



Characterizations of solution sets of set-valued generalized pseudoinvex optimization problems

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Abstract

We study the Stampacchia equilibrium-like problems in terms of normal subdifferential for set-valued maps and study their relations with set-valued optimization problems by the scalarization method. Characterizations of the solution sets of generalized pseudoinvex extremum problems are established. ©2016 all rights reserved.

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1. Introduction

Vector variational inequalities (VVI) have become important research directions in vector optimization problems. The concept of vector variational inequality (VVI) in finite-dimensional spaces was proposed and studied by Giannessi [11]. Then VVIs, generalizations and applications have been extensively considered and studied. Several authors have investigated the relationships between the VVI and one of vector optimization problem (VOP), vector complementarity problem, vector equilibrium problem, etc. For details we refer to the references [4–9, 29]. In [26] Santos et al. considered scalarized variational-like inequalities defined in terms of Clarke's generalized directional derivative using the scalarization method and proved that each of their solutions is a weak efficient solution of a vector optimization problem (VOP). Alshahrani et al. [1] extended the results in [26] and got some existence results for solutions of nonsmooth variational-like inequalities under dense pseudomonotonicity.

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Characterizations of the solution set are useful for knowing better the behavior of solution methods for programs that have multiple optimal solutions. We note that Mangasarian [17] initially presented several characterizations of the solution set of differentiable convex extremum problem and applied them to study monotonic linear complementarity problem. Jeyakumar and Yang [13] extended the above results to the case of differentiable pseudolinear programs. Furthermore, Yang [27] studied the minimization of a differentiable pseudoinvex function and presented some characterizations of the solution sets of pseudoinvex extremum problem. Liu et al. [16] established similar results for pseudoinvex programs with Dini directional derivative. Then, Ansari and Rezaie [2] derived some properties of a class of generalized pseudolinear functions and presented some characterizations of the solution set of a generalized pseudolinear optimization problem. Recently, Zhao et al. [31] provided a characterization for the solution sets of nondifferentiable optimization problems in terms of the basic properties of locally Lipschitz pseudoinvex functions by modifying some results in [18]. Huang and Yao [12] established several characterizations of the nonemptiness and compactness for the weakly efficient solution set of a convex vector set-valued optimization problem. Furthermore, by using the scalarization method, Oveisiha and Zafarani [24] considered Stampacchia variational-like inequalities in terms of normal subdifferential for set-valued maps and established their relationships with set-valued optimization problems. Also, they derived some characterizations of the solution sets of pseudoinvex extremum problems. Further some interesting results can be found in [3, 14, 15, 19, 22, 23].

In this present paper, we study the scalarized Stampacchia equilibrium-like problems (SELPs) and set-valued optimization problems. We will study Stampacchia equilibrium-like problems in terms of normal subdifferential for set-valued maps and study their relationships with set-valued optimization problems by employing this scalarized method. Characterizations of the solution set for optimization problem of a generalized K -pseudoinvex set-valued map will be presented. The paper is organized as follows. In Section 2, we give some basic definitions and preliminary results. In Section 3, we study the relationship between scalarized Stampacchia equilibrium-like problems (SELPs) and scalarized optimization problems (SOPs). In the final section, we will establish some characterizations of the solution set of set-valued generalized K -pseudoinvex and generalized K -pseudoconvex programs.

2. Preliminaries

Let X be a Banach space and x^* its topological dual space. The norm in X and x^* will be denoted by $\|\cdot\|$. We denote by $\langle \cdot, \cdot \rangle$, $[x, y]$, and (x, y) the duality pairing between X and x^* , the line segment for $x, y \in X$, and the interior of $[x, y]$, respectively. Furthermore, we define $[x, y)$ and $(x, y]$ to be $(x, y) \cup \{x\}$ and $(x, y) \cup \{y\}$, respectively. Next, we recall some concepts of subdifferentials and coderivatives that we will need in the following sections.

Definition 2.1. Let X be a normed vector space, Ω be a nonempty subset of X , $x \in \Omega$ and $\varepsilon > 0$. The set of ε -normals to Ω at x is

$$\widehat{N}_\varepsilon(x; \Omega) := \{x^* \in X^* : \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq \varepsilon\},$$

where $u \xrightarrow{\Omega} x$ means that u strongly converges to x with $u \in \Omega$.

If $\varepsilon = 0$, the above set is denoted by $\widehat{N}(x; \Omega)$ and called regular normal cone to Ω at x . Let $\bar{x} \in \Omega$, the basic normal cone to Ω at \bar{x} be

$$N(\bar{x}; \Omega) := \limsup_{x \rightarrow \bar{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x; \Omega).$$

Let $f : X \rightarrow \overline{\mathbf{R}}$ be finite at $\bar{x} \in X$. The regular (Fréchet) subdifferential and basic (limiting) subdifferential due to [24] of f at \bar{x} is defined by the following:

$$\widehat{\partial}f(\bar{x}) := \{x^* \in X^* : (x^*, -1) \in \widehat{N}((\bar{x}, f(\bar{x})); \text{epi}f)\},$$

$$\partial_L f(\bar{x}) := \{x^* \in X^* : (x^*, -1) \in N(\bar{x}, f(\bar{x}); \text{epi}f)\}.$$

If the Banach space X is Asplund, i.e., every continuous convex function defined on X is Fréchet differentiable on a dense set of points, we have

$$\partial_L f(\bar{x}) = \limsup_{x \xrightarrow{f} \bar{x}} \widehat{\partial} f(x),$$

where $x \xrightarrow{f} \bar{x}$ means that $x \rightarrow \bar{x}$ with $f(x) \rightarrow f(\bar{x})$.

