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A general composite steepest-descent method for hierarchical fixed point problems of strictly pseudocontractive mappings in Hilbert spaces

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Abstract

In this paper, we propose general composite implicit and explicit steepest-descent schemes for hierarchical fixed point problems of strictly pseudocontractive mappings in a real Hilbert space. These composite steepest-descent schemes are based on the well-known viscosity approximation method, hybrid steepest-descent method and strongly positive bounded linear operator approach. We obtain some strong convergence theorems under suitable conditions. Our results supplement and develop the corresponding ones announced by some authors recently in this area.

Keywords: General composite steepest-descent method, strictly pseudocontractive mapping, hierarchical fixed point problem, demiclosedness principle, nonexpansive mapping, fixed point.
2010 MSC: 49J30, 47H09, 47J20, 49M05.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $T : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T and by $\mathbf{R}$ the set of all real numbers. A mapping $A : H \to H$ is called $\bar{\gamma}$-strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

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A mapping $F : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Fx - Fy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$ then F is called a nonexpansive mapping; if $L \in [0, 1)$ then F is called a contraction. A mapping $T : C \to C$ is called k-strictly pseudocontractive (or a k-strict pseudocontraction) if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

In particular, if $k = 0$, then T is a nonexpansive mapping. The mapping T is pseudocontractive if and only if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

Note that the class of strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudocontractive. The mapping T is also said to be pseudocontractive if $k = 1$. Obviously, the class of strictly pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudocontractive mappings. The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. Recently, many authors have been devoting the study of the problem of finding fixed points of pseudocontractive mappings; see e.g., [4, 7, 13] and the references therein.

Let $F : C \to H$ be a nonlinear mapping on C. The variational inequality problem (VIP) associated with the set C and the mapping F is stated as follows: find $x^* \in C$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathrm{VI}(C, F)$.

The VIP (1.1) was first discussed by Lions [9] and now is well-known; there are a lot of different approaches towards solving VIP (1.1) in finite-dimensional and infinite-dimensional spaces, and the research is intensively continued. The VIP (1.1) has many applications in computational mathematics, mathematical physics, operations research, mathematical economics, optimization theory, and other fields; see, e.g., [5, 16, 18, 28]. It is well-known that, if F is a strongly monotone and Lipschitz continuous mapping on C, then VIP (1.1) has a unique solution. In the literature, the recent research work shows that variational inequalities like VIP (1.1) cover several topics, for example, monotone inclusions, convex optimization and quadratic minimization over fixed point sets; see [11, 14, 23, 27] for more details.

If we take $C = \mathrm{Fix}(T) \neq \emptyset$ and $A = I - S$ where $T : H \to H$ is one nonexpansive mapping with fixed points and $S : H \to H$ is another nonexpansive mapping (not necessarily with fixed points), then problem (1.1) becomes the VIP of finding $x^* \in \mathrm{Fix}(T)$ such that

$$\langle (I - S)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T),$$

introduced first by Maingé and Moudafi in [10, 15], which is called a hierarchical fixed point problem. Subsequently, this problem is extended to some hierarchical fixed point problems with constraints; see e.g., [3].

In particular, whenever $\mathrm{Fix}(S) \neq \emptyset$, all elements of $\mathrm{Fix}(S)$ are solutions of the last VIP. If S is a ρ-contraction (i.e., $\|Sx - Sy\| \leq \rho\|x - y\|$ for some $\rho \in (0, 1)$) the set of solutions of the last VIP is a singleton and it is well-known as a VIP defined over the fixed-point set, which was first introduced by Moudafi [14] and then developed by several authors [3, 11, 23, 27].

Variational inequalities like the last VIP cover several topics recently investigated in the literature as monotone inclusions, convex optimization and quadratic minimization over fixed point sets; see e.g., [1, 14, 20, 21] and the references therein.

In 2001, Yamada [25] introduced the following hybrid steepest-descent method for solving the VIP (1.1) with $C = \mathrm{Fix}(S)$

$$x_{n+1} = (I - \lambda_n \mu F)Sx_n, \quad \forall n \geq 0, \tag{1.2}$$

where $S : H \to H$ is a nonexpansive mapping with $\mathrm{Fix}(S) \neq \emptyset$, $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$ (i.e., $\|Fx - Fy\| \leq \kappa \|x - y\|$ and $\langle Fx - Fy, x - y\rangle \geq \eta \|x - y\|^2\ \forall x, y \in H$), and $0 < \mu < \frac{2\eta}{\kappa^2}$, and then proved that under appropriate conditions, the sequence $\{x_n\}$ generated by (1.2) converges strongly to the unique solution of VIP (1.1) with $C = \mathrm{Fix}(S)$.

In 2010, by combining Yamada's hybrid steepest-descent method and Marino and Xu's hybrid viscosity approximation method [11], Tian [20] introduced the following general iterative scheme

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)Tx_n, \quad \forall n \geq 0,$$

where $T : H \to H$ is a nonexpansive mapping and $f : H \to H$ is a contractive mapping with constant $\alpha \in (0, 1)$. His results improve and complement the corresponding results of Marino and Xu [11]. In [21], Tian also considered the following general iterative scheme

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F)Tx_n, \quad \forall n \geq 0,$$

where $T : H \to H$ is a nonexpansive mapping and $V : H \to H$ is a Lipschitzian mapping with constant $l \geq 0$. In particular, the results in [21] extend Tian's results [20] from the contractive mapping f to the Lipschitzian mapping V.

In 2011, Ceng et al. [1] also introduced the following iterative method

$$x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)Tx_n], \quad \forall n \geq 0, \tag{1.3}$$

where $F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$, $T : C \to C$ is a nonexpansive mapping, $V : C \to H$ is an l-Lipschitzian mapping with constant $l \geq 0$ and $0 < \mu < \frac{2\eta}{\kappa^2}$. They proved that, under mild conditions, the sequence $\{x_n\}$ generated by (1.3) converges strongly to a point $\tilde{x} \in \mathrm{Fix}(T)$ which is the unique solution to the VIP

$$\langle (\mu F - \gamma V)\tilde{x}, p - \tilde{x}\rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T).$$

It is worth pointing out that they changed the domain of mapping F and thus imposed the projection P_C on Tian's iterative scheme in [21].

In 2011, Ceng et al. [2] introduced one general composite implicit scheme that in an implicit way generates a net $\{x_t\}_{t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau - \gamma\alpha}\})}$

$$x_t = (I - \theta_t A)Tx_t + \theta_t[Tx_t - t(\mu F Tx_t - \gamma f(x_t))], \tag{1.4}$$

and also proposed another general composite explicit scheme that generates a sequence $\{x_n\}$ in an explicit way

$$\begin{cases} y_n = (I - \alpha_n \mu F)Tx_n + \alpha_n \gamma f(x_n), \\ x_{n+1} = (I - \beta_n A)Tx_n + \beta_n y_n, \quad \forall n \geq 0, \end{cases} \tag{1.5}$$

where $x_0 \in H$ is an arbitrary initial guess, $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$, $T : H \to H$ is a nonexpansive mapping, $A : H \to H$ is a $\bar{\gamma}$-strongly positive bounded linear operator, and $f : H \to H$ is an α-contractive mapping with $\alpha \in (0, 1)$. They proved that, under appropriate conditions, the net $\{x_t\}$ and the sequence $\{x_n\}$ generated by (1.4) and (1.5), respectively, converge strongly to the same point $\tilde{x} \in \mathrm{Fix}(T)$, which is the unique solution to the VIP

$$\langle (A - I)\tilde{x}, p - \tilde{x}\rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T).$$

Their results supplement and develop the corresponding ones of Yamada [25], Marino and Xu [11], and Tian [20].

Very recently, inspired by Ceng et al. [2], Jung [8] introduced one general composite implicit scheme that generates a net $\{x_t\}_{t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau - \gamma l}\})}$ in an implicit way

$$x_t = (I - \theta_t A)T_t x_t + \theta_t[t\gamma V x_t + (I - t\mu F)T_t x_t], \tag{1.6}$$

and also proposed another general composite explicit scheme that generates a sequence $\{x_n\}$ in an explicit way

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n x_n, \\ x_{n+1} = (I - \beta_n A) T_n x_n + \beta_n y_n, \quad \forall n \geq 0, \end{cases} \tag{1.7}$$

where $x_0 \in H$ is an arbitrary initial guess and the following conditions are satisfied:

- $T : H \to H$ is a k-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$;
- A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $\bar{\gamma} \in (1, 2)$;
- $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with $0 < \mu < \frac{2\eta}{\kappa^2}$;
- $V : H \to H$ is an l-Lipschitzian mapping with $0 \leq \gamma l < \tau$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
- $T_t : H \to H$ is a mapping defined by $T_t x = \lambda_t x + (1 - \lambda_t) T x, t \in (0, 1)$, for $0 \leq k \leq \lambda_t \leq \lambda < 1$ and $\lim_{t \to 0} \lambda_t = \lambda$;
- $T_n : H \to H$ is a mapping defined by $T_n x = \lambda_n x + (1 - \lambda_n) T x$ for $0 \leq k \leq \lambda_n \leq \lambda < 1$ and $\lim_{n \to \infty} \lambda_n = \lambda$;
- $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$ and $\{\theta_t\}_{t \in (0, \min\{1, \frac{2 - \bar{\gamma}}{\tau - \gamma l}\})} \subset (0, 1)$.

Jung [8] proved that, under the weaker control conditions than previous ones, the net $\{x_t\}$ and the sequence $\{x_n\}$ generated by (1.6) and (1.7), respectively, converge strongly to the same point $\tilde{x} \in \mathrm{Fix}(T)$, which is the unique solution to the VIP

$$\langle (A - I)\tilde{x}, p - \tilde{x} \rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T).$$

His results extend and improve Ceng et al.'s corresponding ones [2] from the nonexpansive mapping T to the strictly pseudocontractive mapping T and from the contractive mapping f to the Lipschitzian mapping V.

