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# A regularity of split-biquaternionic-valued functions in Clifford analysis

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**Abstract**

We examine corresponding Cauchy-Riemann equations by using the non-commutativity for the product on split-biquaternions. Additionally, we describe the regularity of functions and properties of their differential equations on split-biquaternions. We investigate representations and calculations of the derivatives of functions of split-biquaternionic variables. 

*Keywords:* Cauchy-Riemann equations, regular function, split-quaternion, Clifford analysis.
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## 1. Introduction

The algebraic properties of $\mathbb{H}$ contribute to the various fields of analysis, where $\mathbb{H} := \{q = x_0 + e_1x_1 + e_2x_2 + e_3x_3 \mid x_r \in \mathbb{R}$ being the set of real numbers $(r = 0, 1, 2, 3)\}$ is the set of quaternions which has the imaginary base $e_1$, $e_2$, and $e_3$ such that $e_1^2 = e_2^2 = e_3^2 = -1$, $e_1e_2 = e_3 = -e_2e_1$, $e_2e_3 = e_1 = -e_3e_2$ and $e_3e_1 = e_2 = -e_1e_3$. Since quaternions are non-commutative to each other, there are two ways to define the limits of a difference quotient for holomorphy (see [10]): let $U$ be a domain in $\mathbb{H}$ and $f : U \to \mathbb{H}$ be a function such that $f(q) = f_0 + e_1f_1 + e_2f_2 + e_3f_3$ and $f_r = f_r(x_0, x_1, x_2, x_3)$ $(r = 0, 1, 2, 3)$ are real-valued functions. For $\Delta x_r \neq 0$ $(r = 0, 1, 2, 3)$,

$$\lim_{\Delta q\to 0}(\Delta q)^{-1}\{f(q+\Delta q) - f(q)\} = \lim_{\Delta q\to 0}\frac{\Delta x_0\Delta f - e_1\Delta x_1\Delta f - e_2\Delta x_2\Delta f - e_3\Delta x_3\Delta f}{(\Delta x_0)^2 + (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2} \tag{1.1}$$

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and

$$\lim_{\Delta q\to 0}\{f(q+\Delta q)-f(q)\}(\Delta q)^{-1}=\lim_{\Delta q\to 0}\frac{\Delta f\Delta x_0-\Delta f\Delta x_1e_1-\Delta f\Delta x_2e_2-\Delta f\Delta x_3e_3}{(\Delta x_0)^2+(\Delta x_1)^2+(\Delta x_2)^2+(\Delta x_3)^2}, \tag{1.2}$$

where

$$\Delta f := f(q+\Delta q)-f(q)=\Delta f_0+e_1\Delta f_1+e_2\Delta f_2+e_3\Delta f_3$$

and

$$\Delta f_r = f_r(x_0+\Delta x_0, x_1+\Delta x_1, x_2+\Delta x_2, x_3+\Delta x_3)-f_r(x_0,x_1,x_2,x_3) \qquad (r=0,1,2,3),$$

by allowing $\Delta q=\Delta x_0+e_1\Delta x_1+e_2\Delta x_2+e_3\Delta x_3$ with $\Delta x_r\in\mathbb{R}$ $(r=0,1,2,3)$ to approach 0. When these limits exist, by setting $\Delta q$ equal to $\Delta x_0$, $e_1\Delta x_1$, $e_2\Delta x_2$, and $e_3\Delta x_3$, we can obtain equations to get the derivatives from each above equation. The existence of the limits (1.1) and (1.2) give the Cauchy-Riemann equations for a quaternionic-valued function as follows:

$$\begin{cases}\dfrac{\partial f_0}{\partial x_0}=\dfrac{\partial f_1}{\partial x_1}=\dfrac{\partial f_2}{\partial x_2}=\dfrac{\partial f_3}{\partial x_3},\\ \dfrac{\partial f_1}{\partial x_0}=-\dfrac{\partial f_0}{\partial x_1}=-\dfrac{\partial f_3}{\partial x_2}=\dfrac{\partial f_2}{\partial x_3},\\ \dfrac{\partial f_2}{\partial x_0}=\dfrac{\partial f_3}{\partial x_1}=-\dfrac{\partial f_0}{\partial x_2}=-\dfrac{\partial f_1}{\partial x_3},\\ \dfrac{\partial f_3}{\partial x_0}=-\dfrac{\partial f_2}{\partial x_1}=\dfrac{\partial f_1}{\partial x_2}=-\dfrac{\partial f_0}{\partial x_3},\end{cases}$$

and

$$\begin{cases}\dfrac{\partial f_0}{\partial x_0}=\dfrac{\partial f_1}{\partial x_1}=\dfrac{\partial f_2}{\partial x_2}=\dfrac{\partial f_3}{\partial x_3},\\ \dfrac{\partial f_1}{\partial x_0}=-\dfrac{\partial f_0}{\partial x_1}=\dfrac{\partial f_3}{\partial x_2}=-\dfrac{\partial f_2}{\partial x_3},\\ \dfrac{\partial f_2}{\partial x_0}=-\dfrac{\partial f_3}{\partial x_1}=-\dfrac{\partial f_0}{\partial x_2}=\dfrac{\partial f_1}{\partial x_3},\\ \dfrac{\partial f_3}{\partial x_0}=\dfrac{\partial f_2}{\partial x_1}=-\dfrac{\partial f_1}{\partial x_2}=-\dfrac{\partial f_0}{\partial x_3},\end{cases}$$

respectively, which are useful in the theories of polymorphic functions in a quaternion analysis. For instance, a function $f$ of quaternion variables is holomorphic which has continuously differential components if and only if $f$ satisfies the equations $\overline{\partial_q}f=0$ and it has a derivative $\partial_q f$ of $f$, where the differential operator

$$\overline{\partial_q}:=\frac{1}{2}\Big(\frac{\partial}{\partial x_0}+e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}+e_3\frac{\partial}{\partial x_3}\Big)$$

and

$$\partial_q=\frac{1}{2}\Big(\frac{\partial}{\partial x_0}-e_1\frac{\partial}{\partial x_1}-e_2\frac{\partial}{\partial x_2}-e_3\frac{\partial}{\partial x_3}\Big)$$

used to a quaternion analysis.