We remark that Mean-value theorems are important and useful tools in nonsmooth analysis. We have the following mean-value theorem for limiting subdifferential.

Theorem 2.2 ([21]). *Let X be an Asplund space and f be Lipschitz continuous on an open set containing $[x, y]$ in X . Then there exist $c \in [x, y]$ and $x^* \in \partial_L f(c)$ such that*

$$\langle x^*, y - x \rangle \geq f(y) - f(x).$$

Let $\Omega \subseteq X$ be a nonempty set. The map $\eta : \Omega \times \Omega \rightarrow X$ is said to be skew if for all $x, y \in \Omega$,

$$\eta(x, y) + \eta(y, x) = 0.$$

Definition 2.3. Let x be an arbitrary point of Ω . The set Ω is said to be invex at x w.r.t. η if for all $y \in \Omega$,

$$x + t\eta(y, x) \in K \quad \text{for all } t \in [0, 1].$$

Ω is said to be invex w.r.t. η if Ω is invex at every point $x \in \Omega$ w.r.t. η .

Throughout this section, unless otherwise specified, we assume that $\Omega \subseteq X$ is an invex set w.r.t. $\eta : \Omega \times \Omega \rightarrow X$. Inspired by Theorem 2.2, we give the following definition of the mean-value condition for limiting subdifferential $\partial_L f$ w.r.t. ϕ .

Definition 2.4. Let X be an Asplund space, $\Omega \subseteq X$ be invex w.r.t. η and $\phi : X^* \times \Omega \times \Omega \rightarrow \mathbf{R}$. Let x and y be points in Ω and suppose that $f : \Omega \rightarrow \mathbf{R}$ is Lipschitz continuous on an open set containing the line segment $[x, y]$. Then f is said to satisfy the mean-value condition for limiting subdifferential $\partial_L f$ w.r.t. ϕ if there exist $z \in [x, y]$ and $\xi^* \in \partial_L f(z)$ such that

$$\phi(\xi^*, x, y) \geq f(y) - f(x).$$

A set-valued mapping $F : X \rightarrow 2^Y$ between Banach spaces with the range space Y partially ordered by a nonempty, closed and convex cone K is given. Denoting the ordering relation on Y by “ \leq_K ”, we have

$$y_1 \leq_K y_2 \quad \text{if and only if} \quad y_2 - y_1 \in K.$$

Let $\text{dom}F := \{x \in X : F(x) \neq \emptyset\}$, $\text{gr}F := \{(x, y) : x \in \text{dom}F, y \in F(x)\}$, and $\text{epi}F := \{(x, y) : x \in X, y \in F(x) + K\}$.

Definition 2.5 ([21]). Let $F : X \rightarrow 2^Y$ be a set-valued mapping between Banach spaces and $(\bar{x}, \bar{y}) \in \text{gr}F$. Then the Fréchet coderivative of F at (\bar{x}, \bar{y}) is the set-valued mapping $\widehat{D}^*F(\bar{x}, \bar{y}) : Y^* \rightarrow 2^{X^*}$ given by

$$\widehat{D}^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* : (x^*, -y^*) \in \widehat{N}((\bar{x}, \bar{y}); \text{gr}F)\},$$

and furthermore, the normal coderivative of F at (\bar{x}, \bar{y}) is the set-valued mapping $D_N^*F(\bar{x}, \bar{y}) : Y^* \rightarrow 2^{X^*}$ given by

$$D_N^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* : (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gr}F)\}.$$

If $F = f : X \rightarrow Y$ is single-valued, we denote its Fréchet and normal coderivatives at $(\bar{x}, f(\bar{x}))$ by $\widehat{D}^*f(\bar{x})$ and $D_N^*f(\bar{x})$, respectively.

By employing the coderivative of the epigraphical multifunction, Bao and Mordukhovich [3] defined appropriate extensions of the subdifferential notion from extended-real-valued functions to vector-valued and set-valued maps with values in partially ordered spaces.

Definition 2.6 ([3]). Let $F : X \rightarrow 2^Y$ be a set-valued mapping. Then the epigraphical multifunction $E_F : X \rightarrow 2^Y$ is defined by

$$E_F(x) := \{y \in Y : y \in F(x) + K\}.$$

The Fréchet and normal subdifferentials of F at the point $(\bar{x}, \bar{y}) \in \text{epi}F$ in the direction $y^* \in Y^*$ are defined, respectively, by

$$\begin{aligned}\widehat{\partial}F(\bar{x}, \bar{y})(y^*) &:= \widehat{D}^*E_F(\bar{x}, \bar{y})(y^*), \\ \partial F(\bar{x}, \bar{y})(y^*) &:= D_N^*E_F(\bar{x}, \bar{y})(y^*).\end{aligned}$$

Definition 2.7. Let $F : \Omega \subseteq X \rightarrow 2^Y$ with $\text{dom}F \neq \emptyset$ and B_Y be the closed unit ball of Y .

