Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 6294–6301
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On modified degenerate Changhee polynomials and numbers

Jongkyum Kwon[a], Jin-Woo Park[b,*]

[a]*Department of Mathematics Education and RINS, Gyeongsang National University, JinJu, 52828, Republic of Korea.*
[b]*Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.*

Communicated by Y. J. Cho

Abstract

The Changhee polynomials and numbers are introduced in [D. S. Kim, T. Kim, J.-J. Seo, Adv. Studies Theor. Phys., **7** (2013), 993–1003], and some interesting identities and properties of these polynomials are found by many researcher. In this paper, we consider the modified degenerate Changhee polynomials and derive some new and interesting identities and properties of those polynomials.

Keywords: p-adic invariant integral on $\mathbb{Z}_p$, degenerate Changhee polynomials, modified degenerate Changhee polynomials.
2010 MSC: 05A10, 11B68, 11S80.

1. Introduction and preliminaries

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is defined normally as $|p|_p = \frac{1}{p}$.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [2–4, 7, 7, 9, 10, 12, 13, 16]).} \tag{1.1}$$

*Corresponding author
Email addresses: mathkjk26@hanmail.net (Jongkyum Kwon), a0417001@knu.ac.kr (Jin-Woo Park)

Received 2016-10-29

If we put $f_n(x) = f(x+n)$, then, by (1.1), we can derive the following very useful integral identity

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{1.2}$$

In particular, if $n = 1$, then

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{1.3}$$

The Stirling numbers of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l \ (x \geq 0),$$

and the Stirling numbers of the second kind is defined by the generating function to be

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!},$$

(see [1, 17]). Note that

$$(\log(x+1))^n = n! \sum_{l=n}^{\infty} S_1(l,n) \frac{x^l}{l!}, \ (n \geq 0),$$

(see [1, 17]).

As is well-known, Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \ \text{(see [4–7, 9, 10, 12, 13, 16])}.$$

In the special case, $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the Euler numbers of order r.

From (1.1), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!} &= \left(\frac{2}{e^t+1}\right)^r e^{xt} \\ &= e^{xt} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r), \end{aligned} \tag{1.4}$$

and by (1.4), we have

$$E_n^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r), \ (n \geq 0), \tag{1.5}$$

(see [4–6, 8–10, 12, 13, 16]).

In [10], authors defined the Changhee polynomials as follows

$$\sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!} = \frac{2}{2+t}(1+t)^x,$$

and, in [13], authors defined the *modified degenerate Euler of order* r polynomials as follows:

$$\sum_{n=0}^{\infty} \xi_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} = \left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}\right)^r (1+\lambda)^{\frac{t}{\lambda}x}.$$

Recently, Changhee numbers and polynomials have been introduced by Kim et al. in [10], and by many mathematicians, which were generalized and obtained many new and interesting properties (see [5, 7, 11, 12, 14–16]). In this paper, we consider the modified degenerate Changhee polynomials and numbers by using the p-adic invariant integral, and derive some new and interesting identities and properties of those polynomials.

2. Modified degenerate Changhee polynomials and numbers

From now on, we assume that $t \in \mathbb{C}$ with $|t|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$.

The modified degenerate Changhee polynomials are defined by the generating function to be

$$\frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} = \sum_{n=0}^{\infty} MCh_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.1}$$

In the special case, $x = 0$, $MCh_{n,\lambda} = MCh_{n,\lambda}(0)$ are called modified degenerate Changhee numbers.

Note that

$$\begin{aligned}\lim_{\lambda\to 0}\frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} &= \frac{2}{1+t}(1+t)^x \\ &= \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}.\end{aligned}$$

Since

$$\begin{aligned}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)} &= e^{\log(1+\lambda)\frac{x+y}{\lambda}\log(1+t)} \\ &= \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n (x+y)^n \left(\log(1+t)\right)^n \frac{1}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n (x+y)^n \frac{1}{n!} n! \sum_{l=n}^{\infty} S_1(l,n)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m (x+y)^m S_1(n,m)\frac{t^n}{n!},\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} MCh_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} \\ &= \left(\sum_{n=0}^{\infty} MCh_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)x^l\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)x^l MCh_{n-m,\lambda}\frac{t^n}{n!},\end{aligned} \tag{2.3}$$

by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$MCh_{n,\lambda}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)MCh_{n-m,\lambda}x^l.$$

Note that, by (1.3), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-1}(y) &= \frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)} \\ &= \sum_{n=0}^{\infty} MCh_{n,\lambda}(x)\frac{t^n}{n!},\end{aligned} \tag{2.4}$$

and, by (2.2) and (1.5), we get

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-1}(y) &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m (x+y)^m S_1(n,m)d\mu_{-1}(y)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)E_m(x)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain the following theorem.

