-factorizations per iteration		[1, 1]
Number of matrix-vector multiplications per iteration		[7, 13]
Number of vector-vector multiplications per iteration		[11, 20]
	Steps (iter=1)	
$\|\|\mathbf{F}(\boldsymbol{\phi})\|\|_\infty$	1	$5.85e+00$
	4	$1.94e+01$
	Steps (iter=2)	
	1	$5.97e+00$
	3	$9.01e-03$
	7	$2.85e-12$
Simulation Time (sec)		12.52

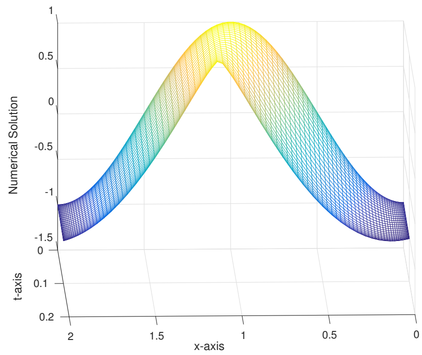

Figure 2: 1-D nonlinear HJ equation (5.2): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 1.0/\pi^2$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 30$, $n_x = 100$.

Table 5: 1-D nonlinear HJ equation (5.3): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 1.41/\pi^2$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 20$, $n_x = 100$.

Iterative methods		EEAF
Problem size		2000
Number of iterations		1
Number of steps		33
Theoretical convergence-order		96
Number of function evaluations per iteration		33
Solutions of system of linear equations per iteration		98
Number of Jacobian evaluations per iteration		2
Number of Jacobian LU-factorizations per iteration		1
Number of matrix-vector multiplications per iteration		65
Number of vector-vector multiplications per iteration		98
	Steps (iter=1)	
$\|\|\mathbf{F}(\boldsymbol{\phi})\|\|_\infty$	1	$4.31e-02$
	5	$1.96e-03$
	15	$5.62e-06$
	20	$2.46e-07$
	25	$5.60e-09$
	30	$5.06e-11$
	33	$8.09e-12$
Simulation Time (sec)		17.166

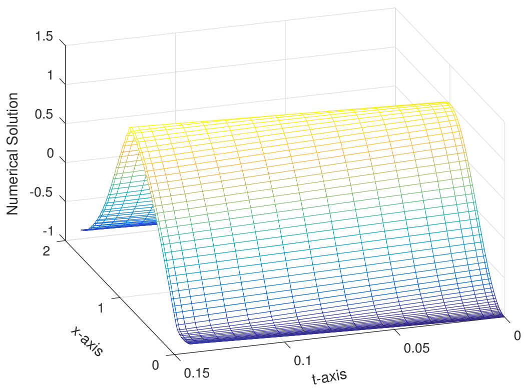

Figure 3: 1-D nonlinear HJ equation (5.3): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 1.41/\pi^2$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 20$, $n_x = 100$.

Table 6: 2-D nonlinear HJ equation (5.4): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 0.1$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 10$, $n_x = 20$, $n_y = 20$.

Iterative methods		EEAF
Problem size		4000
Number of iterations		1
Number of steps		16
Theoretical convergence-order		45
Number of function evaluations per iteration		16
Solutions of system of linear equations per iteration		47
Number of Jacobian evaluations per iteration		2
Number of Jacobian LU-factorizations per iteration		1
Number of matrix-vector multiplications per iteration		31
Number of vector-vector multiplications per iteration		47
	Steps (iter=1)	
$\|\|\mathbf{F}(\boldsymbol{\phi})\|\|_\infty$	1	$2.33e-01$
	3	$1.61e-02$
	5	$8.75e-05$
	9	$1.01e-07$
	15	$1.74e-12$
	16	$4.30e-13$
Simulation Time (sec)		41.154

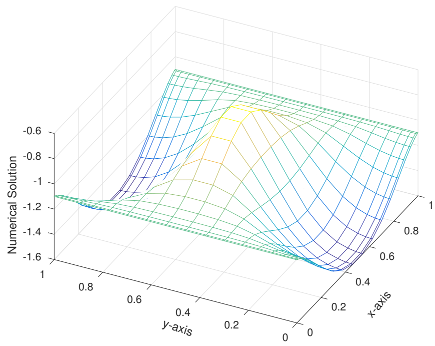

Figure 4: 2-D nonlinear HJ equation (5.4): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 0.1$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 10$, $n_x = 20$, $n_y = 20$.