(i) F is said to be K -invex w.r.t. η if for any $x, y \in \Omega$ and any $t \in [0, 1]$ one has

$$(1-t)F(x) + tF(y) \subseteq F(x + t\eta(y, x)) + K.$$

In particular, a single-valued function $F : \Omega \rightarrow Y$ is said to be K -invex w.r.t. η if for any $x, y \in \Omega$ and any $t \in [0, 1]$,

$$F(x + t\eta(y, x)) \leq_K (1-t)F(x) + tF(y).$$

(ii) F is said to be Lipschitz around $\bar{x} \in \text{dom}F$ [25] if there are a neighborhood U of \bar{x} and $\ell \geq 0$ such that

$$F(x) \subseteq F(u) + \ell\|x - u\|B_Y \quad \text{for all } x, u \in \Omega \cap U.$$

(iii) F is said to be epi-Lipschitz around $\bar{x} \in \text{dom}F$ [25] if E_F is Lipschitz around this point.

Let K be a closed and convex pointed cone in Y , then we define K^+ by

$$K^+ := \{y^* \in Y^* : \langle y^*, k \rangle \geq 0 \quad \forall k \in K\}.$$

We associate with F and $y^* \in Y^*$ the marginal function

$$f_{y^*}(x) := \inf\{y^*(y) : y \in F(x)\},$$

and the minimum set

$$M_{y^*}(x) := \{y \in F(x) : f_{y^*}(x) = y^*(y)\}.$$

Throughout the rest of this paper, we assume that $\text{gr}F$ is closed, and for all $x \in \text{dom}F$ and $y^* \in K^+$, $M_{y^*}(x)$ is nonempty.

Lemma 2.8 ([23]). Suppose that $F : \Omega \subseteq X \rightarrow 2^Y$ is a set-valued map and $\bar{x} \in \text{dom}F$. If F is epi-Lipschitz around \bar{x} and $y^* \in K^+$, then the scalar-function f_{y^*} is locally Lipschitz at \bar{x} .

Theorem 2.9 ([23]). Let X, Y be Asplund spaces, $F : X \rightarrow 2^Y$ and $y^* \in K^+$. Suppose that $\bar{x} \in \text{dom}F$ and $\bar{y} \in M_{y^*}(\bar{x})$.

- (i) If F is Lipschitz around \bar{x} , then $\partial_L f_{y^*}(\bar{x}) \subseteq D_N^*F(\bar{x}, \bar{y})(y^*)$.
- (ii) If F is epi-Lipschitz around \bar{x} , then $\partial_L f_{y^*}(\bar{x}) \subseteq \partial F(\bar{x}, \bar{y})(y^*)$.

Definition 2.10. Let $\phi : X^* \times \Omega \times \Omega \rightarrow \mathbf{R}$ be a function, and $f : \Omega \rightarrow \mathbf{R}$ be a locally Lipschitz function.

(a) f is said to be generalized pseudoinvex w.r.t. ϕ on Ω if for any $x, y \in \Omega$ and any $\xi \in \partial_L f(x)$, one has

$$\phi(\xi, x, y) \geq 0 \quad \Rightarrow \quad f(y) \geq f(x).$$

(b) $\partial_L f$ is said to be invariant pseudomonotonic w.r.t. ϕ on Ω if for any $x, y \in \Omega$ and any $\zeta \in \partial_L f(x), \xi \in \partial_L f(y)$, one has

$$\phi(\zeta, x, y) \geq 0 \quad \Rightarrow \quad \phi(\xi, y, x) \leq 0.$$

(c) f is said to be prequasiinvex w.r.t. η on Ω if for any $x, y \in \Omega$ and any $t \in [0, 1]$, one has

$$f(x + t\eta(y, x)) \leq \max\{f(x), f(y)\}.$$

Remark 2.11. If we put $\phi(\xi, x, y) = \langle \xi, \eta(y, x) \rangle$ for all $(\xi, x, y) \in X^* \times \Omega \times \Omega$, then Definition 2.10 (a) and (b) reduce to Definition 2.6 in [24], i.e., the pseudoinvexity of f w.r.t. η and invariant pseudomonotonicity of $\partial_L f$ w.r.t. η , respectively.

Inspired by Condition C in [19], we introduce the new one, which will be used in the sequel.

Condition C. Let $\Omega \subseteq X$ be an invex set w.r.t. $\eta : \Omega \times \Omega \rightarrow X$. Then η is said to satisfy Condition C w.r.t. $\phi : X^* \times \Omega \times \Omega \rightarrow \mathbf{R}$ if for all $x, y \in \Omega$ and $t \in [0, 1]$,

(a) $\eta(x, x + t\eta(y, x)) = -t\eta(y, x)$ and $\phi(\xi, x + t\eta(y, x), x) = -t\phi(\xi, x, y), \forall \xi \in X^*$;

(b) $\eta(y, x + t\eta(y, x)) = (1 - t)\eta(y, x)$ and $\phi(\xi, x + t\eta(y, x), y) = (1 - t)\phi(\xi, x, y), \forall \xi \in X^*$.