Theorem 2.2. *For each* $n\in\mathbb{N}\cup\{0\}$,

$$\sum_{n=0}^{\infty} MCh_{n,\lambda}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-1}(y),$$

and

$$MCh_{n,\lambda}(x) = \sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)E_m(x).$$

By replacing t as e^t-1 in (2.1), we have

$$\begin{aligned}\frac{2}{1+(1+\lambda)^{\frac{t}{\lambda}}}(1+\lambda)^{\frac{t}{\lambda}x} &= \sum_{n=0}^{\infty} MCh_{n,\lambda}(x)\frac{1}{n!}\left(e^t-1\right)^m\\ &= \sum_{n=0}^{\infty} MCh_{n,\lambda}(x)\frac{1}{n!}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} MCh_{m,\lambda}(x)S_2(n,m)\frac{t^n}{n!},\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+(e^t-1))}}(1+\lambda)^{\frac{x}{\lambda}\log\left(1+(e^t-1)\right)} &= \frac{2}{1+(1+\lambda)^{\frac{t}{x}}}(1+\lambda)^{\frac{xt}{\lambda}}\\ &= \sum_{n=0}^{\infty}\xi_{n,\lambda}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we obtain the following corollary.

Corollary 2.3. *For each nonnegative integer* n,

$$\xi_{n,\lambda}(x) = \sum_{m=0}^{n} MCh_{m,\lambda}(x)S_2(n,m).$$

By (1.3), we note that

$$\begin{aligned}2 &= \int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y+1}{\lambda}\log(1+t)}d\mu_{-1}(y) + \int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ &= \sum_{n=0}^{\infty} MCh_{n,\lambda}(1)\frac{t^n}{n!} + \sum_{n=0}^{\infty} MCh_{n,\lambda}\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(MCh_{n,\lambda}(1) + MCh_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

By (2.8), we obtain the following theorem.

Theorem 2.4. *For each positive integer n, we have*

$$MCh_{0,\lambda} = 1,\ MCh_{n,\lambda}(1) + MCh_{n,\lambda} = 2\delta_{0,n},$$

where $\delta_{i,j}$ is the Kronecker's symbols.

For each $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$, by (1.2), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y+n}{\lambda}\log(1+t)}d\mu_{-1}(y)+&\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ &=2\sum_{a=0}^{n-1}(-1)^a(1+\lambda)^{\frac{a}{\lambda}\log(1+t)}\\ &=\sum_{l=0}^{\infty}\left(2\sum_{m=0}^{l}\sum_{a=0}^{n-1}(-1)^a a^m\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(l,m)\right)\frac{t^l}{l!},\end{aligned} \tag{2.9}$$

and

$$\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y+n}{\lambda}\log(1+t)}d\mu_{-1}(y)+\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)=\sum_{l=0}^{\infty}\left(MCh_{l,\lambda}(n)+MCh_{l,\lambda}\right)\frac{t^l}{l!}. \tag{2.10}$$

Hence, by (2.9) and (2.10), we obtain the following theorem.

Theorem 2.5. *For each nonnegative odd integer n and each nonnegative integer l, we have*

$$MCh_{l,\lambda}(n) + MCh_{l,\lambda} = 2\sum_{m=0}^{l}\sum_{a=0}^{n-1}(-1)^a a^m\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(l,m).$$

From now on, we consider the modified degenerate Changhee polynomials of order r are defined as the generating function to be

$$\sum_{n=0}^{\infty}MCh_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x_1+\cdots+x_r+x}{\lambda}\log(1+t)}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \tag{2.11}$$

When $x=0$, $MCh_{n,\lambda}^{(r)} = MCh_{n,\lambda}^{(r)}(0)$ are called modified degenerate Changhee numbers of order r.