In this paper, we first introduce one general composite implicit steepest-descent scheme for solving a hierarchical fixed point problem of a k-strictly pseudocontractive mapping $T : H \to H$

$$x_t = (I - \theta_t A) T_t x_t + \theta_t [V x_t - t(\mu F V x_t - \gamma T_t x_t)],$$

where $\lim_{t \to 0} \theta_t = 0$ and $l = 1$. It is proven that as $t \to 0$, $\{x_t\}$ converges strongly to a point $\tilde{x} \in \mathrm{Fix}(T)$, which is the unique solution in $\mathrm{Fix}(T)$ to the VIP

$$\langle (A - V)\tilde{x}, p - \tilde{x} \rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{1.8}$$

On the other hand, we also propose another general composite explicit steepest-descent scheme for solving a hierarchical fixed point problem for a k-strictly pseudocontractive mapping $T : H \to H$

$$\begin{cases} y_n = \alpha_n \gamma T_n x_n + (I - \alpha_n \mu F) V x_n, \\ x_{n+1} = (I - \beta_n A) T_n x_n + \beta_n y_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$ and $l = 1$. It is proven that under mild conditions, $\{x_n\}$ converges strongly to the same point $\tilde{x} \in \mathrm{Fix}(T)$, which is the unique solution in $\mathrm{Fix}(T)$ to the VIP (1.8).

The above general composite steepest-descent schemes are based on the well-known viscosity approximation method (see e.g., [14, 23]), hybrid steepest-descent method (see, e.g., [24, 25]) and strongly positive bounded linear operator approach [11]. Our results supplement and develop the corresponding ones announced by some authors recently in this area, e.g., Ceng et al. [2] and Jung [8].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

The following properties of projections are useful and pertinent to our purpose.

Proposition 2.1 ([6]). *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \le 0,\;\; \forall y \in C;$
(ii) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2,\;\; \forall y \in C;$
(iii) $\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2,\;\; \forall y \in H.$

Consequently, P_C is nonexpansive and monotone.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.2 ([19]). *Let X be a real inner product space. Then there holds the following inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.3 ([19]). *Let H be a real Hilbert space. Then the followings hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H$;
(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0, 1]$ *with* $\lambda + \mu = 1$;
(c) *If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

It is clear that, in a real Hilbert space H, $T : C \to C$ is k-strictly pseudocontractive if and only if the following inequality holds:

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2 - \frac{1-k}{2}\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

This immediately implies that if T is a k-strictly pseudocontractive mapping, then $I - T$ is $\frac{1-k}{2}$-inverse strongly monotone; for further detail, we refer to [12] and the references therein. It is well-known that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings and that the class of pseudocontractions strictly includes the class of strict pseudocontractions.

Lemma 2.4 ([12, Proposition 2.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a mapping.*

(i) *If T is a k-strictly pseudocontractive mapping, then T satisfies the Lipschitzian condition*

$$\|Tx - Ty\| \le \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C.$$

(ii) *If T is a k-strictly pseudocontractive mapping, then the mapping $I - T$ is semiclosed at* 0*, that is, if* $\{x_n\}$ *is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-T)x_n \to 0$, then $(I-T)\tilde{x} = 0$.*

(iii) *If T is k-(quasi-)strict pseudocontraction, then the fixed point set* $\mathrm{Fix}(T)$ *of T is closed and convex so that the projection $P_{\mathrm{Fix}(T)}$ is well-defined.*

Lemma 2.5 ([26])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma+\delta)k \le \gamma$. Then*

$$\|\gamma(x-y) + \delta(Tx - Ty)\| \le (\gamma+\delta)\|x-y\|, \quad \forall x, y \in C.$$

Lemma 2.6 ([6, demiclosedness principle])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive self-mapping on C with* $\mathrm{Fix}(S) \neq \emptyset$*. Then $I - S$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-S)x_n\}$ strongly converges to some y, it follows that $(I-S)x = y$. Here I is the identity operator of H.*

Lemma 2.7. *Let $F : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition* 2.1 (i)*) implies*

$$u \in \mathrm{VI}(C, F) \quad \Leftrightarrow \quad u = P_C(u - \lambda F u), \quad \lambda > 0.$$

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0, 1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to C$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C,$$

where $F : C \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C; that is, F satisfies the conditions:

$$\|Fx - Fy\| \le \kappa\|x-y\| \quad \text{and} \quad \langle Fx - Fy, x-y\rangle \ge \eta\|x-y\|^2,$$

for all $x, y \in C$.

Lemma 2.8 ([24, Lemma 3.1])**.** *T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\|T^\lambda x - T^\lambda y\| \le (1 - \lambda\tau)\|x-y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Lemma 2.9 ([22, Lemma 2.1])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \le (1-\omega_n)a_n + \omega_n\delta_n + r_n, \quad \forall n \ge 0,$$

where $\{\omega_n\}, \{\delta_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\omega_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \omega_n = \infty$*;*

(ii) *either* $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} \omega_n|\delta_n| < \infty$*;*

(iii) $r_n \ge 0$ *for all* $n \ge 0$*, and* $\sum_{n=1}^{\infty} r_n < \infty$*.*

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10 ([11])**.** *Assume that A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.*

Let LIM be a Banach limit. According to time and circumstances, we use $\mathrm{LIM}_n a_n$ instead of $\mathrm{LIM} a$ for every $a = \{a_n\} \in l^\infty$. The following properties are well-known:

(i) for all $n \geq 1, a_n \leq c_n$ implies $\text{LIM}_n a_n \leq \text{LIM}_n c_n$;

(ii) $\text{LIM}_n a_{n+N} = \text{LIM}_n a_n$ for any fixed positive integer N;

(iii) $\liminf_{n\to\infty} a_n \leq \text{LIM}_n a_n \leq \limsup_{n\to\infty} a_n$ for all $\{a_n\} \in l^\infty$.

The following lemma was given in [17, Proposition 2]).

Lemma 2.11 ([11])**.** *Let $a \in \mathbb{R}$ be a real number and let a sequence $\{a_n\} \in l^\infty$ satisfy the condition $\text{LIM}_n a_n \leq a$ for all Banach limit* LIM. *If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

3. Main results

Let H be a real Hilbert space. Throughout this section, we always assume the following:

- $T : H \to H$ is a k-strictly pseudocontractive mapping with $\text{Fix}(T) \neq \emptyset$;
- $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with $0 < \mu < \frac{2\eta}{\kappa^2}$;
- A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $\bar{\gamma} \in (1, 2)$;
- $V : H \to H$ is an l-Lipschitzian mapping with constant $l \geq 0$;
- $0 \leq \gamma l < \tau$ with $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
- $T_t : H \to H$ is a mapping defined by $T_t x = \lambda_t x + (1 - \lambda_t)Tx, t \in (0, 1)$, for $0 \leq k \leq \lambda_t \leq \lambda < 1$ and $\lim_{t\to 0} \lambda_t = \lambda$;
- $T_n : H \to H$ is a mapping defined by $T_n x = \lambda_n x + (1 - \lambda_n)Tx$ for $0 \leq k \leq \lambda_n \leq \lambda < 1$ and $\lim_{n\to\infty} \lambda_n = \lambda$;
- $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$ and $\{\theta_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})} \subset (0, 1)$.

It can be readily seen from Lemma 2.5 that T_t and T_n are nonexpansive. Moreover, it is clear that $\text{Fix}(T) = \text{Fix}(T_t) = \text{Fix}(T_n)$.

In this section, we introduce the first general composite implicit steepest-descent scheme that generates a net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})}$ in an implicit manner:

$$x_t = (I - \theta_t A)T_t x_t + \theta_t[V x_t - t(\mu F V x_t - \gamma T_t x_t)], \tag{3.1}$$

where $V : H \to H$ is nonexpansive with $l = 1$ and $0 \leq \gamma < \tau$. We prove the strong convergence of $\{x_t\}$ as $t \to 0$ to a fixed point $\tilde{x}$ of T (i.e., $\tilde{x} \in \text{Fix}(T)$), which is a unique solution to the VIP

$$\langle (A - V)\tilde{x}, p - \tilde{x}\rangle \geq 0, \quad \forall p \in \text{Fix}(T). \tag{3.2}$$

For arbitrarily given $x_0 \in H$, we also propose the second general composite explicit steepest-descent scheme, which generates a sequence $\{x_n\}$ in an explicit way:

$$\begin{cases} y_n = \alpha_n \gamma T_n x_n + (I - \alpha_n \mu F)V x_n, \\ x_{n+1} = (I - \beta_n A)T_n x_n + \beta_n y_n, \quad \forall n \geq 0, \end{cases} \tag{3.3}$$

and establish the strong convergence of $\{x_n\}$ as $n \to \infty$ to a fixed point $\tilde{x}$ of T (i.e., $\tilde{x} \in \text{Fix}(T)$), which is also the unique solution to the VIP (3.2).