Remark 2.12. Obviously, if we put $\eta(y, x) = y - x$ and $\phi(\xi, x, y) = \langle \xi, \eta(y, x) \rangle$ for all $(\xi, x, y) \in X^* \times \Omega \times \Omega$, then η satisfies Condition C w.r.t. ϕ , where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X^* and X . Moreover, it can be easily seen that

$$\eta(x + t\eta(y, x), x) = t\eta(y, x) \quad \text{for all } t \in [0, 1] \text{ and all } x, y \in \Omega.$$

Let \mathcal{H} be a Hausdorff metric on the collection $CB(X)$ of all nonempty, closed and bounded subsets of a normed space X , induced by a metric d in terms of $d(a, b) = \|a - b\|$, which is defined by

$$\mathcal{H}(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\right\}$$

for A and B in $CB(X)$. Note that (see [22]) if A and B are compact sets in X , then for each $a \in A$, there exists $b \in B$ such that $\|a - b\| \leq \mathcal{H}(A, B)$.

Definition 2.13. Let X and Y be two real Banach spaces and $K \subseteq X$ be an invex set w.r.t. η . A compact-valued multifunction $T : K \rightarrow 2^{\mathcal{L}(X, Y)}$ is said to be \mathcal{H} -hemicontinuous if the mapping $t \mapsto T(x + t\eta(y, x))$ is continuous at 0^+ , where $\mathcal{L}(X, Y)$ is the collection of all continuous linear operators of X into Y and $CB(\mathcal{L}(X, Y))$ is equipped with the metric topology induced by \mathcal{H} .

Motivated by Theorem 2.3 in [24], we now state and prove the following result.

Theorem 2.14. Let X be an Asplund space, $\eta : \Omega \times \Omega \rightarrow X$ be continuous in the second variable such that Condition C w.r.t. ϕ holds, and $f : \Omega \rightarrow \mathbf{R}$ be locally Lipschitz and pre-quasiinvex w.r.t. η . Suppose the following conditions are satisfied:

(i) f satisfies the mean-value condition for limiting subdifferential $\partial_L f$ w.r.t. ϕ ;

(ii) $\partial_L f : \Omega \rightarrow 2^{X^*}$ is \mathcal{H} -hemicontinuous with compact values;

(iii) for each $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous.

If f is generalized pseudoinvex w.r.t. ϕ , then $\partial_L f$ is invariant pseudomonotonic w.r.t. ϕ .

Proof. We first claim that for any $x, y \in \Omega$, $x \neq y$ and any $\zeta \in \partial_L f(x)$,

$$\phi(\zeta, x, y) > 0 \quad \Rightarrow \quad f(y) > f(x).$$

Indeed, let $\phi(\zeta, x, y) > 0$ for all $\zeta \in \partial_L f(x)$. Then, we choose sequences $\{x_n\} \subseteq \Omega$ and $\{t_n\} \subset (0, 1)$ such that $x_n \rightarrow x$ and $t_n \rightarrow 0^+$. Utilizing the mean-value condition of f for limiting subdifferential $\partial_L f$ w.r.t. ϕ , from Condition C we know that for each $t_n \in (0, 1)$, there exist $t'_n \in (0, t_n]$ and $\xi_n^* \in \partial_L f(x_n + t'_n \eta(y, x_n))$ such that

$$t_n \phi(\xi_n^*, x_n, y) = \phi(\xi_n^*, x_n, x_n + t_n \eta(y, x_n)) \leq f(x_n + t_n \eta(y, x_n)) - f(x_n). \quad (2.1)$$

Also, by Nadler's result [26], there exists $\zeta_n \in \partial_L f(x)$ such that

$$\|\xi_n^* - \zeta_n\| \leq \mathcal{H}(\partial_L f(x_n + t'_n \eta(y, x_n)), \partial_L f(x)).$$

Since $\eta : \Omega \times \Omega \rightarrow X$ is continuous in the second variable and $\partial_L f : \Omega \rightarrow 2^{X^*}$ is \mathcal{H} -hemicontinuous with nonempty compact values, we know that

$$\|x_n + t'_n \eta(y, x_n) - x\| \leq \|x_n - x\| + t'_n \|\eta(y, x_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and hence

$$\|\xi_n^* - \zeta_n\| \leq \mathcal{H}(\partial_L f(x_n + t'_n \eta(y, x_n)), \partial_L f(x)) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.2)$$

From the compactness of $\partial_L f(x)$, without loss of generality we may assume that $\zeta_n \rightarrow \zeta^* \in \partial_L f(x)$. So, from (2.2) it follows that $\xi_n^* \rightarrow \zeta^*$, which together with $\phi(\zeta, x, y) > 0$ for all $\zeta \in \partial_L f(x)$, leads to $\phi(\zeta^*, x, y) > 0$. Note that $x_n \rightarrow x$ and $\xi_n^* \rightarrow \zeta^*$. Since $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous, we deduce that $\{\phi(\xi_n^*, x_n, y)\}$ converges to $\phi(\zeta^*, x, y)$, and thus there exists $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$, $\phi(\xi_n^*, x_n, y) > 0$. Consequently, by using (2.1) we obtain $f(x_n + t_n \eta(y, x_n)) > f(x_n)$ for all $n \geq n_0$. The pre-quasiinvexity of f w.r.t. η together with the previous inequality implies that for any $t \in (0, 1)$, we have

$$f(x_n + t \eta(y, x_n)) \leq \max\{f(y), f(x_n)\} = f(y).$$

Hence, by the continuity of f and the function $x \mapsto \eta(y, x)$, we conclude that $f(x + t \eta(y, x)) \leq f(y)$, for all $t \in (0, 1)$. Again in terms of Condition C, for each $t \in (0, 1)$ there exist $\theta_t \in (0, t]$ and $\xi_t^* \in \partial_L f(x + \theta_t \eta(y, x))$ such that

$$t \phi(\xi_t^*, x, y) = \phi(\xi_t^*, x, x + t \eta(y, x)) \leq f(x + t \eta(y, x)) - f(x).$$

