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A three step iterative algorithm for common solutions of quasi-variational inclusions and fixed point problems of pseudocontractions

Hui-Ying Hu*, Jin-Zuo Chen

Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China.

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Abstract

In this paper, quasi-variational inclusions and fixed point problems of pseudocontractions are investigated based on a three step iterative process. Some convergence theorems are established in framework of Hilbert spaces. Several special cases are also discussed. The results presented in this paper extend and improve the corresponding results announced by many other authors.

Keywords: Monotone operator, quasi-variational inclusion, strictly pseudocontractive mapping, convex optimization, Hilbert space.
2010 MSC: 47H05, 47H09, 47H10, 47J20, 49M40.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $B : C \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a multi-valued mapping. The so-called quasi-variational inclusion problem is to find a $u \in 2^H$ such that

$$0 \in Bu + Mu. \tag{1.1}$$

In this paper, we use $VI(H, B, M)$ to denote the solution of problem (1.1). A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of this kind of variational inclusions, see for instance [3, 4, 8, 15]. For related work, see [1, 5, 6, 16]. The problem (1.1) includes many problems as special cases.

Next, we consider two special cases of problem (1.1).

*Corresponding author
Email addresses: huiying1117@hotmail.com (Hui-Ying Hu), chanjanegeoger@163.com (Jin-Zuo Chen)

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(1) If $M = \partial\phi : H \to 2^H$, where $\phi : H \to R \cup \{+\infty\}$ is a proper convex lower semi-continuous function and $\partial\phi$ is the sub-differential of ϕ, then problem (1.1) is equivalent to find $u \in H$ such that

$$\langle Bu, v-u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H,$$

which is called the mixed quasi-variational inequality (see [14]).

(2) If $M = \partial\delta_C$, and $\delta_C : H \to [0,\infty]$ is the indicator function of C, i.e.,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then problem (1.1) is equivalent to the classical variational inequality problem, denoted by $VI(C,B)$, which is to find $u \in C$ such that

$$\langle Bu, v-u\rangle \geq 0, \quad \forall v \in C.$$

This problem is called the Hartman-Stampacchia variational inequality (see [10]).

Let $T : C \to C$ be a nonlinear mapping. The iterative scheme of Mann's type for approximating fixed points of T is the following:

$$x_0 \in C, \quad \text{and} \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n,$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $[0,1]$. For two nonlinear mappings S and T, Takahashi and Tamura [17] considered the following iteration procedure

$$x_0 \in C, \quad \text{and} \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)S(\beta_n x_n + (1-\beta_n)Tx_n),$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(S)} \left(\frac{1}{2}\langle Ax - x\rangle - h(x)\right), \tag{1.2}$$

where A is a linear bounded and strongly positive operator, $F(S)$ is the fixed point set of nonexpansive mapping S and h is a potential function for γf, that is, $h'(x) = \gamma f(x)$, for $x \in H$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. Marino and Xu [12] studied the following iterative scheme:

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n\gamma f(x_n), \quad \forall n \geq 0,$$

where f is an α-contractive mapping. They proved $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (\gamma f - A)x^*, x - x^*\rangle \leq 0, \quad \forall x \in F(S),$$

which is the optimality condition for minimization problem (1.2).

Later, Zhang et al. [20] considered problem (1.1). They studied the following iterative scheme:

$$\begin{cases} x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Sy_n, \\ y_n = J_{M,\lambda}(x_n - \lambda Bx_n), \quad \forall n \geq 0, \end{cases}$$

where S is a nonexpansive mapping, B is an inverse-strongly monotone mapping and $J_{M,\lambda}$ is the resolvent operator associated with M. They proved $\{x_n\}$ generated by the above iterative scheme converges strongly to $P_{F(S)\cap VI(H,B,M)}x_0$.

Recently, Meng et al. [13] considered problem (1.1) and they studied the following iterative scheme:

$$\begin{cases} y_n = \kappa T J_{M,\lambda}(x_n - \lambda B x_n) + (1-\kappa) J_{M,\lambda}(x_n - \lambda B x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad \forall n \geq 1, \end{cases}$$

where f is a contractive mapping, T is a strictly pseudocontractive mapping, A is a strongly positive linear bounded operator, B is an inverse-strongly monotone mapping, and $J_{M,\lambda}$ is the resolvent operator associated with M. They proved $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (\gamma f - A)z, \omega - z \rangle \leq 0, \quad \forall \omega \in F(T) \cap VI(H, B, M).$$

Motivated and inspired by the works in this field, the purpose of this paper is to consider the quasi-variational inclusion and fixed point problems of pseudocontractions. A three step iterative algorithm is presented. A strong convergence theorem is demonstrated. We also discuss several special cases. The results in this paper extend and improve the corresponding results announced by many other authors.

2. Preliminaries

Throughout this paper, we assume H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

Definition 2.1. A mapping $A : C \to H$ is called

(i) monotone, if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone, if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C;$$

(iii) ζ-inverse-strongly monotone, if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \zeta \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is easy to see that the projection P_C is 1-inverse-strongly monotone. Inverse-strongly monotone operators have been applied widely in solving practical problems in various fields. It is obvious that if A is ζ-inverse-strongly monotone, then A is monotone and $\frac{1}{\zeta}$-Lipschitz continuous. Moreover, we also have the conclusion that if $0 < \lambda \leq 2\zeta$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$, a unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| := d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.2. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \leq 0, \quad \forall y \in C;$

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C;$

(iii) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall y \in H.$

Consequently, P_C is nonexpansive and monotone.

Definition 2.3. A mapping $T : H \to H$ is said to be

(i) L-Lipschitzian, if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H,$$

in the case of $L = \alpha \in (0, 1)$. T is said to be an α-contractive mapping; in the case of $L = 1$, T is said to be nonexpansive;

(ii) firmly nonexpansive, if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse-strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H,$$

alternatively, T is firmly nonexpansive, if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive;

(iii) T is called a strictly pseudocontractive, if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H.$$

In this case, we say that T is a k-strict pseudocontraction. It is obvious that any inverse-strongly monotone mapping is Lipschitz continuous. Meantime, strictly pseudocontractive mapping can be expressed as

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1 - k}{2}\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H.$$

We know that if T is a k-strict pseudocontractive mapping, then T is Lipschitz continuous with constant $\frac{1+k}{1-k}$, i.e., $\|Tx - Ty\| \leq \frac{1+k}{1-k}\|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of S. It is clear that the class of strict pseudocontractions strictly include the one of nonexpansive mappings.