Now, for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma}\})$, and $\theta_t \in (0, \|A\|^{-1}]$, consider a mapping $Q_t : H \to H$ defined by

$$Q_t x = (I - \theta_t A)T_t x + \theta_t[V x - t(\mu F V x - \gamma T_t x)], \quad \forall x \in H.$$

It is easy to see that Q_t is a contractive mapping with constant $1-\theta_t(\bar{\gamma}-1+t(\tau-\gamma))$. Indeed, by Lemmas 2.8 and 2.10, we have

$$\begin{aligned}\|Q_tx-Q_ty\| &= \|(I-\theta_tA)T_tx+\theta_t[(I-t\mu F)Vx+t\gamma T_tx]-(I-\theta_tA)T_ty-\theta_t[(I-t\mu F)Vy+t\gamma T_ty]\|\\ &\le \|(I-\theta_tA)T_tx-(I-\theta_tA)T_ty\|+\theta_t\|((I-t\mu F)Vx+t\gamma T_tx)-((I-t\mu F)Vy+t\gamma T_ty)\|\\ &\le (1-\theta_t\bar{\gamma})\|T_tx-T_ty\|+\theta_t[\|(I-t\mu F)Vx-(I-t\mu F)Vy\|+t\gamma\|T_tx-T_ty\|]\\ &\le (1-\theta_t\bar{\gamma})\|x-y\|+\theta_t[(1-t\tau)\|x-y\|+t\gamma\|x-y\|]\\ &= (1-\theta_t\bar{\gamma})\|x-y\|+\theta_t(1-t(\tau-\gamma))\|x-y\|\\ &= [1-\theta_t(\bar{\gamma}-1+t(\tau-\gamma))]\|x-y\|.\end{aligned}$$

Since $\bar{\gamma}\in(1,2)$, $\tau-\gamma>0$, and

$$0<t<\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\}\le\frac{2-\bar{\gamma}}{\tau-\gamma},$$

it follows that

$$0<\bar{\gamma}-1+t(\tau-\gamma)<1,$$

which together with $0<\theta_t\le\|A\|^{-1}<1$ yields

$$0<1-\theta_t(\bar{\gamma}-1+t(\tau-\gamma))<1.$$

Hence $Q_t:H\to H$ is a contractive mapping. By the Banach contraction principle, Q_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

We summary the basic properties of $\{x_t\}$. The argument techniques in [7] and [23] are extended to develop the new argument ones for these basic properties by virtue of Lemma 2.8. We include the argument process for the sake of completeness.

Proposition 3.1. *Let $l=1$ and $\{x_t\}$ be defined via (3.1). Then*

(i) *$\{x_t\}$ is bounded for $t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})$;*

(ii) *$\lim_{t\to0}\|x_t-T_tx_t\|=0$ provided $\lim_{t\to0}\theta_t=0$;*

(iii) *$x_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})\to H$ is locally Lipschitzian provided $\theta_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})\to(0,\|A\|^{-1}]$ is locally Lipschitzian, and $\lambda_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})\to[k,\lambda]$ is locally Lipschitzian;*

(iv) *x_t defines a continuous path from $(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})$ into H provided $\theta_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})\to(0,\|A\|^{-1}]$ is continuous, and $\lambda_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})\to[k,\lambda]$ is continuous.*

Proof. (i) Let $p\in\mathrm{Fix}(T)$. Utilizing $\mathrm{Fix}(T)=\mathrm{Fix}(T_t)$ and Lemmas 2.8 and 2.10, from the nonexpansivity of T_t and V we get

$$\begin{aligned}\|x_t-p\| &= \|(I-\theta_tA)T_tx_t+\theta_t((I-t\mu F)Vx_t+t\gamma T_tx_t)-p\|\\ &= \|(I-\theta_tA)T_tx_t-(I-\theta_tA)T_tp+\theta_t((I-t\mu F)Vx_t+t\gamma T_tx_t)-\theta_tAp\|\\ &\le \|(I-\theta_tA)T_tx_t-(I-\theta_tA)T_tp\|+\theta_t\|(I-t\mu F)Vx_t+t\gamma T_tx_t)-Ap\|\\ &\le (1-\theta_t\bar{\gamma})\|T_tx_t-T_tp\|+\theta_t\|(I-t\mu F)Vx_t-(I-t\mu F)Vp+t(\gamma T_tx_t-\mu FVp)+Vp-Ap\|\\ &\le (1-\theta_t\bar{\gamma})\|x_t-p\|+\theta_t[\|(I-t\mu F)Vx_t-(I-t\mu F)Vp\|+t\|\gamma T_tx_t-\mu FVp\|+\|(V-A)p\|]\\ &\le (1-\theta_t\bar{\gamma})\|x_t-p\|+\theta_t[\|(I-t\mu F)Vx_t-(I-t\mu F)Vp\|\\ &\quad+t(\gamma\|T_tx_t-T_tp\|+\|(\gamma I-\mu FV)p\|)+\|(V-A)p\|]\\ &\le (1-\theta_t\bar{\gamma})\|x_t-p\|+\theta_t[(1-t\tau)\|x_t-p\|+t(\gamma\|x_t-p\|+\|(\gamma I-\mu FV)p\|)+\|(V-A)p\|]\\ &= [1-\theta_t(\bar{\gamma}-1+t(\tau-\gamma))]\|x_t-p\|+\theta_t(t\|(\gamma I-\mu FV)p\|+\|(V-A)p\|).\end{aligned}$$

So, it follows that

$$\begin{aligned}\|x_t - p\| &\le \frac{\|(V-A)p\| + t\|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1 + t(\tau - \gamma)} \\ &\le \frac{\|(V-A)p\| + t\|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1} \\ &\le \frac{\|(V-A)p\| + \|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1}.\end{aligned}$$

Hence $\{x_t\}$ is bounded. Since T is k-strictly pseudocontractive, by Lemma 2.4 (i) we know that T is Lipschitzian, $\|Tx_t - p\| \le \frac{1+k}{1-k}\|x-y\|$ for all $x, y \in H$. So, by Lipschitz continuity of the mappings V, T, T_t, and F we deduce that $\{Vx_t\}, \{Tx_t\}, \{T_t x_t\}$, and $\{FVx_t\}$ are bounded.

(ii) By the definition of $\{x_t\}$, we have

$$\begin{aligned}\|x_t - T_t x_t\| &= \|(I - \theta_t A)T_t x_t + \theta_t((I - t\mu F)Vx_t + t\gamma T_t x_t) - T_t x_t\| \\ &= \theta_t\| - AT_t x_t + (I - t\mu F)Vx_t + t\gamma T_t x_t\| \\ &= \theta_t\|Vx_t - AT_t x_t + t(\gamma T_t x_t - \mu FVx_t)\| \\ &\le \theta_t\|Vx_t - AT_t x_t\| + t\|\gamma T_t x_t - \mu FVx_t\| \to 0 \quad \text{as } t \to 0\end{aligned}$$

by the boundedness of $\{Vx_t\}, \{T_t x_t\}$ and $\{FVx_t\}$ in the assertion (i).

(iii) Let $t, t_0 \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma}\})$. Noting that

$$\|T_t x_t - T_{t_0} x_{t_0}\| \le \|T_t x_t - T_t x_{t_0}\| + \|T_t x_{t_0} - T_{t_0} x_{t_0}\| \le \|x_t - x_{t_0}\| + |\lambda_t - \lambda_{t_0}|\|x_{t_0} - Tx_{t_0}\|,$$

we calculate

$$\begin{aligned}\|x_t - x_{t_0}\| &= \|(I - \theta_t A)T_t x_t + \theta_t((I - t\mu F)Vx_t + t\gamma T_t x_t) \\ &\quad - (I - \theta_{t_0} A)T_{t_0} x_{t_0} - \theta_{t_0}((I - t_0\mu F)Vx_{t_0} + t_0\gamma T_{t_0} x_{t_0})\| \\ &\le \|(I - \theta_t A)T_t x_t - (I - \theta_{t_0} A)T_{t_0} x_{t_0}\| \\ &\quad + \|\theta_t((I - t\mu F)Vx_t + t\gamma T_t x_t) - \theta_{t_0}((I - t_0\mu F)Vx_{t_0} + t_0\gamma T_{t_0} x_{t_0})\| \\ &\le \|(I - \theta_t A)T_t x_t - (I - \theta_{t_0} A)T_t x_t\| + \|(I - \theta_{t_0} A)T_t x_t - (I - \theta_{t_0} A)T_{t_0} x_{t_0}\| \\ &\quad + |\theta_t - \theta_{t_0}|\|(I - t\mu F)Vx_t + t\gamma T_t x_t\| \\ &\quad + \theta_{t_0}\|((I - t\mu F)Vx_t + t\gamma T_t x_t) - ((I - t_0\mu F)Vx_{t_0} + t_0\gamma T_{t_0} x_{t_0})\| \\ &\le |\theta_t - \theta_{t_0}|\|A\|\|T_t x_t\| + (1 - \theta_{t_0}\bar{\gamma})\|T_t x_t - T_{t_0} x_{t_0}\| \\ &\quad + |\theta_t - \theta_{t_0}|\|(I - t\mu F)Vx_t + t\gamma T_t x_t\| + \theta_{t_0}\|(t - t_0)\gamma T_t x_t + t_0\gamma(T_t x_t - T_{t_0} x_{t_0}) \\ &\quad - (t - t_0)\mu FVx_t + (I - t_0\mu F)Vx_t - (I - t_0\mu F)Vx_{t_0}\| \\ &\le |\theta_t - \theta_{t_0}|\|A\|\|T_t x_t\| + (1 - \theta_{t_0}\bar{\gamma})\|T_t x_t - T_{t_0} x_{t_0}\| \\ &\quad + |\theta_t - \theta_{t_0}|[\|Vx_t\| + t(\gamma\|T_t x_t\| + \mu\|FVx_t\|)] + \theta_{t_0}[(\gamma\|T_t x_t\| + \mu\|FVx_t\|)|t - t_0| \\ &\quad + t_0\gamma\|T_t x_t - T_{t_0} x_{t_0}\| + (1 - t_0\tau)\|x_t - x_{t_0}\|] \\ &\le |\theta_t - \theta_{t_0}|\|A\|\|T_t x_t\| + (1 - \theta_{t_0}\bar{\gamma})(\|x_t - x_{t_0}\| + |\lambda_t - \lambda_{t_0}|\|x_{t_0} - Tx_{t_0}\|) \\ &\quad + |\theta_t - \theta_{t_0}|[\|Vx_t\| + t(\gamma\|T_t x_t\| + \mu\|FVx_t\|)] + \theta_{t_0}[(\gamma\|T_t x_t\| + \mu\|FVx_t\|)|t - t_0| \\ &\quad + t_0\gamma(\|x_t - x_{t_0}\| + |\lambda_t - \lambda_{t_0}|\|x_{t_0} - Tx_{t_0}\|) + (1 - t_0\tau)\|x_t - x_{t_0}\|] \\ &\le |\theta_t - \theta_{t_0}|\|A\|\|T_t x_t\| + (1 - \theta_{t_0}\bar{\gamma})(\|x_t - x_{t_0}\| + |\lambda_t - \lambda_{t_0}|\|x_{t_0} - Tx_{t_0}\|) \\ &\quad + |\theta_t - \theta_{t_0}|[\|Vx_t\| + t(\gamma\|T_t x_t\| + \mu\|FVx_t\|)] + \theta_{t_0}[(\gamma\|T_t x_t\| + \mu\|FVx_t\|)|t - t_0| \\ &\quad + (1 - t_0(\tau - \gamma))(\|x_t - x_{t_0}\| + |\lambda_t - \lambda_{t_0}|\|x_{t_0} - Tx_{t_0}\|)] \\ &= |\theta_t - \theta_{t_0}|\|A\|\|T_t x_t\| + (1 - \theta_{t_0}(\bar{\gamma} - 1 + t_0(\tau - \gamma))(\|x_t - x_{t_0}\| + |\lambda_t - \lambda_{t_0}|\|x_{t_0} - Tx_{t_0}\|) \\ &\quad + |\theta_t - \theta_{t_0}|[\|Vx_t\| + t(\gamma\|T_t x_t\| + \mu\|FVx_t\|)] + \theta_{t_0}(\gamma\|T_t x_t\| + \mu\|FVx_t\|)|t - t_0|\end{aligned}$$