By using the argument similar to that of $\phi(\xi_n^*, x_n, y) > 0$ for all $n \geq n_0$, we can deduce that $\phi(\xi_t^*, x, y) > 0$ for $t \in (0, 1)$ sufficiently small. Consequently, we have

$$0 < t \phi(\xi_t^*, x, y) = \phi(\xi_t^*, x, x + t \eta(y, x)) \leq f(x + t \eta(y, x)) - f(x) \leq f(y) - f(x),$$

which immediately yields $f(y) > f(x)$.

Next assume to the contrary that there exist $x, y \in K$, $\zeta \in \partial_L f(x)$ and $\xi \in \partial_L f(y)$ such that

$$\phi(\zeta, x, y) \geq 0 \quad \text{but} \quad \phi(\xi, y, x) > 0. \quad (2.3)$$

Since f is generalized pseudoinvex w.r.t. ϕ , by using the first inequality in (2.3), we have $f(y) \geq f(x)$. On the other hand, using the above proven assertion and the second inequality in (2.3), we get $f(x) > f(y)$, which is a contradiction. Therefore, $\partial_L f$ is invariant pseudomonotone w.r.t. ϕ . \square

Theorem 2.15. *Suppose conditions (ii), (iii) in Theorem 2.14 are replaced by the following ones:*

- (ii) $\partial_L f : \Omega \rightarrow 2^{X^*}$ is locally bounded and has closed graph;

(iii) for any $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous in the product topology $w^* \times \tau$, where w^* is the weak* topology in X^* and τ is the norm topology in X .

If other conditions in Theorem 2.14 are not changed, then $\partial_L f$ is invariant pseudomonotonic w.r.t. ϕ .

Proof. Repeating the same argument as in the proof of Theorem 2.14, we deduce that (2.1) holds. Since $\partial_L f$ is locally bounded, there exist a neighborhood of x and a constant $\ell > 0$ such that, for each z in this neighborhood and $\xi \in \partial_L f(z)$, we have $\|\xi\| \leq \ell$. Since $\eta : \Omega \times \Omega \rightarrow X$ is continuous in the second variable, $x_n \rightarrow 0$ and $t'_n \leq t_n \rightarrow 0$ as $n \rightarrow \infty$, we know that $x_n + t'_n \eta(y, x_n) \rightarrow x$ as $n \rightarrow \infty$. This means that for n sufficiently large $\|\xi_n^*\| \leq \ell$. Hence, we may assume, without loss of generality, that $\xi_n^* \xrightarrow{w^*} \zeta^*$. Since the set-valued mapping $\partial_L f(\cdot)$ has closed graph, we get $\zeta^* \in \partial_L f(x)$. Note that for any $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous in the product topology $w^* \times \tau$. So, from $\phi(\zeta, x, y) > 0$ for all $\zeta \in \partial_L f(x)$, it follows that

$$\lim_{n \rightarrow \infty} \phi(\xi_n^*, x_n, y) = \phi(\zeta^*, x, y) > 0.$$

Since the rest of the proof is the same as in the proof of Theorem 2.14, and we omit it. \square

3. Some relations between SELPs and SOPs

In this section, we will establish some relations between Stampacchia equilibrium-like problems and scalarized set-valued optimization problems.

Let $F : X \rightarrow 2^Y$ be a set-valued map between Banach spaces. We consider the following set-valued optimization problem

$$\min F(x), \quad x \in \Omega \subseteq X. \quad (3.1)$$

Definition 3.1 ([10]). A point \bar{x} is said to be a weakly efficient solution of problem (3.1) if there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(\Omega) - \bar{y}) \cap -\text{int}K = \emptyset.$$

We consider the concept of scalarized solution of problem (3.1).

A vector \bar{x} is said to be a scalarized solution of problem (3.1) (\bar{x} is a solution of the SOP) if, for any $y^* \in K^+ \setminus \{0\}$, there exists $\bar{y} \in F(\bar{x})$ such that

$$y^*(\bar{y}) \leq y^*(y) \quad \text{for all } y \in F(\Omega).$$

Next, we consider the following scalarized Stampacchia equilibrium-like problem (SELP) which is to find a vector $\bar{x} \in \Omega$ such that, for any $x \in \Omega$ and $y^* \in K^+ \setminus \{0\}$, there exist $\bar{y} \in M_{y^*}(\bar{x})$ and $x^* \in \partial F(\bar{x}, \bar{y})(y^*)$ such that

$$\phi(x^*, \bar{x}, x) \geq 0,$$

where $\phi : X^* \times \Omega \times \Omega \rightarrow \mathbf{R}$.