Note that, by (1.1) and (2.2),

$$\begin{aligned}\sum_{n=0}^{\infty}MCh_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}&=\left(\frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^r(1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\\ &=\left(\sum_{n=0}^{\infty}MCh_{n,\lambda}\frac{t^n}{n!}\right)^r\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m x^m S_1(n,m)\frac{t^n}{n!}\right)\\ &=\left(\sum_{n=0}^{\infty}\sum_{\substack{n_1,\ldots,n_r\geq 0\\ n_1+\cdots+n_r=n}}MCh_{n_1,\lambda}\cdots MCh_{n_r,\lambda}\frac{t^{n_1}}{n_1!}\cdots\frac{t^{n_r}}{n_r!}\right)\\ &\quad\times\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m x^m S_1(n,m)\frac{t^n}{n!}\right)\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{\substack{n_1,\ldots,n_r\geq 0\\ n_1+\cdots+n_r=m}}\sum_{k=0}^{n-m}\binom{m}{n_1,\cdots,n_r}\binom{n}{m}MCh_{n_1,\lambda}\cdots MCh_{n_r,\lambda}\right.\\ &\quad\left.\times\left(\frac{\log(1+\lambda)}{\lambda}\right)^k x^k S_1(n-m,k)\right)\frac{t^n}{n!},\end{aligned} \tag{2.12}$$

where $\binom{m}{n_1,\dots,n_r}$ are the multinomial coefficients.

In addition, by (1.1) and (2.2), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}&(1+\lambda)^{\frac{x_1+\cdots+x_r+x}{\lambda}\log(1+t)}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^m d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)E_m^{(r)}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.13}$$

By (2.11), (2.12) and (2.13), we obtain the following theorem.

Theorem 2.6. *For each nonnegative integer n, we have*

$$\begin{aligned}MCh_{n,\lambda}^{(r)}(x)=&\sum_{m=0}^{n}\sum_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\cdots+n_r=m}}\sum_{k=0}^{n-m}\binom{m}{n_1,\cdots,n_r}\binom{n}{m}MCh_{n_1,\lambda}\cdots MCh_{n_r,\lambda}\\&\times\left(\frac{\log(1+\lambda)}{\lambda}\right)^k S_1(n-m,k)x^k,\end{aligned}$$

and

$$MCh_{n,\lambda}^{(r)}(x)=\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)E_m^{(r)}(x).$$

By replacing t as e^t-1 in (2.12), we get

$$\begin{aligned}\left(\frac{2}{1+(1+\lambda)^{\frac{t}{\lambda}}}\right)^r(1+\lambda)^{\frac{t}{\lambda}x}&=\sum_{n=0}^{\infty}MCh_{n,\lambda}^{(r)}(x)\frac{1}{n!}\left(e^t-1\right)^m\\&=\sum_{n=0}^{\infty}MCh_{n,\lambda}^{(r)}(x)\frac{1}{n!}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}MCh_{m,\lambda}^{(r)}(x)S_2(n,m)\frac{t^n}{n!},\end{aligned}\tag{2.14}$$

and

$$\begin{aligned}\left(\frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+(e^t-1))}}\right)^r(1+\lambda)^{\frac{x}{\lambda}\log\left(1+(e^t-1)\right)}&=\left(\frac{2}{1+(1+\lambda)^{\frac{t}{x}}}\right)^r(1+\lambda)^{\frac{xt}{\lambda}}\\&=\sum_{n=0}^{\infty}\xi_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.15}$$

By (2.14) and (2.15), we obtain the following theorem.

Theorem 2.7. *For each $n\geq 0$, we have*

$$\xi_{n,\lambda}^{(r)}(x)=\sum_{m=0}^{n}MCh_{m,\lambda}^{(r)}(x)S_2(n,m).$$

By (2.12), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(MCh_{n,\lambda}^{(r)}(x+1)+MCh_{n,\lambda}^{(r)}(x)\right)\frac{t^n}{n!}=&\left(\frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^r(1+\lambda)^{\frac{x+1}{\lambda}\log(1+t)}\\&+\left(\frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^r(1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\\=&2\left(\frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}\right)^{r-1}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\\=&2\sum_{n=0}^{\infty}MCh_{n,\lambda}^{(r-1)}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.8. *For each $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$MCh_{n,\lambda}^{(r)}(x+1)+MCh_{n,\lambda}^{(r)}(x)=2MCh_{n,\lambda}^{(r-1)}(x).$$