$$
\begin{aligned}
&= (1-\theta_{t_0}(\bar{\gamma}-1+t_0(\tau-\gamma))(\|x_t-x_{t_0}\|+|\lambda_t-\lambda_{t_0}|\|x_{t_0}-Tx_{t_0}\|) \\
&\quad +|\theta_t-\theta_{t_0}|[\|A\|\|T_tx_t\|+\|Vx_t\|+t(\gamma\|T_tx_t\|+\mu\|FVx_t\|)]+\theta_{t_0}(\gamma\|T_tx_t\|+\mu\|FVx_t\|)|t-t_0|.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_t-x_{t_0}\| \le &\frac{\|A\|\|T_tx_t\|+\|Vx_t\|+t(\gamma\|T_tx_t\|+\mu\|FVx_t\|)}{\theta_{t_0}(\bar{\gamma}-1+t_0(\tau-\gamma)}|\theta_t-\theta_{t_0}|+\frac{\gamma\|T_tx_t\|+\mu\|FVx_t\|}{\bar{\gamma}-1+t_0(\tau-\gamma)}|t-t_0| \\
&+\frac{(1-\theta_{t_0}(\bar{\gamma}-1+t_0(\tau-\gamma))\|x_{t_0}-Tx_{t_0}\|}{\theta_{t_0}(\bar{\gamma}-1+t_0(\tau-\gamma)}|\lambda_t-\lambda_{t_0}|.
\end{aligned}
$$

Since $\{Vx_t\}, \{T_tx_t\}$, and $\{FVx_t\}$ are bounded, $\theta_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})\to(0,\|A\|^{-1}]$ and $\lambda_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})\to[k,\lambda]$ are locally Lipschitzian, we deduce that $x_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})\to H$ is locally Lipschitzian.

(iv) From the last inequality in (iii), the result follows immediately. □

We prove the following theorem for strong convergence of the net $\{x_t\}$ as $t\to 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.2. *Let $l=1$ and the net $\{x_t\}$ be defined via* (3.1)*. If $\lim_{t\to 0}\theta_t=0$, then x_t converges strongly to a fixed point $\tilde{x}$ of T as $t\to 0$, which solves the VIP* (3.2)*. Equivalently, we have $P_{\mathrm{Fix}(T)}(I+V-A)\tilde{x}=\tilde{x}$.*

Proof. We first show the uniqueness of solutions of the VIP (3.2), which is indeed a consequence of the strong monotonicity of $A-V$. In fact, since A is a $\bar{\gamma}$-strongly positive bounded linear operator with $\bar{\gamma}\in(1,2)$ and V is a nonexpansive mapping with $l=1$, we know that $A-V$ is $(\bar{\gamma}-1)$-strongly monotone with constant $\bar{\gamma}-1\in(0,1)$. Suppose that $\tilde{x}\in\mathrm{Fix}(T)$ and $\hat{x}\in\mathrm{Fix}(T)$ both are solutions to the VIP (3.2). Then we have

$$\langle (A-V)\tilde{x},\tilde{x}-\hat{x}\rangle\le 0, \tag{3.4}$$

and

$$\langle (A-V)\hat{x},\hat{x}-\tilde{x}\rangle\le 0. \tag{3.5}$$

Adding up (3.4) and (3.5) yields

$$\langle (A-V)\tilde{x}-(A-V)\hat{x},\tilde{x}-\hat{x}\rangle\le 0.$$

The strong monotonicity of $A-V$ implies that $\tilde{x}=\hat{x}$ and the uniqueness is proved.

Next, we prove that $x_t\to\tilde{x}$ as $t\to 0$. Observing $\mathrm{Fix}(T)=\mathrm{Fix}(T_t)$, from (3.1), we write, for given $p\in\mathrm{Fix}(T)$,

$$
\begin{aligned}
x_t-p&=(I-\theta_tA)T_tx_t-(I-\theta_tA)T_tp+\theta_t(t\gamma T_tx_t+(I-t\mu F)Vx_t-Vp)+\theta_t(V-A)p \\
&=(I-\theta_tA)(T_tx_t-T_tp)+\theta_t[(I-t\mu F)Vx_t-(I-t\mu F)Vp+t(\gamma T_tx_t-\mu FVp)]+\theta_t(V-A)p \\
&=(I-\theta_tA)(T_tx_t-T_tp)+\theta_t[(I-t\mu F)Vx_t-(I-t\mu F)Vp+t\gamma(T_tx_t-T_tp) \\
&\quad +t(\gamma I-\mu FV)p]+\theta_t(V-A)p.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\|x_t-p\|^2&=\langle (I-\theta_tA)(T_tx_t-T_tp),x_t-p\rangle+\theta_t[\langle (I-t\mu F)Vx_t-(I-t\mu F)Vp,x_t-p\rangle \\
&\quad +t\gamma\langle T_tx_t-T_tp,x_t-p\rangle+t\langle(\gamma I-\mu FV)p,x_t-p\rangle]+\theta_t\langle (V-A)p,x_t-p\rangle \\
&\le (1-\theta_t\bar{\gamma})\|x_t-p\|^2+\theta_t[(1-t\tau)\|x_t-p\|^2+t\gamma\|x_t-p\|^2 \\
&\quad +t\langle(\gamma I-\mu FV)p,x_t-p\rangle]+\theta_t\langle (V-A)p,x_t-p\rangle \\
&=[1-\theta_t(\bar{\gamma}-1+t(\tau-\gamma))]\|x_t-p\|^2+\theta_t(t\langle(\gamma I-\mu FV)p,x_t-p\rangle+\langle (V-A)p,x_t-p\rangle).
\end{aligned}
$$

Therefore,

$$\|x_t-p\|^2\le\frac{1}{\bar{\gamma}-1+t(\tau-\gamma)}(t\langle(\gamma I-\mu FV)p,x_t-p\rangle+\langle (V-A)p,x_t-p\rangle). \tag{3.6}$$

Since the net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})}$ is bounded (due to Proposition 3.1 (i)), we know that if $\{t_n\}$ is a subsequence in $(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma}\})$ such that $t_n\to 0$ and $x_{t_n}\rightharpoonup x^*$, then from (3.6), we obtain $x_{t_n}\to x^*$. Let us show that $x^*\in \mathrm{Fix}(T)$. To this end, define $S: H\to H$ by $Sx=\lambda x+(1-\lambda)Tx$ for all $x\in H$, for $0\le k\le\lambda<1$. Then it is clear that $\mathrm{Fix}(S)=\mathrm{Fix}(T)$. Moreover, by Lemma 2.5 we know that S is nonexpansive. By the definitions of S and T_{t_n} we get

$$\begin{aligned}\|Sx_{t_n}-x_{t_n}\| &\le \|Sx_{t_n}-T_{t_n}x_{t_n}\|+\|T_{t_n}x_{t_n}-x_{t_n}\|\\ &=(\lambda-\lambda_{t_n})\|x_{t_n}-Tx_{t_n}\|+\|T_{t_n}x_{t_n}-x_{t_n}\|\\ &=\frac{\lambda-\lambda_{t_n}}{1-\lambda_{t_n}}\|x_{t_n}-T_{t_n}x_{t_n}\|+\|T_{t_n}x_{t_n}-x_{t_n}\|\\ &=\frac{1+\lambda-2\lambda_{t_n}}{1-\lambda_{t_n}}\|x_{t_n}-T_{t_n}x_{t_n}\|.\end{aligned}$$

So, by Proposition 3.1 (ii) and $\lambda_{t_n}\to\lambda$ as $t_n\to 0$, we have $\lim_{n\to\infty}(I-S)x_{t_n}=0$. Thus, it follows from Lemma 2.6 that $x^*\in\mathrm{Fix}(S)$. In terms of the definition of S, we obtain $x^*\in\mathrm{Fix}(T)$.