Definition 2.4. Let $M : H \to 2^H$ be a multi-valued mapping. $dom(M)$ is the effective domain, that is, $dom(M) = \{x \in H : Mx \neq \emptyset\}$. M is said to be a monotone operator on H, if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in dom(M), u \in Mx, v \in My$. M is said to be maximal, if its graph is not property contained in the graph of any other monotone operator on H. For a maximal monotone operator M on H and $r > 0$, we may define a single-valued operator $J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u)$, for all $u \in H$ which is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping. The resolvent operator $J_{M,\lambda}$ associated with M is single-valued. It is clear that the resolvent $J_{M,\lambda}$ is firmly nonexpansive, that is,

$$\|J_{M,\lambda}x - J_{M,\lambda}y\|^2 \leq \langle x - y, J_{M,\lambda}x - J_{M,\lambda}y\rangle, \quad \forall x, y \in H.$$

Consequently, $J_{M,\lambda}$ is nonexpansive and monotone.

We need the following facts and lemmas for the proof of our results.

In a real Hilbert space H, the following hold:

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle,$$
$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$$
$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2,$$

for all $x, y \in H$ and $y \in [0, 1]$. It is also known that H satisfies the Opial's condition [10], i.e., for any sequence $\{x_n\} \subset H_1$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H$ with $x \neq y$. Hilbert space H satisfies the *Kadec-Klee* property [17], that is, for any sequence $\{x_n\}$ if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$.

Consider the following variational inequality for an inverse strongly monotone mapping B:

$$\text{Find } u \in C \text{ such that } \langle Bu, v-u\rangle \geq 0, \quad \forall v \in C.$$

The set of solutions of the variational inequality is denoted by $VI(C,B)$. It is well-known that

$$u \in VI(C,B) \Longleftrightarrow u = P_C(u-\lambda Bu), \quad \lambda > 0.$$

Lemma 2.5 ([2]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to H$ a k-strictly pseudo-contractive mapping. Define a mapping $J : C \to H$ by $Jx = \alpha x + (1-\alpha)Tx$, for all $x \in C$. Then, as $\alpha \in [k,1)$, J is a non-expansive mapping such that $F(J) = F(T)$.*

Lemma 2.6 ([18]). *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with a constant $l \geq 0$, and let $G : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa, \eta > 0$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle(\mu G-\gamma V)x-(\mu G-\gamma V)y, x-y\rangle \geq (\mu\eta-\gamma l)\|x-y\|^2, \quad \forall x,y \in C.$$

That is, $\mu G - \gamma V$ is strongly monotone with constant $\mu\eta - \gamma l$.

Lemma 2.7 ([20]). *Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by $J_{M,\lambda}(u) = (I+\lambda M)^{-1}(u)$, for all $u \in H$ is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping. The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and nonexpansive for all $\lambda > 0$. $u \in H$ is a solution of variational inclusion* (1.1)*, if and only if $u = J_{M,\lambda}(u-\lambda Bu)$, for all $\lambda > 0$, that is,*

$$VI(H,B,M) = F(J_{M,\lambda}(I-\lambda B)), \quad \forall \lambda > 0.$$

Lemma 2.8 ([19]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a ξ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma+\delta)\xi \leq \gamma$. Then*

$$\|\gamma(x-y)+\delta(Tx-Ty)\| \leq (\gamma+\delta)\|x-y\|, \quad \forall x,y \in C.$$

Lemma 2.9 ([11]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.10 ([10]). *Each Hilbert space H satisfies the Opial condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$, holds for every $y \in H$ with $y \neq x$.*

Lemma 2.11 ([9]). (Demiclosedness Principle). *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then $I-T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ imply that $x = Tx$.*

Lemma 2.12 ([7]). *Let H be a real Hilbert space and $M : H \to 2^H$ be a maximal monotone mapping and $P : H \to H$ be a hemi-continuous bounded monotone mapping with $D(M) = H$. Then, mapping $M + P : H \to 2^H$ is maximal monotone.*

3. Main result

In this section, we introduce and analyze a three step iterative algorithm for common solutions of quasi-variational inclusion and fixed point problems. We prove the strong convergence of the proposed algorithm to the unique solution of variational inequality under some suitable conditions. And many known results are the special cases of our result.

In the rest of the paper, unless otherwise specified, we assume that C is a nonempty, closed, and convex subset of a real Hilbert space H.

Assumption 3.1.

(a) $M : H \to 2^H$ *is a maximal monotone operator with* $dom(M) \subset C$*;*
(b) $J_{M,\lambda} : H \to H$ *defined by* $J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u)$*, for all* $u \in H$ *is the resolvent operator associated with* M*, where* λ *is any position number and* I *is the identity mapping;*
(c) $T : H \to H$ *is a* ξ*-strictly pseudo-contractive mapping;*
(d) $B : C \to H$ *is an* r*-inverse-strongly monotone mapping;*
(e) $G : C \to C$ *is a* κ*-Lipschitzian and* η*-strongly monotone mapping with constant* $\kappa, \eta > 0$*;*
(f) $V : C \to C$ *is an* l*-Lipschitzian mapping with constant* $l > 0$*;*
(g) *Constant* $\mu > 0$ *and* $\gamma \geq 0$ *satisfy* $0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 \leq \gamma l < \tau$*, where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.

We propose the following three step iterative algorithm for finding common solutions of $F(T) \cap VI(H, B, M)$.

Algorithm 3.2. *For an arbitrarily chosen* $x_1 \in C$*, let the iterative sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = \varsigma T J_{M,\lambda}(x_n - \lambda B x_n) + (1 - \varsigma) J_{M,\lambda}(x_n - \lambda B x_n), \\ z_n = \beta_n x_n + \gamma_n y_n + \delta_n T y_n, \\ x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) z_n, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\varsigma \in (0, 1 - \xi]$ *and* $\lambda \in (0, 2r]$.

Next, we give two special cases of Algorithm 3.2.

(1) If $\delta_n = 0$, then Algorithm 3.2 reduces to the following three step iterative algorithm.