Particularly, if we put $\phi(\xi, x, y) = \langle \xi, \eta(y, x) \rangle$ for all $(\xi, x, y) \in X^* \times \Omega \times \Omega$, then the SELP reduces to the SVLI considered in [24]. In this case, if $F = f : X \rightarrow Y$ is a vector-valued function, this nonsmooth variational-like inequality was studied by Santos et al. [26] and Alshahrani et al. [1].

Definition 3.2. Suppose that $F : \Omega \subseteq X \rightarrow 2^Y$. F is said to be generalized K -pseudoinvex w.r.t. $\phi : X^* \times \Omega \times \Omega \rightarrow \mathbf{R}$ if, for any $x_1, x_2 \in \Omega$, $y^* \in K^+ \setminus \{0\}$, $y_1 \in M_{y^*}(x_1)$, $y_2 \in M_{y^*}(x_2)$, and $\xi_1 \in \partial F(x_1, y_1)(y^*)$, one has

$$\phi(\xi_1, x_1, x_2) \geq 0 \quad \Rightarrow \quad y^*(y_2) \geq y^*(y_1).$$

Remark 3.3.

- (i) If $F = f : \Omega \subseteq X \rightarrow \mathbf{R}$ is a real-valued function and $K = \mathbf{R}_+$, then the generalized K -pseudoinvexity w.r.t. ϕ reduces to the generalized pseudoinvexity w.r.t. ϕ in Definition 2.10. In addition, if we put $\phi(\xi, x, y) = \langle \xi, \eta(y, x) \rangle$ for all $(\xi, x, y) \in X^* \times \Omega \times \Omega$, then Definition 3.2 reduce to Definition 3.2 in [7], i.e., the K -pseudoinvexity w.r.t. η .
- (ii) If $F : \Omega \subseteq X \rightarrow 2^Y$ is epi-Lipschitz and generalized K -pseudoinvex w.r.t. ϕ , then Theorem 2.9 implies that, for any $y^* \in K^+ \setminus \{0\}$, f_{y^*} is generalized pseudoinvex w.r.t. ϕ .
- (iii) If we put $\phi(\xi, x, y) = \langle \xi, y - x \rangle$ for all $(\xi, x, y) \in X^* \times \Omega \times \Omega$, then F is said to be K -pseudoinvex.

Lemma 3.4 ([24]). *Every solution of the SOP is a weakly efficient solution of problem (3.1).*

The following result shows that a solution of the SELP is also a weakly efficient solution of problem (3.1).

Proposition 3.5. *Let $F : \Omega \subseteq X \rightarrow 2^Y$ be generalized K -pseudoinvex w.r.t. ϕ . If \bar{x} is a solution of the SELP, then it is a solution of the SOP and hence, a weakly efficient solution of problem (3.1).*

Proof. Suppose that \bar{x} is a solution of the SELP, but not a solution of the SOP. Then there exists $y^* \in K^+ \setminus \{0\}$ such that, for any $\bar{y} \in F(\bar{x})$,

$$y^*(y) < y^*(\bar{y}) \quad \text{for some } y \in F(x), x \in \Omega. \quad (3.2)$$

Since $\bar{x} \in \Omega$ is a solution of the SELP, there exist \bar{y} and x^* such that $\bar{y} \in M_{y^*}(\bar{x})$, $x^* \in \partial F(\bar{x}, \bar{y})(y^*)$ and

$$\phi(x^*, \bar{x}, x) \geq 0 \quad \text{for all } x \in \Omega, y^* \in K^+ \setminus \{0\}.$$

Now, the generalized K -pseudoinvex of F w.r.t. ϕ implies that

$$y^*(y) \geq y^*(\bar{y}) \quad \text{for all } y \in M_{y^*}(x).$$

Therefore, we obtain

$$y^*(y) \geq y^*(\bar{y}) \quad \text{for all } y \in F(x),$$

which contradicts (3.2). Hence, \bar{x} is a solution of the SOP, and from Lemma 3.4 we deduce that \bar{x} is a weakly efficient solution of (3.1). \square

Theorem 3.6. *Let X, Y be Asplund spaces and $F : \Omega \subseteq X \rightarrow 2^Y$ be epi-Lipschitz. Let $\eta : \Omega \times \Omega \rightarrow X$ be continuous in the second variable such that Condition C w.r.t. ϕ holds. Suppose the following conditions are satisfied:*

- (i) for each $y^* \in K^+ \setminus \{0\}$, f_{y^*} satisfies the mean-value condition for limiting subdifferential $\partial_L f_{y^*}$ w.r.t. ϕ ;
- (ii) $\partial_L f_{y^*}(\cdot) : \Omega \rightarrow 2^{X^*}$ is \mathcal{H} -hemicontinuous with compact values;
- (iii) for each $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous.

If \bar{x} is a solution of the SOP, then it is a solution of the SELP.