Finally, let us show that x^* is a solution of the VIP (3.2). As a matter of fact, since

$$x_t=(I-\theta_tA)T_tx_t+\theta_t((I-t\mu F)Vx_t+t\gamma T_tx_t),$$

we have

$$x_t-T_tx_t=\theta_t(V-A)T_tx_t+\theta_t(Vx_t-VT_tx_t+t(\gamma T_tx_t-\mu FVx_t)).$$

Since T_t is nonexpansive, $I-T_t$ is monotone. So, from the monotonicity of $I-T_t$, it follows that, for $p\in\mathrm{Fix}(T)=\mathrm{Fix}(T_t)$,

$$\begin{aligned}0&\le\langle (I-T_t)x_t-(I-T_t)p,x_t-p\rangle=\langle (I-T_t)x_t,x_t-p\rangle\\ &=\theta_t\langle (V-A)T_tx_t,x_t-p\rangle+\theta_t\langle Vx_t-VT_tx_t,x_t-p\rangle+\theta_tt\langle(\gamma T_tx_t-\mu FVx_t),x_t-p\rangle\\ &=\theta_t\langle (V-A)x_t,x_t-p\rangle+\theta_t\langle (V-A)T_tx_t-(V-A)x_t,x_t-p\rangle+\theta_t\langle Vx_t-VT_tx_t,x_t-p\rangle\\ &\quad+\theta_tt\langle(\gamma T_tx_t-\mu FVx_t),x_t-p\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\langle (A-V)x_t,x_t-p\rangle&\le\langle (V-A)T_tx_t-(V-A)x_t,x_t-p\rangle+\langle Vx_t-VT_tx_t,x_t-p\rangle\\ &\quad+t\langle(\gamma T_tx_t-\mu FVx_t),x_t-p\rangle\\ &\le\|(V-A)T_tx_t-(V-A)x_t\|\|x_t-p\|+\|Vx_t-VT_tx_t\|\|x_t-p\|\\ &\quad+t\|\gamma T_tx_t-\mu FVx_t\|\|x_t-p\|\\ &\le(1+\|A\|)\|T_tx_t-x_t\|\|x_t-p\|+\|x_t-T_tx_t\|\|x_t-p\|\\ &\quad+t(\gamma\|T_tx_t\|+\mu\|FVx_t\|)\|x_t-p\|\\ &=(2+\|A\|)\|T_tx_t-x_t\|\|x_t-p\|+t\|(\gamma\|T_tx_t\|+\mu\|FVx_t\|)\|x_t-p\|.\end{aligned}\tag{3.7}$$

Now, replacing t in (3.7) with t_n and letting $n\to\infty$, noticing the boundedness of $\{\gamma\|T_{t_n}x_{t_n}\|+\mu\|FVx_{t_n}\|\}$ and the fact that $(T_{t_n}-I)x_{t_n}\to 0$ as $n\to\infty$ (due to Proposition 3.1 (ii)), we obtain

$$\langle (A-V)x^*,x^*-p\rangle\le 0.$$

That is, $x^*\in\mathrm{Fix}(T)$ is a solution of the VIP (3.2); hence $x^*=\tilde{x}$ by uniqueness. In summary, we have proven that each cluster point of $\{x_t\}$ (as $t\to 0$) equals $\tilde{x}$. Consequently, $x_t\to\tilde{x}$ as $t\to 0$.

The VIP (3.2) can be rewritten as

$$\langle (I+V-A)\tilde{x}-\tilde{x},\tilde{x}-p\rangle\ge 0,\quad \forall p\in\mathrm{Fix}(T).$$

Recalling Proposition 2.1 (i), the last inequality is equivalent to the fixed point equation

$$P_{\mathrm{Fix}(T)}(I+V-A)\tilde{x}=\tilde{x}.$$

□

Taking $F = \frac{1}{2}I,\ \mu = 2$ and $\gamma = 1$ in Theorem 3.2, we get the following.

Corollary 3.3. *Let $l = 1,\ 0 \le \gamma < 1$ and $\{x_t\}$ be defined by*

$$x_t = (I - \theta_t A)T_t x_t + \theta_t(Vx_t + t(\gamma T_t x_t - Vx_t)).$$

If $\lim_{t\to 0}\theta_t = 0$, then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of T, which is the unique solution of the VIP (3.2).

Now, we prove the following result in order to establish the strong convergence of the sequence $\{x_n\}$ generated by the composite explicit steepest-descent scheme (3.3).

Theorem 3.4. *Let $l = 1$ and $\{x_n\}$ be the sequence generated by the explicit scheme* (3.3)*, where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following condition:*

(C1) $\{\alpha_n\} \subset [0,1],\ \{\beta_n\} \subset (0,1]$ *and* $\alpha_n \to 0,\ \beta_n \to 0$ *as* $n \to \infty$.

Let LIM *be a Banach limit. Then*

$$\mathrm{LIM}_n\langle (A - V)\tilde{x}, \tilde{x} - x_n\rangle \le 0,$$

where $\tilde{x} = \lim_{t\to 0^+} x_t$ with x_t being defined by

$$x_t = (I - \theta_t A)Sx_t + \theta_t(Vx_t - t(\mu FVx_t - \gamma Sx_t)), \tag{3.8}$$

and $S : H \to H$ is defined by $Sx = \lambda x + (1-\lambda)Tx$ for $0 \le k \le \lambda < 1$.

Proof. First, note that from the condition (C1), without loss of generality, we may assume that $0 < \beta_n \le \|A\|^{-1}$ for all $n \ge 0$.

Let $\{x_t\}$ be the net generated by (3.8). Since S is a nonexpansive mapping on H, by Theorem 3.2 with $T_t = S$, there exists $\lim_{t\to 0} x_t \in \mathrm{Fix}(S) = \mathrm{Fix}(T)$. Denote it by $\tilde{x}$. Moreover, $\tilde{x}$ is the unique solution of the VIP (3.2). From Proposition 3.1 (i) with $T_t = S$, we know that $\{x_t\}$ is bounded and so are the nets $\{Sx_t\}$ and $\{FVx_t\}$.

Now, let us show that $\{x_n\}$ is bounded. To this end, take $p \in \mathrm{Fix}(T) = \mathrm{Fix}(T_n)$. Simple calculations show that

$$\begin{aligned} x_{n+1} - p &= (I - \beta_n A)T_n x_n + \beta_n y_n - p \\ &= (I - \beta_n A)T_n x_n + \beta_n(\alpha_n \gamma T_n x_n + (I - \alpha_n \mu F)Vx_n) - p \\ &= (I - \beta_n A)T_n x_n - (I - \beta_n A)T_n p + \beta_n(\alpha_n\gamma T_n x_n + (I - \alpha_n\mu F)Vx_n - Vp) + \beta_n(V - A)p \\ &= (I - \beta_n A)(T_n x_n - T_n p) + \beta_n[(I - \alpha_n\mu F)Vx_n - (I - \alpha_n\mu F)Vp + \alpha_n(\gamma T_n x_n - \mu FVp)] \\ &\quad + \beta_n(V - A)p \\ &= (I - \beta_n A)(T_n x_n - T_n p) + \beta_n[(I - \alpha_n\mu F)Vx_n - (I - \alpha_n\mu F)Vp + \alpha_n\gamma(T_n x_n - T_n p) \\ &\quad + \alpha_n(\gamma I - \mu FV)p] + \beta_n(V - A)p, \end{aligned}$$

which together with Lemmas 2.8 and 2.10, implies that

$$\begin{aligned} \|x_{n+1} - p\| &\le \|(I - \beta_n A)(T_n x_n - T_n p)\| + \beta_n[\|(I - \alpha_n\mu F)Vx_n - (I - \alpha_n\mu F)Vp\| + \alpha_n\gamma\|T_n x_n - T_n p\| \\ &\quad + \alpha_n\|(\gamma I - \mu FV)p\|] + \beta_n\|(V - A)p\| \\ &\le (1 - \beta_n\bar{\gamma})\|T_n x_n - T_n p\| + \beta_n[(1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\gamma\|x_n - p\| + \alpha_n\|(\gamma I - \mu FV)p\|] \\ &\quad + \beta_n\|(V - A)p\| \\ &\le (1 - \beta_n\bar{\gamma})\|x_n - p\| + \beta_n[(1 - \alpha_n(\tau - \gamma))\|x_n - p\| + \alpha_n\|(\gamma I - \mu FV)p\|] + \beta_n\|(V - A)p\| \\ &= [1 - \beta_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))]\|x_n - p\| + \beta_n[\|(V - A)p\| + \alpha_n\|(\gamma I - \mu FV)p\|] \\ &= [1 - \beta_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))]\|x_n - p\| \end{aligned}$$

$$
\begin{aligned}
&+ \beta_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))\frac{\|(V - A)p\| + \alpha_n\|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1 + \alpha_n(\tau - \gamma)} \\
&\le \max\{\|x_n - p\|, \frac{\|(V - A)p\| + \alpha_n\|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1 + \alpha_n(\tau - \gamma)}\} \\
&\le \max\{\|x_n - p\|, \frac{\|(V - A)p\| + \alpha_n\|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1}\} \\
&\le \max\{\|x_n - p\|, \frac{\|(V - A)p\| + \|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1}\}.
\end{aligned}
$$

By the induction

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|(V - A)p\| + \|(\gamma I - \mu FV)p\|}{\bar{\gamma} - 1}\}, \quad \forall n \ge 0.$$

This implies that $\{x_n\}$ is bounded and so are $\{Tx_n\}, \{T_n x_n\}, \{FVx_n\}, \{Vx_n\}$, and $\{y_n\}$. Thus, utilizing the control condition (C1), we get

$$\|x_{n+1} - T_n x_n\| = \beta_n\|y_n - AT_n x_n\| \to 0 \quad \text{as } n \to \infty,$$

and

$$
\begin{aligned}
\|Sx_t - x_{n+1}\| &\le \|Sx_t - Sx_n\| + \|Sx_n - T_n x_n\| + \|T_n x_n - x_{n+1}\| \\
&\le \|x_t - x_n\| + |\lambda - \lambda_n|\|x_n - Tx_n\| + \|T_n x_n - x_{n+1}\| \\
&= \|x_t - x_n\| + e_n,
\end{aligned}
\tag{3.9}
$$

where $e_n = |\lambda - \lambda_n|\|x_n - Tx_n\| + \|T_n x_n - x_{n+1}\| \to 0$ as $n \to \infty$. Also observing that A is strongly positive, we have

$$\langle Ax_t - Ax_n, x_t - x_n\rangle = \langle A(x_t - x_n), x_t - x_n\rangle \ge \bar{\gamma}\|x_t - x_n\|^2. \tag{3.10}$$