Algorithm 3.3. *For an arbitrarily chosen* $x_1 \in C$*, let the iterative sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = \varsigma T J_{M,\lambda}(x_n - \lambda B x_n) + (1 - \varsigma) J_{M,\lambda}(x_n - \lambda B x_n), \\ z_n = \beta_n x_n + (1 - \beta_n) y_n, \\ x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) z_n, \quad \forall n \geq 1, \end{cases}$$

where $\varsigma \in (0, 1 - \xi]$ *and* $\lambda \in (0, 2r]$.

(2) If $\beta_n = 0$, $V = f$ a contraction, $G = A$ a strongly positive linear bounded operator, then Algorithm 3.3 reduces to the following iterative algorithm.

Algorithm 3.4. *For an arbitrarily chosen* $x_1 \in C$*, let the iterative sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = \varsigma T J_{M,\lambda}(x_n - \lambda B x_n) + (1 - \varsigma) J_{M,\lambda}(x_n - \lambda B x_n), \\ x_{n+1} = \alpha_n \gamma f x_n + (I - \alpha_n A) y_n, \quad \forall n \geq 1, \end{cases}$$

where $\varsigma \in (0, 1 - \xi]$ *and* $\lambda \in (0, 2r]$ *(This is just the Algorithm in* [13]*).*

The following result provides the convergence of the sequence generated by Algorithm 3.2.

Theorem 3.5. *In addition to Assumption* 3.1*, suppose that* $\Omega = F(T) \cap VI(H, B, M) \neq \emptyset$*. And* $\{\alpha_n\}, \{\beta_n\}$*,* $\{\gamma_n\}$ *and* $\{\delta_n\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:*

(c1) $\lim_{n\to\infty} \alpha_n = 0$*, and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(c2) $\beta_n + \gamma_n + \delta_n = 1$*,* $(\gamma_n + \delta_n)\xi \leq \gamma_n$*, for all* $n \geq 1$*, and* $\beta_n \subset [a, b] \subset (0, 1)$*;*
(c3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$*, and* $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$*;*
(c4) $\lim_{n\to\infty} \delta_n = 0$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.2 *converges strongly to a point $z \in \Omega$, which is the unique solution of the following variational inequality:*

$$\langle (\gamma V - \mu G)z, \omega - z\rangle \leq 0, \quad \forall \omega \in \Omega. \tag{3.2}$$

Proof. By putting $S = \varsigma T + (1-\varsigma)I$, from Lemma 2.5, we see that S is nonexpansive with $F(S) = F(T)$. From Lemma 2.6, we know that $\gamma V - \mu G$ is strongly monotone. From the strong monotonicity of $\mu G - \gamma V$, we can easily get the uniqueness of solution of variational inequality (3.2). Suppose $z_1 \in \Omega$ and $z_2 \in \Omega$ both are solutions to (3.2). It follows that

$$\langle (\gamma V - \mu G)z_2, z_2 - z_1\rangle \geq 0,$$

and

$$\langle (\gamma V - \mu G)z_1, z_1 - z_2\rangle \geq 0.$$

By adding up the two inequalities, we see that

$$\langle (\gamma V - \mu G)z_1 - (\gamma V - \mu G)z_2, z_1 - z_2\rangle \geq 0.$$

The strong monotonicity of $\gamma V - \mu G$ implies that $z_1 = z_2$ and the uniqueness is proved. Below we use z to denote the unique solution of (3.2).

We divide the rest of proof into several steps.

Step 1. We prove that $\{x_n\}$ is bounded.

From the condition on λ, we can see that the mapping $I - \lambda B$ is nonexpansive. By taking $p \in \Omega$, we find from Lemma 2.7 that $p = J_{M,\lambda}(p - \lambda Bp)$. It follows that

$$\begin{aligned}\|y_n - p\| &\leq \|SJ_{M,\lambda}(x_n - \lambda Bx_n) - p\| \\ &\leq \|J_{M,\lambda}(x_n - \lambda Bx_n) - J_{M,\lambda}(p - \lambda Bp)\| \\ &\leq \|x_n - p\|.\end{aligned}$$

By utilizing (3.1), Lemma 2.8 and the above inequality, we have

$$\begin{aligned}\|z_n - p\| &= \|\beta_n x_n + \gamma_n y_n + \delta_n Ty_n - p\| \\ &= \|\beta_n(x_n - p) + \gamma_n(y_n - p) + \delta_n(Ty_n - p)\| \\ &\leq \beta_n\|x_n - p\| + \|\gamma_n(y_n - p) + \delta_n(Ty_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (\gamma_n + \delta_n)\|y_n - p\| \\ &\leq \beta_n\|x_n - p\| + (\gamma_n + \delta_n)\|x_n - p\| \\ &= \|x_n - p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(\gamma Vx_n - \mu Gp) + (I - \alpha_n\mu G)z_n - (I - \alpha_n\mu G)p\| \\ &\leq (1 - \alpha_n\tau)\|z_n - p\| + \alpha_n\gamma\|Vx_n - Vp\| + \alpha_n\|\gamma Vp - \mu Gp\| \\ &\leq (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\gamma l\|x_n - p\| + \alpha_n\|\gamma Vp - \mu Gp\| \\ &= [1 - \alpha_n(\tau - \gamma l)]\|x_n - p\| + \alpha_n(\tau - \gamma l)\frac{\|\gamma Vp - \mu Gp\|}{\tau - \gamma l} \\ &\leq \max\left\{\|x_n - p\|, \frac{\|\gamma Vp - \mu Gp\|}{\tau - \gamma l}\right\}.\end{aligned}$$

By the induction, we obtain

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma Vp - \mu Gp\|}{\tau - \gamma l}\right\}, \quad \forall n \geq 1.$$

Thus, $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$ and $\{z_n\}$.