Now, we consider the modified degenerate Changhee polynomials of the second kind are defined as the generating function to be

$$\sum_{n=0}^{\infty}\widehat{MCh}_{n,\lambda}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{x-y}{\lambda}\log(1+t)}d\mu_{-1}(y).\tag{2.17}$$

By (1.1), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{-y+x}{\lambda}\log(1+t)}d\mu_{-1}(y)=&\frac{2}{1+(1+\lambda)^{\frac{-1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\\=&\frac{2}{1+(1+\lambda)^{\frac{1}{\lambda}\log(1+t)}}(1+\lambda)^{\frac{x+1}{\lambda}\log(1+t)}\\=&\sum_{n=0}^{\infty}MCh_{n,\lambda}(x+1)\frac{t^n}{n!},\end{aligned}\tag{2.18}$$

and, by (2.2),

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{-y+x}{\lambda}\log(1+t)}d\mu_{-1}(y)=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\int_{\mathbb{Z}_p}(x-y)^m d\mu_{-1}(y)\frac{t^n}{n!}\\=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)(-1)^m E_m(-x)\frac{t^n}{n!}.\end{aligned}\tag{2.19}$$

Thus, by (2.17), (2.18) and (2.19), we have the following theorem.

Theorem 2.9. *For each $n \geq 0$, we have*

$$\widehat{MCh}_{n,\lambda}(x)=MCh_{n,\lambda}(x+1),$$

and

$$\widehat{MCh}_{n,\lambda}(x)=\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)(-1)^m E_m(-x).$$

Acknowledgment

Dedicated to Professor Yeol Je Cho on the occasion of his 65th birthday.

References

[1] L. Comtet, *Advanced combinatorics*, The art of finite and infinite expansions, Revised and enlarged edition, D. Reidel Publishing Co., Dordrecht, (1974). 1

[2] T. Kim, *On a q-analogue of the p-adic log gamma functions and related integrals*, J. Number Theory, **76** (1999), 320–329. 1.1

[3] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., **9** (2002), 288–299.

[4] T. Kim, *Note on the Euler numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **17** (2008), 131–136. 1.1, 1, 1

[5] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys., **16** (2009), 484–491. 1

[6] T. Kim, Y. H. Kim, *Generalized q-Euler numbers and polynomials of higher order and some theoretic identities*, J. Inequal. Appl., **2010** (2010), 6 pages. 1

[7] T. Kim, D. S. Kim, *A note on nonlinear Changhee differential equations*, Russ. J. Math. Phys., **23** (2016), 88–92. 1.1, 1, 1

[8] D. S. Kim, T. Kim, *Generalized Boole numbers and polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, **110** (2016), 823–839. 1

[9] D. S. Kim, T. Kim, *On degenerate Bell numbers and polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, **2016** (2016), 12 pages. 1.1, 1

[10] D. S. Kim, T. Kim, J.-J. Seo, *A note on Changhee polynomials and numbers*, Adv. Studies Theor. Phys., **7** (2013), 993–1003. 1.1, 1, 1

[11] T. Kim, T. Mansour, S. H. Rim, J.-J. Seo, *A note on q-Changhee polynomials and numbers*, Adv. Studies Theor. Phys., **8** (2014), 35–41. 1

[12] H.-I. Kwon, T. Kim, J.-J. Seo, *A note on degenerate Changhee numbers and polynomials*, Proc. Jangjeon Math. Soc., **18** (2015), 295–3056. 1.1, 1, 1

[13] H.-I. Kwon, T. Kim, J.-J. Seo, *Modified degenerate Euler polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **26** (2016), 203–209. 1.1, 1, 1

[14] J.-W. Park, *On the twisted q-Changhee polynomials of higher order*, J. Comput. Anal. Appl., **20** (2016), 424–431. 1

[15] F. Qi, L.-C. Jang, H.-I. Kwon, *Some new and explicit identities related with the Appell-type degenerate q-Changhee polynomials*, Adv. Difference Equ., **2016** (2016), 8 pages.

[16] S.-H. Rim, J.-W. Park, S.-S. Pyo, J.-K. Kwon, *The n-th twisted Changhee polynomials and numbers*, Bull. Korean Math. Soc., **52** (2015), 741–749. 1.1, 1, 1

[17] S. Roman, *The umbral calculus*, Pure and Applied Mathematics, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, (2005). 1