Furthermore, by (3.8), we have

$$
\begin{aligned}
x_t - x_{n+1} &= (I - \theta_t A)Sx_t + \theta_t(t\gamma Sx_t + (I - t\mu F)Vx_t) - x_{n+1} \\
&= (I - \theta_t A)Sx_t - (I - \theta_t A)x_{n+1} + \theta_t(t\gamma Sx_t + (I - t\mu F)Vx_t - Ax_{n+1}) \\
&= (I - \theta_t A)Sx_t - (I - \theta_t A)x_{n+1} + \theta_t[(I - t\mu F)Vx_t - (I - t\mu F)Vx_{n+1} \\
&\quad + t(\gamma Sx_t - \mu FVx_{n+1}) + (V - A)x_{n+1}].
\end{aligned}
$$

Applying Lemma 2.2, we have

$$
\begin{aligned}
\|x_t - x_{n+1}\|^2 &\le \|(I - \theta_t A)Sx_t - (I - \theta_t A)x_{n+1}\|^2 + 2\theta_t\langle(I - t\mu F)Vx_t \\
&\quad - (I - t\mu F)Vx_{n+1}, x_t - x_{n+1}\rangle \\
&\quad + 2\theta_t t\langle\gamma Sx_t - \mu FVx_{n+1}, x_t - x_{n+1}\rangle + 2\theta_t\langle(V - A)x_{n+1}, x_t - x_{n+1}\rangle \\
&\le (1 - \theta_t\bar{\gamma})^2\|Sx_t - x_{n+1}\|^2 + 2\theta_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle \\
&\quad - 2\theta_t t\mu\langle FVx_t - FVx_{n+1}, x_t - x_{n+1}\rangle + 2\theta_t t\|\gamma Sx_t - \mu FVx_{n+1}\|\|x_t - x_{n+1}\| \\
&\quad + 2\theta_t\langle(V - A)x_{n+1}, x_t - x_{n+1}\rangle \\
&\le (1 - \theta_t\bar{\gamma})^2\|Sx_t - x_{n+1}\|^2 + 2\theta_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle \\
&\quad + 2\theta_t t\mu\|FVx_t - FVx_{n+1}\|\|x_t - x_{n+1}\| + 2\theta_t t\|\gamma Sx_t - \mu FVx_{n+1}\|\|x_t - x_{n+1}\| \\
&\quad + 2\theta_t\langle(V - A)x_{n+1}, x_t - x_{n+1}\rangle \\
&= (1 - \theta_t\bar{\gamma})^2\|Sx_t - x_{n+1}\|^2 + 2\theta_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle \\
&\quad + 2\theta_t t(\mu\|FVx_t - FVx_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\
&\quad + 2\theta_t\langle(V - A)x_{n+1}, x_t - x_{n+1}\rangle.
\end{aligned}
\tag{3.11}
$$

Utilizing (3.9) and (3.10) in (3.11), we obtain

$$\begin{aligned}
&\|x_t - x_{n+1}\|^2 \\
&\quad \le (1-\theta_t\bar{\gamma})^2\|Sx_t - x_{n+1}\|^2 + 2\theta_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle \\
&\qquad + 2\theta_t t(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\
&\qquad + 2\theta_t\langle (V-A)x_{n+1}, x_t - x_{n+1}\rangle \\
&\quad \le (1-\theta_t\bar{\gamma})^2(\|x_t - x_n\| + e_n)^2 + 2\theta_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle \\
&\qquad + 2\theta_t t(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| + 2\theta_t\langle (V-A)x_{n+1}, x_t - x_{n+1}\rangle \\
&\quad = (\theta_t^2\bar{\gamma} - 2\theta_t)\bar{\gamma}\|x_t - x_n\|^2 + \|x_t - x_n\|^2 + (1-\theta_t\bar{\gamma})^2(2\|x_t - x_n\|e_n + e_n^2) \\
&\qquad + 2\theta_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle + 2\theta_t t(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\
&\qquad + 2\theta_t\langle (V-A)x_{n+1}, x_t - x_{n+1}\rangle \\
&\quad \le (\theta_t^2\bar{\gamma} - 2\theta_t)\langle Ax_t - Ax_n, x_t - x_n\rangle + \|x_t - x_n\|^2 + (1-\theta_t\bar{\gamma})^2(2\|x_t - x_n\|e_n + e_n^2) \\
&\qquad + 2\theta_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle + 2\theta_t t(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\
&\qquad + 2\theta_t\langle (V-A)x_{n+1}, x_t - x_{n+1}\rangle \\
&\quad = \theta_t^2\bar{\gamma}\langle Ax_t - Ax_n, x_t - x_n\rangle + \|x_t - x_n\|^2 + (1-\theta_t\bar{\gamma})^2(2\|x_t - x_n\|e_n + e_n^2) \\
&\qquad + 2\theta_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle + 2\theta_t t(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\
&\qquad + 2\theta_t[\langle (V-A)x_{n+1}, x_t - x_{n+1}\rangle - \langle Ax_t - Ax_n, x_t - x_n\rangle] \\
&\quad = \theta_t^2\bar{\gamma}\langle A(x_t - x_n), x_t - x_n\rangle + \|x_t - x_n\|^2 + (1-\theta_t\bar{\gamma})^2(2\|x_t - x_n\|e_n + e_n^2) \\
&\qquad + 2\theta_t\langle Vx_t - Vx_{n+1}, x_t - x_{n+1}\rangle + 2\theta_t t(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\
&\qquad + 2\theta_t[\langle (V-A)x_t, x_t - x_{n+1}\rangle + \langle (V-A)x_{n+1} - (V-A)x_t, x_t - x_{n+1}\rangle \\
&\qquad - \langle A(x_t - x_n), x_t - x_n\rangle] \\
&\quad = \theta_t^2\bar{\gamma}\langle A(x_t - x_n), x_t - x_n\rangle + \|x_t - x_n\|^2 + (1-\theta_t\bar{\gamma})^2(2\|x_t - x_n\|e_n + e_n^2) \\
&\qquad + 2\theta_t t(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\
&\qquad + 2\theta_t[\langle (V-A)x_t, x_t - x_{n+1}\rangle + \langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle - \langle A(x_t - x_n), x_t - x_n\rangle].
\end{aligned} \tag{3.12}$$

Applying the Banach limit LIM to (3.12), together with $\lim_{n\to\infty} e_n = 0$, we have

$$\begin{aligned}
\mathrm{LIM}_n\|x_t - x_{n+1}\|^2 \le{}& \theta_t^2\bar{\gamma}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle + \mathrm{LIM}_n\|x_t - x_n\|^2 \\
&+ 2\theta_t t\mathrm{LIM}_n(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\
&+ 2\theta_t[\mathrm{LIM}_n\langle (V-A)x_t, x_t - x_{n+1}\rangle + \mathrm{LIM}_n\langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle \\
&- \mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle].
\end{aligned} \tag{3.13}$$

Using the property $\mathrm{LIM}_n a_n = \mathrm{LIM}_n a_{n+1}$ of the Banach limit in (3.13), we obtain

$$\begin{aligned}
\mathrm{LIM}_n\langle (A-V)x_t, x_t - x_n\rangle &= \mathrm{LIM}_n\langle (A-V)x_t, x_t - x_{n+1}\rangle \\
&\le \frac{\theta_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle \\
&\quad + \frac{1}{2\theta_t}[\mathrm{LIM}_n\|x_t - x_n\|^2 - \mathrm{LIM}_n\|x_t - x_{n+1}\|^2] \\
&\quad + t\mathrm{LIM}_n(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\| \\
&\quad + \mathrm{LIM}_n\langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle - \mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle \\
&= \frac{\theta_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle \\
&\quad + t\mathrm{LIM}_n(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma Sx_t - \mu FVx_{n+1}\|)\|x_t - x_{n+1}\|.
\end{aligned} \tag{3.14}$$

Since

$$\theta_t\langle A(x_t - x_n), x_t - x_n\rangle \le \theta_t\|A\|\|x_t - x_n\|^2 \le \theta_t\|A\|K^2 \to 0 \quad \text{as } t \to 0, \tag{3.15}$$

where $\|x_t - x_n\| + \|\gamma S x_t - \mu F V x_n\| \le K$,

$$t\|x_t - x_{n+1}\|^2 \to 0 \quad \text{and} \quad t\|\gamma S x_t - \mu F V x_{n+1}\|\|x_t - x_{n+1}\| \to 0 \quad \text{as } t \to 0, \tag{3.16}$$

we conclude from (3.14)-(3.16) that

$$\begin{aligned}\text{LIM}_n\langle (A-V)\tilde{x}, \tilde{x} - x_n\rangle &\le \limsup_{t\to 0}\text{LIM}_n\langle (A-V)x_t, x_t - x_n\rangle \\ &\le \limsup_{t\to 0}\frac{\theta_t\bar{\gamma}}{2}\text{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle \\ &\quad + \limsup_{t\to 0} t\text{LIM}_n(\mu\kappa\|x_t - x_{n+1}\| + \|\gamma S x_t - \mu F V x_{n+1}\|)\|x_t - x_{n+1}\| \\ &= 0.\end{aligned}$$

This completes the proof. □

Now, using Theorem 3.4, we establish the strong convergence of the sequence $\{x_n\}$ generated by the general composite explicit steepest-descent scheme (3.3) to a fixed point $\tilde{x}$ of T (i.e., $\tilde{x} \in \text{Fix}(T)$), which is also the unique solution of the VIP (3.2).