Step 2. We prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Note that

$$\begin{aligned}\|y_{n+1}-y_n\| &\le \|J_{M,\lambda}(x_{n+1}-\lambda Bx_{n+1})-J_{M,\lambda}(x_n-\lambda Bx_n)\|\\ &\le \|(x_{n+1}-\lambda Bx_{n+1})-(x_n-\lambda Bx_n)\|\\ &\le \|x_{n+1}-x_n\|.\end{aligned}\tag{3.3}$$

Furthermore, define $z_n=\beta_n x_n+(1-\beta_n)w_n$, for all $n\ge 1$. It follows that

$$\begin{aligned}w_{n+1}-w_n &= \frac{z_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{z_n-\beta_n x_n}{1-\beta_n}\\ &= \frac{\gamma_{n+1}y_{n+1}+\delta_{n+1}Ty_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n y_n+\delta_n Ty_n}{1-\beta_n}\\ &= \frac{\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Ty_{n+1}-Ty_n)}{1-\beta_{n+1}}\\ &\quad+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)y_n+\left(\frac{\delta_{n+1}}{1-\beta_{n+1}}-\frac{\delta_n}{1-\beta_n}\right)Ty_n.\end{aligned}\tag{3.4}$$

Since $(\gamma_n+\delta_n)\xi\le\gamma_n$ for all $u\ge 1$, by utilizing Lemma 2.8, we have

$$\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Ty_{n+1}-Ty_n)\|\le(\gamma_{n+1}+\delta_{n+1})\|y_{n+1}-y_n\|.\tag{3.5}$$

Hence, it follows from (3.3), (3.4), (3.5) that

$$\begin{aligned}\|w_{n+1}-w_n\| &\le \frac{\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Ty_{n+1}-Ty_n)\|}{1-\beta_{n+1}}\\ &\quad+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\|y_n\|+\left|\frac{\delta_{n+1}}{1-\beta_{n+1}}-\frac{\delta_n}{1-\beta_n}\right|\|Ty_n\|\\ &\le \frac{(\gamma_{n+1}+\delta_{n+1})}{1-\beta_{n+1}}\|y_{n+1}-y_n\|+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|(\|y_n\|+\|Ty_n\|)\\ &= \|y_{n+1}-y_n\|+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|(\|y_n\|+\|Ty_n\|)\\ &\le \|x_{n+1}-x_n\|+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|(\|y_n\|+\|Ty_n\|).\end{aligned}\tag{3.6}$$

In the meantime, simple calculation shows that

$$z_{n+1}-z_n=\beta_n(x_{n+1}-x_n)+(1-\beta_n)(w_{n+1}-w_n)+(\beta_{n+1}-\beta_n)(x_{n+1}-w_{n+1}).$$

So, it follows from (3.6) that

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \beta_n\|x_{n+1}-x_n\|+(1-\beta_n)\|w_{n+1}-w_n\|+|\beta_{n+1}-\beta_n|\|x_{n+1}-w_{n+1}\|\\ &\le \beta_n\|x_{n+1}-x_n\|+(1-\beta_n)\left\{\|x_{n+1}-x_n\|+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|(\|y_n\|+\|Ty_n\|)\right\}\\ &\quad+|\beta_{n+1}-\beta_n|\|x_{n+1}-w_{n+1}\|\\ &\le \|x_{n+1}-x_n\|+\frac{|\gamma_{n+1}-\gamma_n|(1-\beta_n)+\gamma_n|\beta_{n+1}-\beta_n|}{1-\beta_{n+1}}(\|y_n\|+\|Ty_n\|)\\ &\quad+|\beta_{n+1}-\beta_n|\|x_{n+1}-w_{n+1}\|\\ &\le \|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|\frac{\|y_n\|+\|Ty_n\|}{1-b}+|\beta_{n+1}-\beta_n|\left(\|x_{n+1}-w_{n+1}\|+\frac{\|y_n\|+\|Ty_n\|}{1-b}\right)\\ &\le \|x_{n+1}-x_n\|+M_0\left(|\gamma_{n+1}-\gamma_n|+|\beta_{n+1}-\beta_n|\right),\end{aligned}$$

where $\sup_{n\geq 1}\left\{\frac{\|y_n\|+\|Ty_n\|}{1-b}+\|x_{n+1}-w_{n+1}\|+\frac{\|y_n\|+\|Ty_n\|}{1-b}\right\}\leq M_0$ for some $M_0>0$. Next, we estimate

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n\gamma Vx_n+(I-\alpha_n\mu G)z_n-\alpha_{n-1}\gamma Vx_{n-1}-(I-\alpha_{n-1}\mu G)z_{n-1}\|\\ &= \|\alpha_n\gamma(Vx_n-Vx_{n-1})+(\alpha_n-\alpha_{n-1})\gamma Vx_{n-1}+(I-\alpha_n\mu G)z_n-(I-\alpha_n\mu G)z_{n-1}\\ &\quad +(I-\alpha_n\mu G)z_{n-1}-(I-\alpha_{n-1}\mu G)z_{n-1}\|\\ &\leq \alpha_n\gamma l\|x_n-x_{n-1}\|+(1-\tau\alpha_n)\|z_n-z_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\gamma\|Vx_{n-1}\|+\mu\|Gz_{n-1}\|)\\ &\leq \alpha_n\gamma l\|x_n-x_{n-1}\|+(1-\tau\alpha_n)\{\|x_n-x_{n-1}\|+M_0(|\gamma_n-\gamma_{n-1}|+|\beta_n-\beta_{n-1}|)\}\\ &\quad +|\alpha_n-\alpha_{n-1}|(\gamma\|Vx_{n-1}\|+\mu\|Gz_{n-1}\|)\\ &\leq (1-(\tau-\gamma l)\alpha_n)\|x_n-x_{n-1}\|+M_0(|\gamma_n-\gamma_{n-1}|+|\beta_n-\beta_{n-1}|)\\ &\quad +|\alpha_n-\alpha_{n-1}|(\gamma\|Vx_{n-1}\|+\mu\|Gz_{n-1}\|)\\ &\leq (1-(\tau-\gamma l)\alpha_n)\|x_n-x_{n-1}\|+M_1(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|),\end{aligned}$$

where $\sup_{n\geq 1}\{M_0+\gamma\|Vx_{n-1}\|+\mu\|Gz_{n-1}\|\}\leq M_1$ for some $M_1>0$. It follows by conditions (c1)-(c4) and Lemma 2.9 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.7}$$

Step 3. We show that $\lim_{n\to\infty}\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|=0$ and $\lim_{n\to\infty}\|x_n-y_n\|=0$.