Herein, we consider functions of a split-biquaternionic variable and research two analogous definitions of a holomorphic function. There are many studies about relations between holomorphy and the Cauchy-Riemann equations on quaternions and split-quaternions. Libine [9] approached the split-quaternions as a real form, introduced the notion of regular functions and gave two different analogues of the Cauchy-Fueter formula valid for different classes of functions. Masrouri et al. [10] studied the theory of mathematical analysis over split quaternions, formulated in a closely analogous condition for analyticity of functions of a split quaternion variable. Nôno [11] studied the properties of quaternions and the definition of hyper-holomorphic functions of quaternion variables. Kajiwara et al. [2] gave a basic estimate for inhomogeneous Cauchy-Riemann systems in quaternion analysis. They applied the theory to a closed, densely defined operator and a priori estimate for the adjoint operator in Hilbert space and biconvex domains. Kim et. al.

[4, 7] obtained some results regarding the regularity of functions on the reduced quaternion field, and on the form of (dual) split quaternions, defined by differential operators in Clifford analysis. In addition, Kim and Shon [5, 6] researched corresponding Cauchy-Riemann systems and the properties of functions with values in special quaternions (such as reduced or split-quaternions) by using a regular function with values in dual split-quaternions and gave properties and calculations of functions of bicomplex variables with the commutative multiplication rule [8]. Kim [3] studied the corresponding inverse of functions of multidual complex variables in Clifford analysis.

Now, we give the two different analogous ways of defining a holomorphic function of a split-biquaternionic variable. We research the corresponding Cauchy-Riemann equations on split-biquaternions by comparing the left-side with the right-side calculations together. Also, we give regularities of functions of split-biquaternionic variables and properties of their differential equations on split-biquaternions.

## 2. Preliminaries

The split-biquaternions are the complex Clifford algebra which are elements of the following set, denoted by $\mathcal{S}_{\mathbb{C}}$,

$$\mathcal{S}_{\mathbb{C}} := \{Z = z_0 + z_1 i + z_2 j + z_3 ij : z_r = x_r + \mathbf{i} y_r \in \mathbb{C}, \quad x_r, y_r \in \mathbb{R}\},$$

where $\mathbf{i} = \sqrt{-1}$ and $\mathbb{C}$ is the set of complex numbers. Moreover, $i$, $j$ are non-commutative base elements with $i^2 = -1$, $j^2 = 1$, and $ij = -ji$, which are commutative with $\mathbf{i}$. The set $\mathcal{S}_{\mathbb{C}}$ is isomorphic to $\mathbb{C}^4$. For two split-biquaternions $Z = z_0 + z_1 i + z_2 j + z_3 ij$ and $W = w_0 + w_1 i + w_2 j + w_3 ij$, where $w_k \in \mathbb{C}$, addition and multiplication are given by

$$Z + W = (z_0 + w_0) + (z_1 + w_1)i + (z_2 + w_2)j + (z_3 + w_3)ij,$$

and

$$\begin{aligned} ZW = {} & (z_0 w_0 - z_1 w_1 + z_2 w_2 + z_3 w_3) + (z_0 w_1 + z_1 w_0 - z_2 w_3 + z_3 w_2)i \\ & + (z_0 w_2 - z_1 w_3 + z_2 w_0 + z_3 w_1)j + (z_0 w_3 + z_1 w_2 - z_2 w_1 + z_3 w_0)ij. \end{aligned}$$

Consider a conjugation $Z^\dagger$ of $\mathcal{S}_{\mathbb{C}}$ such that $Z^\dagger = z_0 - z_1 i - z_2 j - z_3 ij$. Then we obtain the following form

$$ZZ^\dagger = Z^\dagger Z = z_0^2 + z_1^2 - z_2^2 - z_3^2 \in \mathbb{C}.$$

We give a modulus $M(Z)$ of $Z \in \mathcal{S}_{\mathbb{C}}$ as

$$M(Z) := (x_0^2 - y_0^2 + x_1^2 - y_1^2 - x_2^2 + y_2^2 - x_3^2 + y_3^2)^2 + 4(x_0 y_0 + x_1 y_1 - x_2 y_2 - x_3 y_3)^2.$$

From the conjugation of $\mathcal{S}_{\mathbb{C}}$, we have the inverse element of $Z$ in $\mathcal{S}_{\mathbb{C}}$ such that

$$Z^{-1} = \frac{Z^\dagger}{ZZ^\dagger} \quad (x_r \neq y_r, \quad r = 0, 1, 2, 3).$$

There is a representation of the base of the split-quaternions as $2 \times 2$ matrices over $\mathbb{R}$. We correspond the following relations (see [1]):

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad j \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad ij \leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we can map $\mathcal{S}_{\mathbb{C}}$ to the real $2 \times 2$ matrices as follows:

$$z_0 + z_1 i + z_2 j + z_3 ij \mapsto \begin{pmatrix} z_0 - z_3 & -z_1 + z_2 \\ z_1 + z_2 & z_0 + z_3 \end{pmatrix}.$$

Thus, we also have

$$\det_{\dagger}\begin{bmatrix} z_0 - z_3 & -z_1 + z_2 \\ z_1 + z_2 & z_0 + z_3 \end{bmatrix} = z_0^2 + z_1^2 - z_2^2 - z_3^2 \in \mathbb{C}.$$

Consider the following function

$$F : U \subset \mathbb{C}^4 \ \to \ \mathcal{S}_{\mathbb{C}}$$

and

$$F(z_0, z_1, z_2, z_3) = f_0 + f_1 i + f_2 j + f_3 ij,$$

where $U$ is open in $\mathbb{C}^4$ and $f_r : \mathbb{C}^4 \to \mathbb{C}$; $f_r = f_r(z_0, z_1, z_2, z_3)$ $(r = 0, 1, 2, 3)$ are complex-valued functions. We try to construct the analogous definition of holomorphy. The first method considers split-biquaternionic-valued differential operators and the second utilizes a difference quotient (see [9] and [10]).