Theorem 3.5. *Let $l = 1$ and $\{x_n\}$ be the sequence generated by the explicit scheme* (3.3)*, where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:*

(C1) $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$ *and* $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$*;*

(C2) $\sum_{n=0}^{\infty} \beta_n = \infty$.

If $\{x_n\}$ is weakly asymptotically regular (i.e., $x_{n+1} - x_n \rightharpoonup 0$), then $\{x_n\}$ converges strongly to $\tilde{x} \in \text{Fix}(T)$, which is the unique solution of the VIP (3.2).

Proof. First, note that from the condition (C1), without loss of generality, we may assume that $0 \le \alpha_n < \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma}\}$ and $0 < \beta_n \le \|A\|^{-1}$ for all $n \ge 0$. In this case, we obtain $0 < \beta_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma)) < 1$ for all $n \ge 0$.

Let x_t be defined by (3.8), that is,

$$x_t = (I - \theta_t A) S x_t + \theta_t(V x_t - t(\mu F V x_t - \gamma S x_t)),$$

for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma}\})$, where $Sx = \lambda x + (1-\lambda)Tx$ for $0 \le k \le \lambda < 1$, and $\lim_{t\to 0} x_t := \tilde{x} \in \text{Fix}(S) = \text{Fix}(T)$ (due to Theorem 3.1). Then $\tilde{x}$ is the unique solution of the VIP (3.2).

We divide the rest of the proof into several steps.

Step 1. We see that

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|(\gamma I - \mu F V)p\| + \|(V - A)p\|}{\bar{\gamma} - 1}\}, \quad \forall n \ge 0$$

for all $p \in \text{Fix}(T)$ as in the proof of Theorem 3.4. Hence $\{x_n\}$ is bounded and so are $\{T x_n\}, \{T_n x_n\}$, $\{F V x_n\}, \{V x_n\}$, and $\{y_n\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle (V - A)\tilde{x}, x_n - \tilde{x}\rangle \le 0$. To this end, put

$$a_n := \langle (A - V)\tilde{x}, \tilde{x} - x_n\rangle, \quad \forall n \ge 0.$$

Then, by Theorem 3.4 we get $\text{LIM}_n a_n \le 0$ for any Banach limit LIM. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \limsup_{j\to\infty}(a_{n_j+1} - a_{n_j})$$

and $x_{n_j} \rightharpoonup v \in H$. This implies that $x_{n_j+1} \rightharpoonup v$ since $\{x_n\}$ is weakly asymptotically regular. Therefore, we have

$$w - \lim_{j\to\infty}(\tilde{x} - x_{n_j+1}) = w - \lim_{j\to\infty}(\tilde{x} - x_{n_j}) = (\tilde{x} - v),$$

and so

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}\langle (A - V)\tilde{x}, (\tilde{x} - x_{n_j+1}) - (\tilde{x} - x_{n_j})\rangle = 0.$$

Then, by Lemma 2.11, we obtain $\limsup_{n\to\infty} a_n \le 0$, that is,

$$\limsup_{n\to\infty}\langle (V - A)\tilde{x}, x_n - \tilde{x}\rangle = \limsup_{n\to\infty}\langle (A - V)\tilde{x}, \tilde{x} - x_n\rangle \le 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. Indeed, by using (3.3) and $T_n\tilde{x} = \tilde{x}$, we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \|(I - \beta_n A)(T_n x_n - T_n\tilde{x}) + \beta_n[(I - \alpha_n\mu F)Vx_n - (I - \alpha_n\mu F)V\tilde{x} + \alpha_n\gamma(T_n x_n - T_n\tilde{x}) \\
&\quad + \alpha_n(\gamma I - \mu FV)\tilde{x}] + \beta_n(V - A)\tilde{x}\|^2 \\
&= \|(I - \beta_n A)(T_n x_n - T_n\tilde{x}) + \beta_n[(I - \alpha_n\mu F)Vx_n - (I - \alpha_n\mu F)V\tilde{x} + \alpha_n\gamma(T_n x_n - T_n\tilde{x})] \\
&\quad + \beta_n[\alpha_n(\gamma I - \mu FV)\tilde{x} + (V - A)\tilde{x}]\|^2 \\
&\le \|(I - \beta_n A)(T_n x_n - T_n\tilde{x}) + \beta_n[(I - \alpha_n\mu F)Vx_n - (I - \alpha_n\mu F)V\tilde{x} + \alpha_n\gamma(T_n x_n - T_n\tilde{x})]\|^2 \\
&\quad + 2\beta_n[\alpha_n\langle(\gamma I - \mu FV)\tilde{x}, x_{n+1} - \tilde{x}\rangle + \langle (V - A)\tilde{x}, x_{n+1} - \tilde{x}\rangle] \\
&\le [\|(I - \beta_n A)(T_n x_n - T_n\tilde{x})\| + \beta_n(\|(I - \alpha_n\mu F)Vx_n - (I - \alpha_n\mu F)V\tilde{x}\| \\
&\quad + \alpha_n\gamma\|T_n x_n - T_n\tilde{x}\|)]^2 + 2\beta_n[\alpha_n\|(\gamma I - \mu FV)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle] \\
&\le [(1 - \beta_n\bar{\gamma})\|T_n x_n - T_n\tilde{x}\| + \beta_n((1 - \alpha_n\tau)\|x_n - \tilde{x}\| + \alpha_n\gamma\|T_n x_n - T_n\tilde{x}\|)]^2 \\
&\quad + 2\beta_n[\alpha_n\|(\gamma I - \mu FV)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle] \\
&\le [(1 - \beta_n\bar{\gamma})\|x_n - \tilde{x}\| + \beta_n((1 - \alpha_n\tau)\|x_n - \tilde{x}\| + \alpha_n\gamma\|x_n - \tilde{x}\|)]^2 \\
&\quad + 2\beta_n[\alpha_n\|(\gamma I - \mu FV)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle] \\
&= [1 - \beta_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))]^2\|x_n - \tilde{x}\|^2 \\
&\quad + 2\beta_n[\alpha_n\|(\gamma I - \mu FV)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle] \\
&\le [1 - \beta_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))]\|x_n - \tilde{x}\|^2 \\
&\quad + 2\beta_n[\alpha_n\|(\gamma I - \mu FV)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle] \\
&\le [1 - \beta_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))]\|x_n - \tilde{x}\|^2 \\
&\quad + 2\beta_n[\alpha_n\|(\gamma I - \mu FV)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle] \\
&= [1 - \beta_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))]\|x_n - \tilde{x}\|^2 \\
&\quad + \beta_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma)) \cdot \frac{2}{\bar{\gamma} - 1 + \alpha_n(\tau - \gamma)}[\alpha_n\|(\gamma I - \mu FV)\tilde{x}\|\|x_{n+1} - \tilde{x}\| \\
&\quad + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle] \\
&= (1 - \omega_n)\|x_n - \tilde{x}\|^2 + \omega_n\delta_n,
\end{aligned}$$

where $\omega_n = \beta_n(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma))$ and

$$\delta_n = \frac{2}{\bar{\gamma} - 1 + \alpha_n(\tau - \gamma)}[\alpha_n\|(\gamma I - \mu FV)\tilde{x}\|\|x_{n+1} - \tilde{x}\| + \langle (A - V)\tilde{x}, \tilde{x} - x_{n+1}\rangle].$$

It can be readily seen from Step 2 and conditions (C1) and (C2) that $\omega_n \to 0$, $\sum_{n=0}^{\infty}\omega_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. By Lemma 2.9 with $r_n = 0$, we conclude that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. This completes the proof. □

Corollary 3.6. *Let $l = 1$ and $\{x_n\}$ be the sequence generated by the explicit scheme* (3.3)*. Assume that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions* (C1) *and* (C2) *in Theorem* 3.5*. If $\{x_n\}$ is asymptotically regular (i.e., $x_{n+1} - x_n \to 0$), then $\{x_n\}$ converges strongly to $\tilde{x} \in \mathrm{Fix}(T)$, which is the unique solution of the VIP* (3.2)*.*

Putting $\mu = 2$, $F = \frac{1}{2}I$ and taking $\gamma \in [0, 1)$ (due to $\tau = 1$) in Theorem 3.5, we obtain the following.

Corollary 3.7. *Let $l = 1$ and $\{x_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} y_n = \alpha_n \gamma T_n x_n + (1 - \alpha_n) V x_n, \\ x_{n+1} = (I - \beta_n A) T_n x_n + \beta_n y_n, \quad \forall n \geq 0. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (C1) *and* (C2) *in Theorem* 3.5*. If $\{x_n\}$ is weakly asymptotically regular (i.e., $x_{n+1} - x_n \rightharpoonup 0$), then $\{x_n\}$ converges strongly to $\tilde{x} \in \mathrm{Fix}(T)$, which is the unique solution of the VIP* (3.2)*.*

Putting $\alpha_n = 0$ for all $n \geq 0$ in Corollary 3.7, we get the following.

Corollary 3.8. *Let $l = 1$ and $\{x_n\}$ be generated by the following iterative scheme:*

$$x_{n+1} = (I - \beta_n A) T_n x_n + \beta_n V x_n, \quad \forall n \geq 0.$$

Assume that the sequence $\{\beta_n\}$ satisfies the conditions (C1) *and* (C2) *in Theorem* 3.5 *with $\alpha_n = 0$ for all $n \geq 0$. If $\{x_n\}$ is weakly asymptotically regular (i.e., $x_{n+1} - x_n \rightharpoonup 0$), then $\{x_n\}$ converges strongly to $\tilde{x} \in \mathrm{Fix}(T)$, which is the unique solution of the VIP* (3.2)*.*

Remark 3.9. If $\{\alpha_n\}, \{\beta_n\}$ in Corollary 3.6 and $\{\lambda_n\}$ in T_n satisfy conditions (C2) and

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; or

(C4) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\lim_{n\to\infty} \frac{\beta_n}{\beta_{n+1}} = 1$ or, equivalently, $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}} = 0$, and $\lim_{n\to\infty} \frac{\beta_n - \beta_{n+1}}{\beta_{n+1}} = 0$; or,

(C5) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $|\beta_{n+1} - \beta_n| \leq o(\beta_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (the perturbed control condition);

(C6) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

then the sequence $\{x_n\}$ generated by (3.3) is asymptotically regular.