Since $x_{n+1}-z_n=\alpha_n(\gamma Vx_n-\mu Gz_n)$, which implies from the restriction imposed on $\{\alpha_n\}$ that

$$\lim_{n\to\infty}\|x_{n+1}-z_n\|=0. \tag{3.8}$$

By combining (3.8) with (3.7), we can easily get

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.9}$$

Since $p\in\Omega=F(T)\cap VI(H,M,B)$, one has

$$\begin{aligned}\|y_n-p\|^2 &= \|SJ_{M,\lambda}(x_n-\lambda Bx_n)-p\|^2\leq\|J_{M,\lambda}(x_n-\lambda Bx_n)-p\|^2\\ &\leq \|(x_n-\lambda Bx_n)-(p-\lambda Bp)\|^2\\ &\leq \|x_n-p\|^2-\lambda(2r-\lambda)\|Bx_n-Bp\|^2.\end{aligned}$$

It follows from (3.1) and the above inequality that

$$\begin{aligned}\|z_n-p\|^2 =&\|\beta_nx_n+\gamma_ny_n+\delta_nTy_n-p\|^2\\ =&\|\beta_n(x_n-p)+(1-\beta_n)\left(\frac{\gamma_ny_n+\delta_nTy_n}{1-\beta_n}-p\right)\|^2\\ =&\beta_n\|x_n-p\|^2+(1-\beta_n)\left\|\frac{\gamma_ny_n+\delta_nTy_n}{1-\beta_n}-p\right\|^2-\beta_n(1-\beta_n)\left\|\frac{\gamma_ny_n+\delta_nTy_n}{1-\beta_n}-x_n\right\|^2\\ =&\beta_n\|x_n-p\|^2+(1-\beta_n)\left\|\frac{\gamma_n(y_n-p)+\delta_n(Ty_n-p)}{1-\beta_n}\right\|^2-\beta_n(1-\beta_n)\|\frac{z_n-x_n}{1-\beta_n}\|^2\\ \leq&\beta_n\|x_n-p\|^2+(1-\beta_n)\frac{(\gamma_n+\delta_n)^2\|y_n-p\|^2}{(1-\beta_n)^2}-\beta_n(1-\beta_n)\frac{\|z_n-x_n\|^2}{(1-\beta_n)^2}\\ =&\beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2-\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2\\ \leq&\beta_n\|x_n-p\|^2+(1-\beta_n)\{\|x_n-p\|^2-\lambda(2r-\lambda)\|Bx_n-Bp\|^2\}-\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2\\ =&\|x_n-p\|^2-(1-\beta_n)\lambda(2r-\lambda)\|Bx_n-Bp\|^2-\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2.\end{aligned} \tag{3.10}$$

By using (3.1) and (3.10), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n\gamma Vx_n+(I-\alpha_n\mu G)z_n-p\|^2\\ &= \|\alpha_n(\gamma Vx_n-\mu Gp)+(I-\alpha_n\mu G)(z_n-p)\|^2\\ &\le (\alpha_n\|\gamma Vx_n-\mu Gp\|+(1-\alpha_n\tau)\|z_n-p\|)^2\\ &\le \alpha_n\|\gamma Vx_n-\mu Gp\|^2+(1-\alpha_n\tau)\|z_n-p\|^2+2\alpha_n(1-\alpha_n\tau)\|\gamma Vx_n-\mu Gp\|\|z_n-p\|\\ &\le \alpha_n\|\gamma Vx_n-\mu Gp\|^2+\|x_n-p\|^2-(1-\alpha_n\tau)(1-\beta_n)\lambda(2r-\lambda)\|Bx_n-Bp\|^2\\ &\quad -(1-\alpha_n\tau)\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2+2\alpha_n(1-\alpha_n\tau)\|\gamma Vx_n-\mu Gp\|\|z_n-p\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}&(1-\alpha_n\tau)(1-\beta_n)\lambda(2r-\lambda)\|Bx_n-Bp\|^2+(1-\alpha_n\tau)\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2\\ &\le\alpha_n\|\gamma Vx_n-\mu Gp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n(1-\alpha_n\tau)\|\gamma Vx_n-\mu Gp\|\|z_n-p\|\\ &\le\alpha_n\|\gamma Vx_n-\mu Gp\|^2+\|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|)+2\alpha_n(1-\alpha_n\tau)\|\gamma Vx_n-\mu Gp\|\|z_n-p\|.\end{aligned}$$

From (3.7) and the condition (c1), we get

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0. \tag{3.11}$$

On the other hand, since $J_{M,\lambda}$ is firmly nonexpansive, one has

$$\begin{aligned}\|J_{M,\lambda}(x_n-\lambda Bx_n)-p\|^2 &\le \langle (x_n-\lambda Bx_n)-(p-\lambda Bp), J_{M,\lambda}(x_n-\lambda Bx_n)-p\rangle\\ &\le \frac{1}{2}(\|(x_n-\lambda Bx_n)-(p-\lambda Bp)\|^2+\|J_{M,\lambda}(x_n-\lambda Bx_n)-p\|^2\\ &\quad -\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)-\lambda(Bx_n-Bp)\|^2)\\ &\le \frac{1}{2}(\|x_n-p\|^2+\|J_{M,\lambda}(x_n-\lambda Bx_n)-p\|^2\\ &\quad -\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|^2-\lambda^2\|Bx_n-Bp\|^2\\ &\quad +2\lambda\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|\|Bx_n-Bp\|).\end{aligned}$$

Therefore, we arrive at

$$\begin{aligned}\|y_n-p\|^2=\|SJ_{M,\lambda}(x_n-\lambda Bx_n)-p\|^2 &\le \|J_{M,\lambda}(x_n-\lambda Bx_n)-p\|^2\\ &\le \|x_n-p\|^2-\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|^2-\lambda^2\|Bx_n-Bp\|^2\\ &\quad +2\lambda\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|\|Bx_n-Bp\|.\end{aligned}$$

It follows from (3.10) that

$$\begin{aligned}\|z_n-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2-\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\{\|x_n-p\|^2-\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|^2-\lambda^2\|Bx_n-Bp\|^2\\ &\quad +2\lambda\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|\|Bx_n-Bp\|\}-\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2\\ &\le \|x_n-p\|^2-(1-\beta_n)\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|^2-\lambda^2(1-\beta_n)\|Bx_n-Bp\|^2\\ &\quad +2\lambda(1-\beta_n)\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|\|Bx_n-Bp\|-\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2.\end{aligned} \tag{3.12}$$