We give the following Dirac operators $D_Z$ and $D_Z^{\dagger}$:

$$D_Z := \frac{1}{2}\Big(\frac{\partial}{\partial z_0} - i\frac{\partial}{\partial z_1} + j\frac{\partial}{\partial z_2} + ij\frac{\partial}{\partial z_3}\Big) \quad \text{and} \quad D_Z^{\dagger} = \frac{1}{2}\Big(\frac{\partial}{\partial z_0} + i\frac{\partial}{\partial z_1} - j\frac{\partial}{\partial z_2} - ij\frac{\partial}{\partial z_3}\Big),$$

where $\frac{\partial}{\partial z_r} = \frac{\partial}{\partial x_r} - \mathbf{i}\frac{\partial}{\partial y_r}$ and $\frac{\partial}{\partial \overline{z_r}} = \frac{\partial}{\partial x_r} + \mathbf{i}\frac{\partial}{\partial y_r}$ $(r = 0, 1, 2, 3)$ are usual differential operators in complex analysis. Since the Dirac operators are non-commutative for the product on $\mathcal{S}_{\mathbb{C}}$, these operators are acted to functions on either the left-side or right-side, and each calculation provides different results. The product of $D_Z$ and $D_Z^{\dagger}$ gives the Laplacian for complex analysis, denoted by $\Delta_{\mathcal{S}_{\mathbb{C}}}$,

$$\Delta_{\mathcal{S}_{\mathbb{C}}} = \frac{1}{4}\Big(\frac{\partial^2}{\partial z_0^2} + \frac{\partial^2}{\partial z_1^2} - \frac{\partial^2}{\partial z_2^2} - \frac{\partial^2}{\partial z_3^2}\Big).$$

**Definition 2.1.** Let $U$ be an open set in $\mathcal{S}_{\mathbb{C}}$ and let $F : U \ \to \ \mathcal{S}_{\mathbb{C}}$ be a function in $\mathcal{C}^1(U)$ being the class which consists of all differentiable functions whose derivative is continuous. A function $F$ is said to be left-regular if

$$\begin{aligned} D_Z^{\dagger} F =& \frac{1}{2}\Big(\frac{\partial F}{\partial z_0} + i\frac{\partial F}{\partial z_1} - j\frac{\partial F}{\partial z_2} - ij\frac{\partial F}{\partial z_3}\Big) \\ =& \frac{1}{2}\Big\{\Big(\frac{\partial f_0}{\partial z_0} - \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2} - \frac{\partial f_3}{\partial z_3}\Big) + \Big(\frac{\partial f_0}{\partial z_1} + \frac{\partial f_1}{\partial z_0} - \frac{\partial f_2}{\partial z_3} + \frac{\partial f_3}{\partial z_2}\Big)i \\ &+ \Big(-\frac{\partial f_0}{\partial z_2} - \frac{\partial f_1}{\partial z_3} - \frac{\partial f_3}{\partial z_1} + \frac{\partial f_2}{\partial z_0}\Big)j + \Big(-\frac{\partial f_0}{\partial z_3} + \frac{\partial f_1}{\partial z_2} + \frac{\partial f_2}{\partial z_1} + \frac{\partial f_3}{\partial z_0}\Big)ij\Big\} = 0, \end{aligned}$$

for every $Z \in U$. Similarly, we say that $F$ is right-regular if

$$\begin{aligned} F D_Z^{\dagger} =& \frac{1}{2}\Big\{\Big(\frac{\partial f_0}{\partial z_0} - \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2} - \frac{\partial f_3}{\partial z_3}\Big) + \Big(\frac{\partial f_0}{\partial z_1} + \frac{\partial f_1}{\partial z_0} + \frac{\partial f_2}{\partial z_3} - \frac{\partial f_3}{\partial z_2}\Big)i \\ &+ \Big(-\frac{\partial f_0}{\partial z_2} + \frac{\partial f_1}{\partial z_3} + \frac{\partial f_2}{\partial z_0} + \frac{\partial f_3}{\partial z_1}\Big)j + \Big(-\frac{\partial f_0}{\partial z_3} - \frac{\partial f_1}{\partial z_2} - \frac{\partial f_2}{\partial z_1} + \frac{\partial f_3}{\partial z_0}\Big)ij\Big\} = 0, \end{aligned}$$

for every $Z \in U$.