Proof. For any $x \in \Omega$ fixed and $\lambda \in (0, 1]$, set $x(\lambda) = \bar{x} + \lambda\eta(x, \bar{x})$. Since \bar{x} is a solution of the SOP, for any $y^* \in K^+ \setminus \{0\}$, there exists $\bar{y} \in F(\bar{x})$ such that $y^*(\bar{y}) \leq y^*(y)$ for all $y \in F(z)$ and $z \in \Omega$. Therefore, $f_{y^*}(\bar{x}) \leq f_{y^*}(z)$. Since F is epi-Lipschitz, by Lemma 2.8, for any $y^* \in K^+ \setminus \{0\}$, f_{y^*} is Lipschitz continuous on Ω . Also, since η satisfies Condition C w.r.t. ϕ and f_{y^*} satisfies the mean-value condition for limiting subdifferential $\partial_L f_{y^*}$ w.r.t. ϕ , there exist $t \in [0, \lambda)$ and $\xi_t \in \partial_L f_{y^*}(x(t))$ such that

$$\lambda\phi(\xi_t, \bar{x}, x) = \phi(\xi_t, \bar{x}, x(\lambda)) \geq f_{y^*}(x(\lambda)) - f_{y^*}(\bar{x}) \geq 0.$$

Again from Condition C we get

$$\phi(\xi_t, x(t), x) = (1 - t)\phi(\xi_t, \bar{x}, x),$$

and hence

$$\phi(\xi_t, x(t), x) \geq 0. \quad (3.3)$$

Since $\partial_L f_{y^*}(\cdot) : \Omega \rightarrow 2^{X^*}$ is compact-valued, by Nadler's result we know that for each $\xi_t \in \partial_L f_{y^*}(x(t))$ there exists $\tilde{\xi}_t \in \partial_L f_{y^*}(\bar{x})$ such that

$$\|\xi_t - \tilde{\xi}_t\| \leq \mathcal{H}(\partial_L f_{y^*}(x(t)), \partial_L f_{y^*}(\bar{x})).$$

Since $x(t) \rightarrow \bar{x}$ as $t \rightarrow 0$ and $\partial_L f_{y^*}(\cdot) : \Omega \rightarrow 2^{X^*}$ is \mathcal{H} -hemicontinuous, it follows that

$$\|\xi_t - \tilde{\xi}_t\| \leq \mathcal{H}(\partial_L f_{y^*}(x(t)), \partial_L f_{y^*}(\bar{x})) \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

Note that the net $\{\tilde{\xi}_t\}$ lies in the compact set $\partial_L f_{y^*}(\bar{x})$. So, we may assume, without loss of generality, that $\tilde{\xi}_t \rightarrow \bar{\xi} \in \partial_L f_{y^*}(\bar{x})$ as $t \rightarrow 0$. This together with $\|\xi_t - \tilde{\xi}_t\| \rightarrow 0$, implies that $\xi_t \rightarrow \bar{\xi} \in \partial_L f_{y^*}(\bar{x})$ as $t \rightarrow 0$. Since $\phi(\cdot, \cdot, x) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous, $\xi_t \rightarrow \bar{\xi}$ and $x(t) \rightarrow \bar{x}$ as $t \rightarrow 0$, we conclude from (3.3) that

$$\phi(\bar{\xi}, \bar{x}, x) \geq 0.$$

Now, by Theorem 2.9 we obtain $\bar{\xi} \in \partial F(\bar{x}, \bar{y})(y^*)$ and therefore, \bar{x} is a solution of the SELP. \square

Theorem 3.7. *Suppose conditions (ii), (iii) in Theorem 2.14 are replaced by the one:*

(ii) *for any $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous in the $w^* \times \tau$ -topology (see Theorem 2.14). Assume other conditions in Theorem 3.6 are not changed. If \bar{x} is a solution of the SOP, then it is a solution of the SELP.*

Proof. Repeating the same argument as in the proof of Theorem 3.6, we deduce that (3.3) holds. Since $\partial_L f_{y^*}(\cdot)$ is locally bounded (due to Corollary 1.81 in [21]), there exists a neighborhood of \bar{x} and a constant $\ell > 0$ such that, for each z in this neighborhood and $\xi \in \partial_L f_{y^*}(z)$, we have $\|\xi\| \leq \ell$. Since $x(t) \rightarrow \bar{x}$ as $t \rightarrow 0$, for t sufficiently small $\|\xi_t\| \leq \ell$; hence, without loss of generality we may assume that $\xi_t \xrightarrow{w^*} \bar{\xi}$. Since the set-valued mapping $\partial_L f_{y^*}(\cdot)$ has closed graph, we have $\bar{\xi} \in \partial_L f_{y^*}(\bar{x})$. Since $\phi(\cdot, \cdot, x) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous in the product topology $w^* \times \tau$, from (3.3) we get

$$\phi(\bar{\xi}, \bar{x}, x) \geq 0.$$

Now, by Theorem 2.9 we obtain $\bar{\xi} \in \partial F(\bar{x}, \bar{y})(y^*)$ and therefore, \bar{x} is a solution of the SELP. \square

If we let $F = f : X \rightarrow \mathbf{R}^n$, $K = \mathbf{R}_+^n$ and $f_i : X \rightarrow \mathbf{R}$, the components of f which are non-differentiable functions. The definition of limiting subdifferential can be extended to real vector-valued functions. The generalized limiting subdifferential of f at $x \in X$ is the set

$$\partial_L f(x) = \partial_L f_1(x) \times \partial_L f_2(x) \times \cdots \times \partial_L f_n(x).$$

In the rest of this section, let $\Omega \subseteq X$ be invex w.r.t. $\eta : \Omega \times \Omega \rightarrow X$ and $f : \Omega \rightarrow \mathbf{R}^n$. Let $\Phi : \mathcal{L}(X, \mathbf{R}^n) \times \Omega \times \Omega \rightarrow \mathbf{R}^n$ be an equilibrium-like function, that is, $\Phi(u, x, y) + \Phi(u, y, x) = 0$ for all $(u, x, y) \in \mathcal{L}(X, \mathbf{R}^n) \times \Omega \times \Omega$, where $\mathcal{L}(X, \mathbf{R}^n)$ denotes the family of all continuous linear operators from X into \mathbf{R}^n .