By using (3.1) and (3.12), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n\gamma Vx_n+(I-\alpha_n\mu G)z_n-p\|^2\\ &= \|\alpha_n(\gamma Vx_n-\mu Gp)+(I-\alpha_n\mu G)(z_n-p)\|^2\\ &\le (\alpha_n\|\gamma Vx_n-\mu Gp\|+(1-\alpha_n\tau)\|z_n-p\|)^2\\ &\le \alpha_n\|\gamma Vx_n-\mu Gp\|^2+(1-\alpha_n\tau)\|z_n-p\|^2+2\alpha_n(1-\alpha_n\tau)\|\gamma Vx_n-\mu Gp\|\|z_n-p\|\\ &\le \alpha_n\|\gamma Vx_n-\mu Gp\|^2+\|x_n-p\|^2-(1-\alpha_n\tau)(1-\beta_n)\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|^2\\ &\quad -(1-\alpha_n\tau)\lambda^2(1-\beta_n)\|Bx_n-Bp\|^2-(1-\alpha_n\tau)\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2\\ &\quad +(1-\alpha_n\tau)2\lambda(1-\beta_n)\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|\|Bx_n-Bp\|\\ &\quad +2\alpha_n(1-\alpha_n\tau)\|\gamma Vx_n-\mu Gp\|\|z_n-p\|.\end{aligned} \tag{3.13}$$

Then, from (3.13), we get

$$\begin{aligned}&(1-\alpha_n\tau)(1-\beta_n)\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|^2+(1-\alpha_n\tau)\lambda^2(1-\beta_n)\|Bx_n-Bp\|^2\\ &\qquad +(1-\alpha_n\tau)\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2\\ &\quad\le \alpha_n\|\gamma Vx_n-\mu Gp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\qquad +(1-\alpha_n\tau)2\lambda(1-\beta_n)\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|\|Bx_n-Bp\|\\ &\qquad +2\alpha_n(1-\alpha_n\tau)\|\gamma Vx_n-\mu Gp\|\|z_n-p\|\\ &\quad\le \alpha_n\|\gamma Vx_n-\mu Gp\|^2+\|x_{n+1}-x_n\|(\|x_n-p\|\\ &\qquad +\|x_{n+1}-p\|)+2\alpha_n(1-\alpha_n\tau)\|\gamma Vx_n-\mu Gp\|\|z_n-p\|\\ &\qquad +(1-\alpha_n\tau)2\lambda(1-\beta_n)\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|\|Bx_n-Bp\|.\end{aligned}$$

From (3.7), (3.11) and the condition (c1), we get

$$\lim_{n\to\infty}\|x_n-J_{M,\lambda}(x_n-\lambda Bx_n)\|=0. \tag{3.14}$$

Also, observe that

$$z_n-x_n=\gamma_n(y_n-x_n)+\delta_n(Ty_n-x_n),\quad \forall n\ge 1.$$

Hence, we obtain

$$\gamma_n\|y_n-x_n\|\le\|z_n-x_n\|+\delta_n\|Ty_n-x_n\|.$$

So, from (3.9) and the condition (c4), it follows that

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.15}$$

Step 4. We show that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu G)z,x_n-z\rangle\le 0,$$

where $z\in\Omega$ is the unique solution of the variational inequality (3.2).

To show this, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu G)z,x_n-z\rangle=\lim_{i\to\infty}\langle(\gamma V-\mu G)z,x_{n_i}-z\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to ω. Without loss of generality, we can assume that $x_n\rightharpoonup\omega$. Next, we prove $\omega\in\Omega=Fix(T)\cap VI(H,M,B)$. Note that

$$\begin{aligned}\|x_n-Sx_n\|&\le\|x_n-SJ_{M,\lambda}(x_n-\lambda Bx_n)\|+\|SJ_{M,\lambda}(x_n-\lambda Bx_n)-Sx_n\|\\ &\le\|x_n-y_n\|+\|J_{M,\lambda}(x_n-\lambda Bx_n)-x_n\|.\end{aligned}$$

By using (3.14) and (3.15), one has $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. From Lemmas 2.5 and 2.11, one gets $\omega \in F(S) = F(T)$.

Next, we prove $\omega \in VI(H, M, B)$. In fact, since B is r-inverse-strongly monotone, it follows from B is Lipschitz continuous. It follows from Lemma 2.12 that $M + B$ is a maximal monotone operator. Let $(u, v) \in G(M + B)$. That is, $v - Bu \in M(u)$. By setting $t_n = J_{M,\lambda}(x_n - \lambda Bx_n)$, we have $x_n - \lambda Bx_n \in t_n + \lambda Mt_n$, that is,

$$\frac{x_n - t_n}{\lambda} - Bx_n \in Mt_n.$$

By virtue of the maximal monotonicity of $M + B$, we have

$$\left\langle u - t_n, v - Bu - \frac{x_n - t_n}{\lambda} + Bx_n \right\rangle \geq 0.$$

Hence, we have

$$\begin{aligned}\langle u - t_n, v\rangle &\geq \left\langle u - t_n, Bu + \frac{x_n - t_n}{\lambda} - Bx_n \right\rangle\\ &= \left\langle u - t_n, Bu - Bt_n + Bt_n - Bx_n + \frac{x_n - t_n}{\lambda} \right\rangle\\ &\geq \langle u - t_n, Bt_n - Bx_n\rangle + \left\langle u - t_n, \frac{x_n - t_n}{\lambda} \right\rangle.\end{aligned}$$

From (3.14), we have $\langle u - \omega, v\rangle \geq 0$. Since $B + M$ is maximal monotone, this implies that $0 \in (M + B)(\omega)$, that is, $\omega \in VI(H, MB)$, and so $\omega \in \Omega = T(T) \cap VI(H, M, B)$.

Now, since z is the unique solution of the variational inequality (3.2), we conclude

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma V - \mu G)z, x_n - z\rangle &= \lim_{i\to\infty}\langle(\gamma V - \mu G)z, x_{n_i} - z\rangle\\ &= \langle(\gamma V - \mu G)z, \omega - z\rangle \leq 0.\end{aligned}$$

Step 5. Finally, we show that $x_n \to z$, as $n \to \infty$.