**Proposition 2.2.** *Let $F : U \ \to \ \mathcal{S}_{\mathbb{C}}$ be a function in $\mathcal{C}^1(U)$. Then $F$ is left-regular if and only if $F$ satisfies the following partial differential equations (PDEs):*

$$\begin{cases} \dfrac{\partial f_0}{\partial z_0} - \dfrac{\partial f_1}{\partial z_1} - \dfrac{\partial f_2}{\partial z_2} - \dfrac{\partial f_3}{\partial z_3} = 0, \\ \dfrac{\partial f_1}{\partial z_0} + \dfrac{\partial f_0}{\partial z_1} + \dfrac{\partial f_3}{\partial z_2} - \dfrac{\partial f_2}{\partial z_3} = 0, \\ \dfrac{\partial f_2}{\partial z_0} - \dfrac{\partial f_3}{\partial z_1} - \dfrac{\partial f_0}{\partial z_2} - \dfrac{\partial f_1}{\partial z_3} = 0, \\ \dfrac{\partial f_3}{\partial z_0} + \dfrac{\partial f_2}{\partial z_1} + \dfrac{\partial f_1}{\partial z_2} - \dfrac{\partial f_0}{\partial z_3} = 0. \end{cases} \tag{2.1}$$

**Proposition 2.3.** *Let $F : U \to \mathcal{S}_{\mathbb{C}}$ be a function in $\mathcal{C}^1(U)$. Then $F$ is right-regular if and only if $F$ satisfies the following of PDEs:*

$$\begin{cases} \dfrac{\partial f_0}{\partial z_0} - \dfrac{\partial f_1}{\partial z_1} - \dfrac{\partial f_2}{\partial z_2} - \dfrac{\partial f_3}{\partial z_3} = 0, \\ \dfrac{\partial f_1}{\partial z_0} + \dfrac{\partial f_0}{\partial z_1} - \dfrac{\partial f_3}{\partial z_2} + \dfrac{\partial f_2}{\partial z_3} = 0, \\ \dfrac{\partial f_2}{\partial z_0} + \dfrac{\partial f_3}{\partial z_1} - \dfrac{\partial f_0}{\partial z_2} + \dfrac{\partial f_1}{\partial z_3} = 0, \\ \dfrac{\partial f_3}{\partial z_0} - \dfrac{\partial f_2}{\partial z_1} - \dfrac{\partial f_1}{\partial z_2} - \dfrac{\partial f_0}{\partial z_3} = 0. \end{cases}$$

Referring to [12], we offer the following example which shows that $F$ is not a left-regular function in $\mathcal{S}_{\mathbb{C}}$.

**Example 2.4.** Let $A = a + bi + cj + dij \in \mathcal{S}_{\mathbb{C}}$, where $a, b, c, d$ are constants in $\mathcal{S}_{\mathbb{C}}$. Then

$$\begin{aligned} AZ =& (az_0 - bz_1 + cz_2 + dz_3) + (bz_0 + az_1 + dz_2 - cz_3)i \\ &+ (cz_0 + dz_1 + az_2 - bz_3)j + (dz_0 - cz_1 + bz_2 + az_3)ij. \end{aligned}$$

Thus, we have

$$\begin{aligned} D_Z^{\dagger}(AZ) =& (a + bi + cj + dij) + i(-b + ai + dj - cij) - j(c + di + aj + bij) - ij(d - ci - bj + aij) \\ =& - a + bi + cj + dij = -A^{\dagger} \neq 0 \end{aligned}$$

and

$$\begin{aligned} D_Z^{\dagger}(ZA) =& (a + bi + cj + dij) + i(-b + ai - dj + cij) - j(c - di + aj - bij) - ij(d + ci + bj + aij) \\ =& - a - bi - cj - dij = -A \neq 0. \end{aligned}$$

Similarly, using the differential operator $D_Z^{\dagger}$ on right-side, we can show that $AZ$ and $ZA$ are not right-regular.

## 3. Differential function

A corresponding Cauchy-Riemann equations are obtained by allowing $\Delta Z = \Delta z_0 + \Delta z_1 i + \Delta z_2 j + \Delta z_3 ij$ to approach 0 being equivalent to $\Delta z_0 \to 0$, $i\Delta z_1 \to 0$, $j\Delta z_2 \to 0$, and $ij\Delta z_3 \to 0$. Referring to [10], because of non-commutativity of the split-quaternions, we consider two ways which are used to construct analogue of holomorphy. These ways are defined by, for $x_r \neq y_r$ $(r = 0, 1, 2, 3)$,

$$\lim_{\Delta Z \to 0} (\Delta Z)^{-1} \{F(Z + \Delta Z) - F(Z)\} = \lim_{\Delta Z \to 0} \frac{(\Delta z_0)\Delta F - (\Delta z_1)i\Delta F - (\Delta z_2)j\Delta F - (\Delta z_3)k\Delta F}{(\Delta z_0)^2 + (\Delta z_1)^2 - (\Delta z_2)^2 - (\Delta z_3)^2} \tag{3.1}$$

and

$$\lim_{\Delta Z \to 0} \{F(Z + \Delta Z) - F(Z)\}(\Delta Z)^{-1} = \lim_{\Delta Z \to 0} \frac{\Delta F(\Delta z_0) - \Delta F(\Delta z_1)i - \Delta F(\Delta z_2)j - \Delta F(\Delta z_3)k}{(\Delta z_0)^2 + (\Delta z_1)^2 - (\Delta z_2)^2 - (\Delta z_3)^2}, \tag{3.2}$$

where

$$\Delta F := F(Z + \Delta Z) - F(Z) = \Delta f_0 + \Delta f_1 i + \Delta f_2 j + \Delta f_3 k$$

and

$$\Delta f_r(z_0 + \Delta z_0, z_1 + \Delta z_1, z_2 + \Delta z_2, z_3 + \Delta z_3) - f_r(z_0, z_1, z_2, z_3) \quad (r = 0, 1, 2, 3).$$