Table 7: 2-D nonlinear HJ equation (5.5): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 0.8/\pi^2$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 10$, $n_x = 20$, $n_y = 20$.

Iterative methods		EEAF
Problem size		4000
Number of iterations		1
Number of steps		29
Theoretical convergence-order		84
Number of function evaluations per iteration		29
Solutions of system of linear equations per iteration		86
Number of Jacobian evaluations per iteration		2
Number of Jacobian LU-factorizations per iteration		1
Number of matrix-vector multiplications per iteration		57
Number of vector-vector multiplications per iteration		86
	Steps (iter=1)	
$\|\|\mathbf{F}(\boldsymbol{\phi})\|\|_\infty$	1	$3.22e+00$
	3	$3.46e+00$
	13	$5.23e-03$
	21	$1.70e-07$
	23	$1.10e-08$
	25	$6.49e-10$
	29	$2.83e-12$
Simulation Time (sec)		63.284

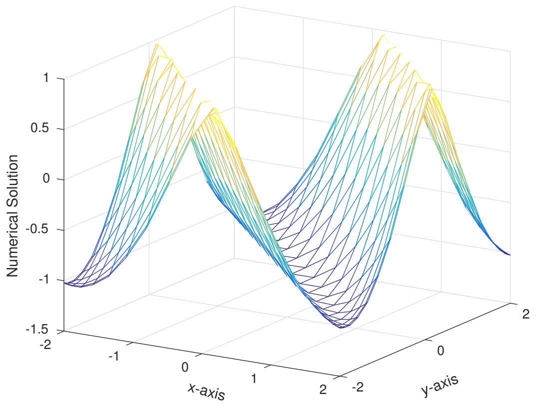

Figure 5: 2-D nonlinear HJ equation (5.5): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 0.8/\pi^2$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 10$, $n_x = 20$, $n_y = 20$.

Table 8: 2-D nonlinear HJ equation (5.6): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 0.8/\pi^2$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 10$, $n_x = 20$, $n_y = 20$.

Iterative methods		EEAF
Problem size		4000
Number of iterations		1
Number of steps		31
Theoretical convergence-order		90
Number of function evaluations per iteration		31
Solutions of system of linear equations per iteration		92
Number of Jacobian evaluations per iteration		2
Number of Jacobian LU-factorizations per iteration		1
Number of matrix-vector multiplications per iteration		61
Number of vector-vector multiplications per iteration		92
	Steps (iter=1)	
$\|\|\mathbf{F}(\boldsymbol{\phi})\|\|_\infty$	1	$8.48e-01$
	2	$9.25e-01$
	7	$6.97e-02$
	30	$1.05e-12$
	31	$9.19e-13$
Simulation Time (sec)		61.81

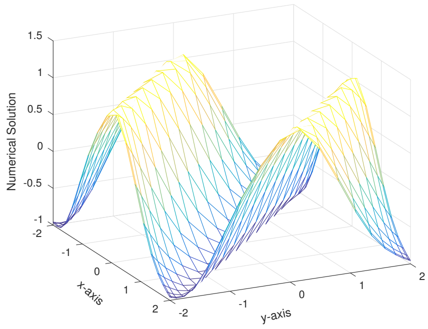

Figure 6: 2-D nonlinear HJ equation (5.6): absolute error in infinity norm of residue $\mathbf{F}(\boldsymbol{\phi})$, $t_f = 0.8/\pi^2$, initial guess $\boldsymbol{\phi} = \mathbf{0}$, $n_t = 10$, $n_x = 20$, $n_y = 20$.

6. Conclusions

The frozen Jacobian multi-step iterative methods are efficient iterative methods for solving the system of nonlinear equations. We have shown that our proposed iterative method EEAF offers convergence when we solve the system of nonlinear equations associated with HJ equations. The frozen Jacobian makes the computation less expensive and we solve repeatably lower and upper triangular systems in the multi-step part that makes the method computationally efficient. Actually the multi-step part gives an increment in the convergence order by paying a reasonable computational cost. For non-stiff problems, a single iteration of our proposed iterative method is enough to get convergence. And in the case of stiff problem, we have observed that single iterative gives us divergence and hence we are obliged to use more number of iterations to get convergence. The non-smoothness of the solution of HJ equations also makes the problem stiff and a successive selection of initial guess may result in the convergence.

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