Now we give only the proof in the case when $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\}$ satisfy the conditions (C2), (C5) and (C6). By Step 1 in the proof of Theorem 3.5, there exists a constant $M > 0$ such that

$$\|x_n - Tx_n\| \leq M, \ \mu\|FVx_n\| + \gamma\|T_n x_n\| \leq M \text{ and } \|A\|\|T_n x_n\| + \|y_n\| \leq M,$$

for all $n \geq 0$. Next, we notice that

$$\begin{aligned} \|T_n x_n - T_{n-1} x_{n-1}\| &\leq \|T_n x_n - T_n x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|x_{n-1} - Tx_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M. \end{aligned}$$

So we obtain, for all $n \geq 0$,

$$\begin{aligned} \|y_n - y_{n-1}\| = \|&\gamma\alpha_n(T_n x_n - T_{n-1} x_{n-1}) + (\alpha_n - \alpha_{n-1})(\gamma T_{n-1} x_{n-1} - \mu FV x_{n-1}) \\ &+ (I - \alpha_n \mu F) V x_n - (I - \alpha_n \mu F) V x_{n-1}\| \end{aligned}$$

$$\begin{aligned}
&\leq \gamma\alpha_n\|T_nx_n - T_{n-1}x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\gamma T_{n-1}x_{n-1} - \mu FVx_{n-1}\| \\
&\quad + \|(I-\alpha_n\mu F)Vx_n - (I-\alpha_n\mu F)Vx_{n-1}\| \\
&\leq \gamma\alpha_n(\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M) + |\alpha_n - \alpha_{n-1}|(\gamma\|T_{n-1}x_{n-1}\| + \mu\|FVx_{n-1}\|) \\
&\quad + (1-\alpha_n\tau)\|x_n - x_{n-1}\| \\
&\leq (1-\alpha_n(\tau-\gamma))(\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M) + |\alpha_n - \alpha_{n-1}|M,
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(I-\beta_nA)T_nx_n + \beta_ny_n - (I-\beta_{n-1}A)T_{n-1}x_{n-1} - \beta_{n-1}y_{n-1}\| \\
&\leq \|(I-\beta_nA)(T_nx_n - T_{n-1}x_{n-1})\| \\
&\quad + |\beta_n - \beta_{n-1}|\|A\|\|T_{n-1}x_{n-1}\| + \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|y_{n-1}\| \\
&\leq (1-\beta_n\bar{\gamma})\|T_nx_n - T_{n-1}x_{n-1}\| + \beta_n[(1-\alpha_n(\tau-\gamma))(\|x_n - x_{n-1}\| \\
&\quad + |\lambda_n - \lambda_{n-1}|M) + |\alpha_n - \alpha_{n-1}|M] + |\beta_n - \beta_{n-1}|(\|A\|\|T_{n-1}x_{n-1}\| + \|y_{n-1}\|) \\
&\leq (1-\beta_n\bar{\gamma})(\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M) + \beta_n[(1-\alpha_n(\tau-\gamma))(\|x_n - x_{n-1}\| \\
&\quad + |\lambda_n - \lambda_{n-1}|M) + |\alpha_n - \alpha_{n-1}|M] + |\beta_n - \beta_{n-1}|M \\
&\leq (1-\beta_n\bar{\gamma})\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M \\
&\quad + \beta_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M + |\alpha_n - \alpha_{n-1}|M + |\beta_n - \beta_{n-1}|M \\
&= (1-\beta_n(\bar{\gamma}-1))\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M + 2|\lambda_n - \lambda_{n-1}|M + |\alpha_n - \alpha_{n-1}|M \\
&\leq (1-\beta_n(\bar{\gamma}-1))\|x_n - x_{n-1}\| + (o(\beta_n) + \sigma_{n-1})M + |\alpha_n - \alpha_{n-1}|M + 2|\lambda_n - \lambda_{n-1}|M.
\end{aligned} \tag{3.17}$$

By taking $a_{n+1} = \|x_{n+1} - x_n\|$, $\omega_n = \beta_n(\bar{\gamma}-1)$, $\omega_n\delta_n = Mo(\beta_n)$, and $r_n = (|\alpha_n - \alpha_{n-1}| + \sigma_{n-1} + 2|\lambda_n - \lambda_{n-1}|)M$, from (3.17) we have

$$a_{n+1} \leq (1-\omega_n)a_n + \omega_n\delta_n + r_n.$$

Consequently, utilizing the conditions (C2), (C5), (C6) and Lemma 2.9, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

In view of this observation, we have the following.

Corollary 3.10. *Let $l = 1$ and $\{x_n\}$ be the sequence generated by the explicit scheme* (3.3)*, where the sequences $\{\alpha_n\}, \{\beta_n\}$, and $\{\lambda_n\}$ satisfy the conditions* (C1), (C2), (C5) *and* (C6) *(or the conditions* (C1), (C2), (C3) *and* (C6)*, or the conditions* (C1), (C2), (C4) *and* (C6)*). Then $\{x_n\}$ converges strongly to $\tilde{x} \in \mathrm{Fix}(T)$, which is the unique solution of the VIP* (3.2).

4. Concluding remarks

We introduced and analyzed one general composite implicit steepest-descent scheme and another general composite explicit steepest-descent scheme for solving the hierarchical fixed point problem (3.2) of a k-strictly pseudocontractive mapping $T : H \to H$ in a real Hilbert space H by virtue of the general composite implicit and explicit schemes for a nonexpansive mapping $T : H \to H$ (see [2]) and the general composite implicit and explicit ones for a strict pseudocontraction $T : H \to H$ (see [8]). Our Theorems 3.2, 3.4, 3.5 and Corollaries 3.3, 3.10 supplement and develop Theorems 3.1 and 3.2 of [2] and Theorems 3.1-3.3 and Corollary 3.5 of [8] in the following aspects.

(i) Ceng et al.'s general composite implicit scheme for a nonexpansive mapping $T : H \to H$ (see (3.1) in [2]) and Jung's general composite implicit one for a strict pseudocontraction $T : H \to H$ (see (3.1) in [8]) are extended to develop the general composite implicit scheme (3.1) for hierarchical fixed point problem (3.2) of a strict pseudocontraction $T : H \to H$. Moreover, Ceng et al.'s general composite explicit scheme for a nonexpansive mapping $T : H \to H$ (see (3.5) in [2]) and Jung's general composite explicit one for a strict pseudocontraction (see (3.3) in [8]) are extended to develop the general composite explicit one (3.3) for hierarchical fixed point problem (3.2) of a strict pseudocontraction $T : H \to H$.

(ii) Our general composite implicit scheme (3.1) is very different both from the general composite implicit one (3.1) in [2] and from the general composite implicit one (3.1) in [8] because the general composite implicit one $x_t = (I - \theta_t A)Tx_t + \theta_t[Tx_t - t(\mu FTx_t - \gamma f(x_t))]$ (see (3.1) in [2]) and the general composite implicit one $x_t = (I - \theta_t A)T_t x_t + \theta_t[t\gamma V x_t + (I - t\mu F)T_t x_t]$ (see (3.1) in [8]) are replaced by our general composite implicit one $x_t = (I - \theta_t A)T_t x_t + \theta_t[V x_t - t(\mu F V x_t - \gamma T_t x_t)]$. In the meantime, our general composite explicit scheme (3.3) is very different both from the general composite explicit one (3.5) in [2] and from the general composite explicit one (3.3) in [8] because the first iterative step $y_n = (I - \alpha_n \mu F)Tx_n + \alpha_n \gamma f(x_n)$ of (3.5) in [2] and the first iterative one $y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F)T_n x_n$ of (3.3) in [8] are replaced by the first iterative step $y_n = \alpha_n \gamma T_n x_n + (I - \alpha_n \mu F)V x_n$ in our general composite explicit scheme (3.3).

(iii) The hierarchical fixed point problem (3.2) of a strict pseudocontraction $T : H \to H$ in our Theorems 3.2, 3.4, 3.5 and Corollaries 3.3, 3.10 is more general and more subtle than the hierarchical fixed point one of a nonexpansive mapping $T : H \to H$ in [2, Theorems 3.1 and 3.2] and the hierarchical fixed point problem of a strictly pseudocontractive mapping $T : H \to H$ in [8, Theorems 3.1-3.3 and Corollary 3.5]. It is worth pointing out that the mapping $A - I$ in the hierarchical fixed point problems in [2, Theorems 3.1 and 3.2] and [8, Theorems 3.1-3.3 and Corollary 3.5] is extended to the mapping $A - V$ in our hierarchical fixed point problem (3.2), where $V : H \to H$ is a nonexpansive mapping.

(iv) The range $0 < \gamma\alpha < \tau = \mu(\eta - \frac{\mu\kappa^2}{2})$ in [2] and the one $0 < \gamma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$ in [8] are relaxed to the case of range $0 < \gamma < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$ with $l = 1$. Moreover, the range $0 < t < \min\{1, \frac{2-\bar{\gamma}}{\tau - \gamma\alpha}\}$ with $\alpha \in (0, 1)$ in [2, Theorems 3.1] and the one $0 < t < \min\{1, \frac{2-\bar{\gamma}}{\tau - \gamma l}\}$ in [8, Theorem 3.1] are relaxed to the case of range $0 < t < \min\{1, \frac{2-\bar{\gamma}}{\tau - \gamma}\}$ with $l = 1$.

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