Such functions are called left- and right-$\mathcal{S}_{\mathbb{C}}$-differentiable, respectively, if each limit exists. By taking the limits as (3.1) and (3.2), we get the derivatives of $F$ in $\mathcal{S}_{\mathbb{C}}$ as follows:

$$\begin{cases} \lim\limits_{\Delta z_0\to 0}(\Delta z_0)^{-1}\{F(Z+\Delta z_0)-F(Z)\} = \dfrac{\partial f_0}{\partial z_0}+\dfrac{\partial f_1}{\partial z_0}i+\dfrac{\partial f_2}{\partial z_0}j+\dfrac{\partial f_3}{\partial z_0}ij, \\ \lim\limits_{i\Delta z_1\to 0}(-i)(\Delta z_1)^{-1}\{F(Z+\Delta z_1)-F(Z)\} = -\dfrac{\partial f_0}{\partial z_1}i+\dfrac{\partial f_1}{\partial z_1}-\dfrac{\partial f_2}{\partial z_1}ij+\dfrac{\partial f_3}{\partial z_1}j, \\ \lim\limits_{j\Delta z_2\to 0}j(\Delta z_2)^{-1}\{F(Z+\Delta z_2)-F(Z)\} = \dfrac{\partial f_0}{\partial z_2}j-\dfrac{\partial f_1}{\partial z_2}ij+\dfrac{\partial f_2}{\partial z_2}-\dfrac{\partial f_3}{\partial z_2}i, \\ \lim\limits_{ij\Delta z_3\to 0}ij(\Delta z_3)^{-1}\{F(Z+\Delta z_3)-F(Z)\} = \dfrac{\partial f_0}{\partial z_3}ij+\dfrac{\partial f_1}{\partial z_3}j+\dfrac{\partial f_2}{\partial z_3}i+\dfrac{\partial f_3}{\partial z_3}, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \lim\limits_{\Delta z_0\to 0}\{F(Z+\Delta z_0)-F(Z)\}(\Delta z_0)^{-1} = \dfrac{\partial f_0}{\partial z_0}+\dfrac{\partial f_1}{\partial z_0}i+\dfrac{\partial f_2}{\partial z_0}j+\dfrac{\partial f_3}{\partial z_0}ij, \\ \lim\limits_{i\Delta z_1\to 0}\{F(Z+\Delta z_1)-F(Z)\}(-i)(\Delta z_1)^{-1} = -\dfrac{\partial f_0}{\partial z_1}i+\dfrac{\partial f_1}{\partial z_1}+\dfrac{\partial f_2}{\partial z_1}ij-\dfrac{\partial f_3}{\partial z_1}j, \\ \lim\limits_{j\Delta z_2\to 0}\{F(Z+\Delta z_2)-F(Z)\}j(\Delta z_2)^{-1} = \dfrac{\partial f_0}{\partial z_2}j+\dfrac{\partial f_1}{\partial z_2}ij+\dfrac{\partial f_2}{\partial z_2}+\dfrac{\partial f_3}{\partial z_2}i, \\ \lim\limits_{ij\Delta z_3\to 0}\{F(Z+\Delta z_3)-F(Z)\}ij(\Delta z_3)^{-1} = \dfrac{\partial f_0}{\partial z_3}ij-\dfrac{\partial f_1}{\partial z_3}j-\dfrac{\partial f_2}{\partial z_3}i+\dfrac{\partial f_3}{\partial z_3}, \end{cases} \tag{3.4}$$

where each $\Delta z_r = \Delta x_r + \mathbf{i}\Delta y_r$ and $(\Delta z_r)^{-1} = \frac{\Delta x_r - \mathbf{i}\Delta y_r}{(\Delta x_r)^2+(\Delta y_r)^2}$ with $x_r \neq 0$ and $y_r \neq 0$ $(r = 0, 1, 2, 3)$, is the form of the inverse element in complex numbers. Dealing with the equations (3.3), we obtain the following system of PDEs:

$$\begin{cases} ll\dfrac{\partial f_0}{\partial z_0} = \dfrac{\partial f_1}{\partial z_1} = \dfrac{\partial f_2}{\partial z_2} = \dfrac{\partial f_3}{\partial z_3}, \\ \dfrac{\partial f_1}{\partial z_0} = -\dfrac{\partial f_0}{\partial z_1} = -\dfrac{\partial f_3}{\partial z_2} = \dfrac{\partial f_2}{\partial z_3}, \\ \dfrac{\partial f_2}{\partial z_0} = \dfrac{\partial f_3}{\partial z_1} = \dfrac{\partial f_0}{\partial z_2} = \dfrac{\partial f_1}{\partial z_3}, \\ \dfrac{\partial f_3}{\partial z_0} = -\dfrac{\partial f_2}{\partial z_1} = -\dfrac{\partial f_1}{\partial z_2} = \dfrac{\partial f_0}{\partial z_3}. \end{cases} \tag{3.5}$$

If we consider the equations (3.4), we also have the following system of PDEs:

$$\begin{cases} \dfrac{\partial f_0}{\partial z_0} = \dfrac{\partial f_1}{\partial z_1} = \dfrac{\partial f_2}{\partial z_2} = \dfrac{\partial f_3}{\partial z_3}, \\ \dfrac{\partial f_1}{\partial z_0} = -\dfrac{\partial f_0}{\partial z_1} = \dfrac{\partial f_3}{\partial z_2} = -\dfrac{\partial f_2}{\partial z_3}, \\ \dfrac{\partial f_2}{\partial z_0} = -\dfrac{\partial f_3}{\partial z_1} = \dfrac{\partial f_0}{\partial z_2} = -\dfrac{\partial f_1}{\partial z_3}, \\ \dfrac{\partial f_3}{\partial z_0} = \dfrac{\partial f_2}{\partial z_1} = \dfrac{\partial f_1}{\partial z_2} = \dfrac{\partial f_0}{\partial z_3}. \end{cases} \tag{3.6}$$