Indeed, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \langle \alpha_n\gamma Vx_n + (I - \alpha_n\mu G)z_n - z, x_{n+1} - z\rangle\\ &= \langle \alpha_n(\gamma Vx_n - \mu Gz) + (I - \alpha_n\mu G)(z_n - z), x_{n+1} - z\rangle\\ &\leq \alpha_n\langle \gamma Vx_n - \mu Gz, x_{n+1} - z\rangle + (1 - \alpha_n\tau)\langle z_n - z, x_{n+1} - z\rangle\\ &\leq \alpha_n\langle \gamma Vx_n - \gamma Vz, x_{n+1} - z\rangle + \alpha_n\langle(\gamma V - \mu G)z, x_{n+1} - z\rangle\\ &\quad + (1 - \alpha_n\tau)\|z_n - z\|\|x_{n+1} - z\|\\ &\leq \alpha_n\gamma l\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle(\gamma V - \mu G)z, x_{n+1} - z\rangle\\ &\quad + (1 - \alpha_n\tau)\|z_n - z\|\|x_{n+1} - z\|\\ &\leq \frac{\alpha_n\gamma l}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle(\gamma V - \mu G)z, x_{n+1} - z\rangle\\ &\quad + \frac{1 - \alpha_n\tau}{2}(\|z_n - z\|^2 + \|x_{n+1} - z\|^2).\end{aligned}$$

It follows from (3.10) that

$$\begin{aligned}\|x_{n+1} - z\|^2 \leq &\frac{\alpha_n\gamma l}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle \gamma Vz - \mu Gz, x_{n+1} - z\rangle\\ &+ \frac{1 - \alpha_n\tau}{2}(\|x_n - z\|^2 - (1 - \beta_n)\lambda(2r - \lambda)\|Bx_n - Bz\|^2\\ &- \frac{\beta_n}{1 - \beta_n}\|z_n - x_n\|^2 + \|x_{n+1} - z\|^2)\end{aligned}$$

$$
\begin{aligned}
=&\frac{1-\alpha_n(\tau-\gamma l)}{2}\|x_n-z\|^2+\frac{1-\alpha_n(\tau-\gamma l)}{2}\|x_{n+1}-z\|^2+\alpha_n\langle(\gamma V-\mu G)z,x_{n+1}-z\rangle\\
&-\frac{1-\alpha_n\tau}{2}(1-\beta_n)\lambda(2r-\lambda)\|Bx_n-Bz\|^2-\frac{1-\alpha_n\tau}{2}\frac{\beta_n}{1-\beta_n}\|z_n-x_n\|^2\\
\leq&\frac{1-\alpha_n(\tau-\gamma l)}{2}\|x_n-z\|^2+\frac{1-\alpha_n(\tau-\gamma l)}{2}\|x_{n+1}-z\|^2+\alpha_n\langle(\gamma V-\mu G)z,x_{n+1}-z\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-z\|^2&\leq\frac{1-\alpha_n(\tau-\gamma l)}{1+\alpha_n(\tau-\gamma l)}\|x_n-z\|^2+\frac{2\alpha_n}{1+\alpha_n(\tau-\gamma l)}\langle(\gamma V-\mu G)z,x_{n+1}-z\rangle\\
&\leq(1-\alpha_n(\tau-\gamma l))\|x_n-z\|^2+\frac{2\alpha_n}{1+\alpha_n(\tau-\gamma l)}\langle(\gamma V-\mu G)z,x_{n+1}-z\rangle.
\end{aligned}
$$

By using Lemma 2.9, we get the desired conclusion immediately. This completes the proof. □

Remark 3.6. Theorem 3.5 improve, extend, supplement and develop Theorem 3.1 in [13] in the following aspects:

(i) The iterative algorithm of [13] is extended to a three step iterative algorithm.
(ii) The contraction mapping f of Theorem 3.1 in [13] is extended to the case of a Lipschitzian mapping V.
(iii) The strongly positive linear bounded operator A of Theorem 3.1 in [13] is extended to the case of the κ-Lipschitzian and η-strongly monotone G.
(iv) If $\beta_n=0,\gamma_n=1,\delta_n=0$, $G=A$ a strongly positive linear bounded operator, $V=f$ a contraction, then the proposed method is an extension and improvement of a method studied in [13].

Corollary 3.7. *In addition to Assumption* 3.1*, suppose that* $\Omega=F(T)\cap VI(H,B,M)\neq\emptyset$*. And* $\{\alpha_n\},\{\beta_n\}$ *and* $\{\gamma_n\}$ *and are sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n=0$*, and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(ii) $\beta_n+\gamma_n+\delta_n=1$*,* $(\gamma_n+\delta_n)\xi\leq\gamma_n$*, for all* $n\geq1$*, and* $\beta_n\subset[a,b]\subset(0,1)$*;*
(iii) $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$*,* $\sum_{n=1}^{\infty}|\beta_n-\beta_{n-1}|<\infty$*, and* $\sum_{n=1}^{\infty}|\gamma_n-\gamma_{n-1}|<\infty$.

Then the sequence $\{x_n\}$ *generated by Algorithm* 3.3 *converges strongly to a point* $z\in\Omega$*, which is the unique solution of the variational inequality* (3.2)*.*

Corollary 3.8. *In addition to Assumption* 3.1*, let* f *be a contraction of* H *into itself with the coefficient* $\alpha(0<\alpha<1)$ *and let* A *be a strongly positive linear bounded self-joint operator with the coefficient* $\overline{\gamma}>0$*. Assume that* $0<\gamma<\overline{\gamma}/\alpha$ *and* $\Omega=F(T)\cap VI(H,B,M)\neq\emptyset$*. And* $\{\alpha_n\}$ *is a sequences in* $(0,1)$ *such that* $\lim_{n\to\infty}\alpha_n=0$*,* $\sum_{n=1}^{\infty}\alpha_n=\infty$*, and* $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$*. Then the sequence* $\{x_n\}$ *generated by Algorithm* 3.4 *converges strongly to a point* $z\in\Omega$*, which is the unique solution of the variational inequality* (3.2)*.*

Remark 3.9. This is exactly the form of Theorem 3.1 of [13].

4. Applications

In this section, we obtain the following results by using a special case of the proposed method for example.