**Example 3.1.** Consider a function

$$\begin{aligned} AZ+K =& (az_0 - bz_1 + cz_2 + dz_3 + k) + (bz_0 + az_1 + dz_2 - cz_3 + l)i \\ &+ (cz_0 + dz_1 + az_2 - bz_3 + m)j + (dz_0 - cz_1 + bz_2 + az_3 + n)ij, \end{aligned}$$

where $A$ and $K$ are arbitrary constants in $\mathcal{S}_{\mathbb{C}}$. Because of the non-commutativity for $i$ and $j$, there dose not exist the limit (3.1) for $(\Delta Z)^{-1}$. However, there exists the limit (3.2) such as

$$\lim_{\Delta Z\to 0}\{A(Z+\Delta Z)+K-AZ-K\}(\Delta Z)^{-1} = \lim_{\Delta Z\to 0}(A\Delta Z)(\Delta Z)^{-1} = A.$$

**Theorem 3.2.** *Let $F : U \to \mathcal{S}_{\mathbb{C}}$ be a function in $\mathcal{C}^1(U)$. Then $F$ is right-$\mathcal{S}_{\mathbb{C}}$-differentiable if and only if $F(Z) = AZ + K$ for constants $A$ and $K$ in $\mathcal{S}_{\mathbb{C}}$.*

*Proof.* Suppose that $F$ is right-$\mathcal{S}_{\mathbb{C}}$-differentiable. In order to obtain the form $F(Z) = AZ+K$, it is sufficient to show that $D_Z(D_Z F) = 0$. The equations (3.6) give that each component of $F$ is continuous with respect to the complex variables $z_r$ $(r = 0, 1, 2, 3)$. From the equations (3.6), the first term of $D_Z(D_Z F)$ has the following expansion:

$$\begin{aligned}
&\frac{1}{4}\Big(\frac{\partial^2 f_0}{\partial z_0^2} + 2\frac{\partial^2 f_1}{\partial z_1 \partial z_0} - \frac{\partial^2 f_0}{\partial z_1^2}\Big) + i\frac{1}{4}\Big(\frac{\partial^2 f_1}{\partial z_0^2} - 2\frac{\partial^2 f_0}{\partial z_1 \partial z_0} + \frac{\partial^2 f_1}{\partial z_1^2}\Big)\\
&\qquad =\frac{1}{4}\Big(\frac{\partial^2 f_1}{\partial z_1 \partial z_0} + \frac{\partial^2 f_3}{\partial z_3 \partial z_0} + \frac{\partial^2 f_3}{\partial z_2 \partial z_1} - \frac{\partial^2 f_2}{\partial z_3 \partial z_1}\Big)\\
&\qquad\quad + i\frac{1}{4}\Big(\frac{\partial^2 f_3}{\partial z_2 \partial z_0} - \frac{\partial^2 f_2}{\partial z_3 \partial z_0} - \frac{\partial^2 f_2}{\partial z_2 \partial z_1} - \frac{\partial^2 f_1}{\partial z_3 \partial z_1}\Big)\\
&\qquad =\frac{1}{4}\Big\{\frac{\partial}{\partial z_1}\Big(\frac{\partial f_1}{\partial z_0} - \frac{\partial f_3}{\partial z_2}\Big) + \frac{\partial}{\partial z_3}\Big(\frac{\partial f_3}{\partial z_0} - \frac{\partial f_2}{\partial z_1}\Big)\Big\}\\
&\qquad\quad + i\frac{1}{4}\Big\{\frac{\partial}{\partial z_2}\Big(\frac{\partial f_3}{\partial z_0} - \frac{\partial f_2}{\partial z_1}\Big) - \frac{\partial}{\partial z_0}\Big(\frac{\partial f_2}{\partial z_3} - \frac{\partial f_0}{\partial z_1}\Big)\Big\} = 0,
\end{aligned}$$

and the second term of $D_Z(D_Z F)$ has the following expansion:

$$\begin{aligned}
&\frac{1}{4}\Big(\frac{\partial^2 f_2}{\partial z_2^2} - 2\frac{\partial^2 f_3}{\partial z_2 \partial z_3} - \frac{\partial^2 f_2}{\partial z_3^2}\Big) + i\frac{1}{4}\Big(\frac{\partial^2 f_3}{\partial z_2^2} + 2\frac{\partial^2 f_2}{\partial z_2 \partial z_3} - \frac{\partial^2 f_3}{\partial z_3^2}\Big)\\
&\qquad =\frac{1}{4}\Big(\frac{\partial^2 f_3}{\partial z_2 \partial z_3} + \frac{\partial^2 f_1}{\partial z_3 \partial z_0} + \frac{\partial^2 f_3}{\partial z_2 \partial z_3} - \frac{\partial^2 f_0}{\partial z_3 \partial z_1}\Big)\\
&\qquad\quad + i\frac{1}{4}\Big(\frac{\partial^2 f_1}{\partial z_2 \partial z_0} - \frac{\partial^2 f_2}{\partial z_3 \partial z_2} - \frac{\partial^2 f_0}{\partial z_2 \partial z_1} - \frac{\partial^2 f_2}{\partial z_3 \partial z_2}\Big)\\
&\qquad =\frac{1}{4}\Big\{\frac{\partial}{\partial z_3}\Big(\frac{\partial f_3}{\partial z_2} - \frac{\partial f_1}{\partial z_0}\Big) + \frac{\partial}{\partial z_3}\Big(\frac{\partial f_3}{\partial z_2} + \frac{\partial f_0}{\partial z_1}\Big)\Big\}\\
&\qquad\quad + i\frac{1}{4}\Big\{\frac{\partial}{\partial z_2}\Big(\frac{\partial f_1}{\partial z_0} + \frac{\partial f_2}{\partial z_3}\Big) - \frac{\partial}{\partial z_2}\Big(\frac{\partial f_0}{\partial z_1} - \frac{\partial f_2}{\partial z_3}\Big)\Big\} = 0.
\end{aligned}$$

Similarly, for another terms, we get that each expansion of all terms of $D_Z(D_Z F)$ is zero, by using the equations (3.6). Thus, $F$ has the form $AZ + K$, where $A$ and $K$ are constants in $\mathcal{S}_{\mathbb{C}}$.

Conversely, if $F(Z) = AZ + K$ is right-$\mathcal{S}_{\mathbb{C}}$-differentiable (see Example 3.1). Therefore, we obtain the result. □

**Corollary 3.3.** *Let $F : U \to \mathcal{S}_{\mathbb{C}}$ be a function in $\mathcal{C}^1(U)$. Then $F$ is left-$\mathcal{S}_{\mathbb{C}}$-differentiable if and only if $F(Z) = ZA + K$, where $A$ and $K$ are in $\mathcal{S}_{\mathbb{C}}$.*

**Example 3.4.** A function $F(Z) = AZ + K$ is not right-$\mathcal{S}_{\mathbb{C}}$-differentiable. However, $F(Z) = ZA + K$ is left-$\mathcal{S}_{\mathbb{C}}$-differentiable.

Let $F : U \to \mathcal{S}_{\mathbb{C}}$ consist of the elements of $\mathcal{C}^2(U)$, which is the class of functions having the first and second derivative of the function both exist and are continuous, and $F$ satisfies at least one of the following equations:

$$D_Z^{\dagger} F = 0, \quad F D_Z^{\dagger} = 0, \quad D_Z^{\dagger} F = 0, \quad \text{and} \quad F D_Z^{\dagger} = 0.$$

Then each component of $F$ satisfies Johns equation (which is referred by [9]):

$$\Delta_{\mathcal{S}_{\mathbb{C}}} f_r = 0 \quad (r = 0, 1, 2, 3).$$

Such functions are said to be pseudo-hyperbolic. From the definition of pseudo-hyperbolic functions, we give a representation of regular functions. Let $\varphi : U \to \mathbb{C}$ be pseudo-hyperbolic. Then $D_Z\varphi$ is both left-and right-regular. Let $F = D_Z\varphi$. Then we have

$$D_Z^\dagger F = D_Z^\dagger(D_Z\varphi) = \Delta_{\mathcal{S}_\mathbb{C}}\varphi = 0$$

and

$$FD_Z^\dagger = (D_Z\varphi)D_Z^\dagger = \Delta_{\mathcal{S}_\mathbb{C}}\varphi = 0.$$

We can also deal with left- and right-$\mathcal{S}_\mathbb{C}$-differentiable functions which are formed by pseudo-hyperbolic.

**Theorem 3.5.** *Let $F : U \to \mathcal{S}_\mathbb{C}$ be comprised of functions in $\mathcal{C}^2(U)$. Suppose that $F$ is left-$\mathcal{S}_\mathbb{C}$-differentiable. Then the components of $F$ are pseudo-hyperbolic.*

*Proof.* Suppose $F(z_0, z_1, z_2, z_3) = f_0 + f_1 i + f_2 j + f_3 ij$ is right-$\mathcal{S}_\mathbb{C}$-differentiable. Then

$$\begin{aligned}\frac{\partial^2 f_0}{\partial z_0^2} + \frac{\partial^2 f_0}{\partial z_1^2} - \frac{\partial^2 f_0}{\partial z_2^2} - \frac{\partial^2 f_0}{\partial z_3^2} &= \frac{\partial}{\partial z_0}\frac{\partial f_1}{\partial z_1} + \frac{\partial}{\partial z_1}\frac{\partial f_0}{\partial z_1} - \frac{\partial}{\partial z_3}\frac{\partial f_1}{\partial z_2} + \frac{\partial}{\partial z_3}\frac{\partial f_0}{\partial z_3} \\ &= \frac{\partial}{\partial z_1}\Big(\frac{\partial f_1}{\partial z_0} + \frac{\partial f_0}{\partial z_1}\Big) + \frac{\partial}{\partial z_3}\Big(\frac{\partial f_1}{\partial z_2} + \frac{\partial f_0}{\partial z_3}\Big) = 0.\end{aligned}$$

A similar process for the other $f_r$ $(r = 1, 2, 3)$ gives the result as desired. □

**Corollary 3.6.** *Let $F : U \to \mathcal{S}_\mathbb{C}$ be comprised of functions in $\mathcal{C}^2(U)$. Suppose that $F$ is right-$\mathcal{S}_\mathbb{C}$-differentiable. Then the components of $F$ are pseudo-hyperbolic.*

*Proof.* A similar method shown in the Theorem 3.5 gives the result as desired. □

From the property shown above, we consider more simple expression of left- and right-$\mathcal{S}_\mathbb{C}$-differentiable functions. We let the idempotent elements, whose symbols is referred by [1], such as

$$j_+ = \frac{1+j}{2} \quad \text{and} \quad j_- = \frac{1-j}{2},$$

which satisfy

$$j_+^2 = j_+, \quad j_-^2 = j_-, \quad j_+ + j_- = 1 \quad \text{and} \quad j_+ - j_- = j.$$

*Remark* 3.7. From the above properties of $j_+$ and $j_-$, we have

$$ij_+ = j_- i \quad \text{and} \quad ij_- = j_+ i.$$

Then we can also represent

$$\begin{aligned} Z &= z_0 + z_1 i + z_2 j + z_3 ij \\ &= \frac{1}{2}\{2z_0 + 2z_1 i + 2z_2 j + 2z_3 ij + (z_0 j - z_0 j + z_2 - z_2 + z_1 j - z_1 j + z_3 - z_3)\} \\ &= (z_0 + z_2)\frac{1+j}{2} + (z_0 - z_2)\frac{1-j}{2} + i\Big\{(z_1 + z_3)\frac{1+j}{2} + (z_1 - z_3)\frac{1-j}{2}\Big\}.\end{aligned}$$