Theorem 4.1. *Let* C *be a nonempty,closed, and convex subset of a real Hilbert space* H*. Let* $M:H\to2^H$ *a maximal monotone operator. Let* $B:C\to H$ *be a* r*-inverse-strongly monotone and let* T *be a nonexpansive mapping on* H*. Let* $G:C\to C$ *be a* κ*-Lipschitzian and* η*-strongly monotone mapping, and let* $V:C\to C$

be an l-Lipschitzian mapping. Assume that $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \le \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$ and $\Omega = F(T) \cap VI(H, B, M) \neq \emptyset$. Let $x_1 \in C$, and sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = TJ_{M,\lambda}(x_n - \lambda Bx_n), \\ z_n = \beta_n x_n + \gamma_n y_n + \delta_n Ty_n, \\ x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) z_n, \quad \forall n \ge 1, \end{cases} \tag{4.1}$$

where $\lambda \in (0, 2r]$. And $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(c1) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c2) $\beta_n + \gamma_n + \delta_n = 1, (\gamma_n + \delta_n)\xi \le \gamma_n$, *for all* $n \ge 1$, *and* $\beta_n \subset [a, b] \subset (0, 1)$;
(c3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *and* $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$;
(c4) $\lim_{n\to\infty} \delta_n = 0$.

Then the sequence $\{x_n\}$ generated by (4.1) converges strongly to a point $z \in \Omega$, which is the unique solution of the variational inequality (3.2).

Theorem 4.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $M : H \to 2^H$ a maximal monotone operator. Let $B : C \to H$ be a r-inverse-strongly monotone and $\Omega = VI(H, B, M) \neq \emptyset$. Let $G : C \to C$ be a κ-Lipschitzian and η-strongly monotone mapping, and let $V : C \to C$ be an l-Lipschitzian mapping. Assume that $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \le \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $x_1 \in C$, and sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) J_{M,\lambda}(x_n - \lambda Bx_n), \\ x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) y_n, \quad \forall n \ge 1, \end{cases} \tag{4.2}$$

where $\lambda \in (0, 2r]$. And $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(c1) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c2) $\beta_n \subset [a, b] \subset (0, 1)$;
(c3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ generated by (4.2) converges strongly to a point $z \in \Omega$, which is the unique solution of the variational inequality (3.2).

If T is ξ-strictly pseudocontractive, then $I - T$ is $\frac{1-\xi}{2}$-inverse-strongly monotone. We are in a position to give a result on common fixed points of a pair of strictly pseudocontractive mappings.

Theorem 4.3. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let T be a ξ-strictly pseudocontractive mapping on H and let S be a $\overline{\xi}$-strictly pseudocontractive mapping on H. Let $G : C \to C$ be a κ-Lipschitzian and η-strongly monotone mapping, and let $V : C \to C$ be an l-Lipschitzian mapping. Assume that $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \le \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$ and $\Omega = F(T) \cap F(S) \neq \emptyset$. Let $x_1 \in C$, and sequence $\{x_n\}$ be generated by*

$$\begin{cases} u_n = \lambda S x_n + (1 - \lambda) x_n, \\ y_n = \kappa T u_n + (1 - \kappa) u_n, \\ z_n = \beta_n x_n + \gamma_n y_n + \delta_n Ty_n, \\ x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) z_n, \quad \forall n \ge 1, \end{cases} \tag{4.3}$$

where $\varsigma \in (0, 1 - \xi]$ and $\lambda \in (0, 1 - \overline{\xi}]$. And $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(c1) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c2) $\beta_n + \gamma_n + \delta_n = 1$, $(\gamma_n + \delta_n)\xi \le \gamma_n$, *for all* $n \ge 1$, *and* $\beta_n \subset [a, b] \subset (0, 1)$;
(c3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$, *and* $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$;
(c4) $\lim_{n\to\infty} \delta_n = 0$.

Then the sequence $\{x_n\}$ *generated by* (4.3) *converges strongly to a point* $z \in \Omega$*, which is the unique solution of the variational inequality* (3.2)*.*

$$\langle (\gamma V - \mu G)z, z - \omega \rangle \geq 0, \quad \forall \omega \in \Omega.$$

Proof. By putting $B := I - S$, we find B is $\frac{1-\bar{\xi}}{2}$-inverse-strongly monotone. We also find $VI(H,B) = F(S)$ and $\lambda S x_n + (1-\lambda)x_n = J_{M,\lambda}(x_n - \lambda S x_n)$. From Theorem 3.5 , we obtain the desired result immediately. □

Let C be a nonempty closed and convex subset of H and $B : C \to H$ be a mapping. Recall that the classical variational inequality is to find an $x \in C$ such that $\langle Bx, y - x\rangle \geq 0$, for all $y \in C$. The solution set of variational inequality is denoted by $VI(C,B)$. It is known that x is a solution to the variational inequality iff x is fixed point of the mapping $P_C(I - \lambda B)$, where I denotes the identity on H. Let i_C be a function defined by

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

It is easy to see that i_C is proper lower and semicontinuous convex function on H, and the subdifferential ∂_{i_C} of i_C is maximal monotone. Define the resolvent $J_{i_C,\lambda} := (I + \lambda \partial_{i_C} x)^{-1}$ of the subdifferential operator ∂_{i_C}. By letting $x = J_{i_C,\lambda} y$, we find that

$$x = J_{i_C,\lambda} y \Longleftrightarrow y \in x + \lambda \partial_{i_C} x \Longleftrightarrow x = P_C y.$$

By putting $M = \partial_{i_C}$ in Theorem 3.5, we find the following results immediately.

Theorem 4.4. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $B : C \to H$ be an r-inverse-strongly monotone and let T be a ξ-strictly pseudocontractive mapping on H. Let $G : C \to C$ be a κ-Lipschitzian and η-strongly monotone mapping, and let $V : C \to C$ be an l-Lipschitzian mapping. Assume that $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$ and $\Omega = F(T) \cap VI(H,B,M) \neq \emptyset$. Let $x_1 \in C$, and sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \zeta T P_C(x_n - \lambda B x_n) + (1-\zeta) P_C(x_n - \lambda B x_n), \\ z_n = \beta_n x_n + \gamma_n y_n + \delta_n T y_n, \\ x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) z_n, \quad \forall n \geq 1, \end{cases} \tag{4.4}$$

where $\zeta \in (0, 1-\xi]$ and $\lambda \in (0, 2r]$. And $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(c1) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c2) $\beta_n + \gamma_n + \delta_n = 1, (\gamma_n + \delta_n)\xi \leq \gamma_n$, *for all* $n \geq 1$, *and* $\beta_n \subset [a,b] \subset (0,1)$;
(c3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *and* $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$;
(c4) $\lim_{n\to\infty} \delta_n = 0$.

Then the sequence $\{x_n\}$ *generated by* (4.4) *converges strongly to a point* $z \in \Omega$*, which is the unique solution of the variational inequality* (3.2)*.*

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