We put $\zeta_+ = z_0 + z_2$, $\zeta_- = z_0 - z_2$, $\eta_+ = z_1 + z_3$ and $\eta_- = z_1 - z_3$. Then

$$Z = \zeta_+ j_+ + \zeta_- j_- + i(\eta_+ j_+ + \eta_- j_-).$$

Also, for the differential operators, let

$$\partial_{\zeta_+} := \frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_2}, \qquad \partial_{\zeta_-} = \frac{\partial}{\partial z_0} - \frac{\partial}{\partial z_2},$$
$$\partial_{\eta_+} := \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_3}, \qquad \partial_{\eta_-} = \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_3}.$$

Then we have

$$D_Z^\dagger = \frac{1}{2}\Big\{(\partial_{\zeta_-} j_+ + \partial_{\zeta_+} j_-) + i(\partial_{\eta_-} j_+ + \partial_{\eta_+} j_-)\Big\}.$$

**Theorem 3.8.** *Let $F : U \subset \mathbb{C}^4 \to \mathcal{S}_{\mathbb{C}}$ be a function in $\mathcal{C}^1(U)$. Then $F$ has the following form:*

$$\begin{aligned} F(z_0, z_1, z_2, z_3) =& \{\varphi_0(\zeta_+, \eta_+) + \varphi_1(\zeta_-, \eta_-)\} + \{\varphi_2(\zeta_-, \eta_-) + \varphi_3(\zeta_+, \eta_+)\}i \\ &+ \{\varphi_0(\zeta_+, \eta_+) - \varphi_1(\zeta_-, \eta_-)\}j + \{\varphi_2(\zeta_-, \eta_-) - \varphi_3(\zeta_+, \eta_+)\}ij, \end{aligned}$$

*where $\varphi_r : U \to \mathbb{C}$ $(r = 0, 1, 2, 3)$ in $\mathcal{C}^1(U)$ if and only if $F$ satisfies $D_Z^\dagger F = 0$.*

*Proof.* We write the function $F = f_0 + f_1 i + f_2 j + f_3 ij$, where $f_0 = \varphi_0 + \varphi_1$, $f_1 = \varphi_2 + \varphi_3$, $f_2 = \varphi_0 - \varphi_1$ and $f_3 = \varphi_2 - \varphi_3$. Since $\varphi_r$ $(r = 0, 1, 2, 3)$ are continuously differentiable on $U$, they content the equations (3.5). So, the function $F$ satisfies the equations (2.1) and then $F$ assures the equation $D_Z^\dagger F = 0$.

Conversely, suppose that $F$ satisfies $D_Z^\dagger F = 0$. Let $\Phi_+ := f_0 + f_2$, $\Phi_- := f_0 - f_2$, $\Psi_+ := f_1 + f_3$ and $\Psi_- := f_1 - f_3$. Then we also write $F = (\Phi_+ j_+ + \Phi_- j_-) + i(\Psi_+ j_+ + \Psi_- j_-)$. Using the properties of $j_+$ and $j_-$, we see that the settings imply the function $F$ satisfies $D_Z^\dagger F = 0$ if and only if $F$ satisfies the following equations:

$$\begin{cases} \partial_{\zeta_-} \Phi_+ j_+ + \partial_{\zeta_+} \Phi_- j_- = \partial_{\eta_+} \Psi_+ j_+ + \partial_{\eta_-} \Psi_- j_-, \\ \partial_{\zeta_+} \Psi_+ j_+ + \partial_{\zeta_-} \Psi_- j_- = -(\partial_{\eta_-} \Phi_+ j_+ + \partial_{\eta_+} \Phi_- j_-). \end{cases}$$

We put $\Phi_+ = \varphi_0$, $\Phi_- = \varphi_1$, $\Psi_+ = \varphi_2$, and $\Psi_- = \varphi_3$. Then $\varphi_r$ $(r = 0, 1, 2, 3)$ satisfy (3.5) and they are continuously differentiable and construct a function such as

$$\begin{aligned} F(z_0, z_1, z_2, z_3) =& \{\varphi_0(\zeta_+, \eta_+) + \varphi_1(\zeta_-, \eta_-)\} + \{\varphi_2(\zeta_-, \eta_-) + \varphi_3(\zeta_+, \eta_+)\}i \\ &+ \{\varphi_0(\zeta_+, \eta_+) - \varphi_1(\zeta_-, \eta_-)\}j + \{\varphi_2(\zeta_-, \eta_-) - \varphi_3(\zeta_+, \eta_+)\}ij. \end{aligned}$$

Thus, the function $F$ has the form as desired. □

**Example 3.9.** Consider a split-biquaternionic-valued function

$$F(z_0, z_1, z_2, z_3) = z_1 z_2 z_3 - z_0 z_2 z_3 i + z_0 z_1 z_3 j + z_0 z_1 z_2 ij.$$

Then $F$ satisfies the equation $D_Z^\dagger F = 0$. However, if we write $F$ as in the above proof, then

$$F = \frac{(\eta_+^2 - \eta_-^2)(\zeta_+ j_+ - \zeta_- j_-)}{4} + i\frac{(\zeta_+^2 \zeta_-^2)(-\zeta_+ j_+ + \zeta_- j_-)}{4}.$$

Hence, we have

$$(\partial_{\eta_+} j_+ + \partial_{\eta_-} j_-)\left\{\frac{(\eta_+^2 - \eta_-^2)(\zeta_+ j_+ - \zeta_- j_-)}{4}\right\} = -2\eta_+\zeta_+ j_+ - 2\eta_-\zeta_- \neq 0.$$

Thus, $F$ is not of the form which has shown in Theorem 3.8.

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