In the following theorems, we derive similar results for real vector-valued functions.

Theorem 3.8. *Let X be an Asplund space and $\Omega \subseteq X$ be invex w.r.t. η that is continuous in the second variable and satisfies Condition C w.r.t. each $\phi_i, i = 1, \dots, n$, where $\Phi(\xi, x, y) = (\phi_1(\xi_1, x, y), \dots, \phi_n(\xi_n, x, y))$ for all $(\xi, x, y) \in \mathcal{L}(X, \mathbf{R}^n) \times \Omega \times \Omega$ with $\xi = (\xi_1, \dots, \xi_n)$. Let $f = (f_1, \dots, f_n) : \Omega \rightarrow \mathbf{R}^n$ be locally Lipschitz on Ω and suppose for $i = 1, \dots, n$ that*

- (i) *each f_i satisfies the mean-value condition for limiting subdifferential $\partial_L f_i$ w.r.t. ϕ_i ;*
- (ii) *each $\partial_L f_i : \Omega \rightarrow 2^{X^*}$ is \mathcal{H} -hemicontinuous with compact values;*

(iii) $\phi_i(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous for each $y \in \Omega$.

If \bar{x} is a weakly efficient solution to problem (3.1), then, for any $x \in \Omega$, one has

$$\Phi(\partial_L f(\bar{x}), \bar{x}, x) \not\subseteq -\text{int}K.$$

Proof. For any $x \in \Omega$ fixed and $\lambda \in (0, 1]$, set $x(\lambda) = \bar{x} + \lambda\eta(x, \bar{x})$. Since \bar{x} is a weakly efficient solution to problem (3.1), we can find sequence $\lambda_j \downarrow 0^+$ and $i \in \{1, \dots, n\}$ such that

$$f_i(x(\lambda_j)) \geq f_i(\bar{x}) \quad \text{for all } j \in \mathbf{N}.$$

Now, by the similar argument to that in the proof of Theorem 3.6, we can obtain a $\bar{\xi}_i \in \partial_L f_i(\bar{x})$ for all $i \in \{1, \dots, n\}$ such that $\phi_i(\bar{\xi}_i, \bar{x}, x) \geq 0$. Hence, $\Phi(\partial_L f(\bar{x}), \bar{x}, x) \not\subseteq -\text{int}K$. \square

Theorem 3.9. Suppose conditions (ii), (iii) in Theorem 3.8 are replaced by the following ones:

- (ii) each $\partial_L f_i : \Omega \rightarrow 2^{X^*}$ is locally bounded and has closed graph;
- (iii) $\phi_i(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous in the $w^* \times \tau$ -topology for each $y \in \Omega$.

Assume other conditions in Theorem 3.8 are not changed. If \bar{x} is a weakly efficient solution to problem (3.1), then, for any $x \in \Omega$, one has

$$\Phi(\partial_L f(\bar{x}), \bar{x}, x) \not\subseteq -\text{int}K.$$

Proof. Since the proof is similar to that of Theorem 3.8, we omit it. \square

Remark 3.10. Theorems 3.8 and 3.9 extend, improve and develop Theorem 3.2 in [24] for equilibrium-like function and Condition C for η w.r.t. each ϕ_i . However, Theorem 3.2 in [24] improves Theorem 5.2 in [25] for limiting subdifferential without any generalized convexity.

4. Some characterizations of the solution sets

In this final section, we consider that \bar{S} to be the set of all scalarized solutions of problem (3.1) and assume that \bar{S} is nonempty. We will derive some characterizations of the solution sets of generalized K -pseudoinvex program.

Theorem 4.1. Let X, Y be Asplund spaces and $F : \Omega \subseteq X \rightarrow 2^Y$ be epi-Lipschitz. Let $\eta : \Omega \times \Omega \rightarrow X$ be continuous in the second variable such that Condition C w.r.t. ϕ holds. Suppose that for each $y^* \in K^+ \setminus \{0\}$, f_{y^*} is pre-quasiinvex w.r.t. η and satisfies the mean-value condition for limiting subdifferential $\partial_L f_{y^*}$ w.r.t. ϕ and $\partial_L f_{y^*}(\cdot) : \Omega \rightarrow 2^{X^*}$ is \mathcal{H} -hemicontinuous with compact values, and that for each $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous. If F is generalized K -pseudoinvex w.r.t. ϕ and x_1, x_2 are solutions of the SOP, then, for any $y^* \in K^+ \setminus \{0\}$, there exist $y_i \in M_{y^*}(x_i)$ and $\xi_i \in \partial F(x_i, y_i)(y^*)$, $i = 1, 2$, such that

$$\phi(\xi_1, x_1, x_2) = \phi(\xi_2, x_2, x_1) = 0.$$

Proof. Since x_1, x_2 are solutions of the SOP, by the proof of Theorem 3.6, for any $y^* \in K^+ \setminus \{0\}$, there exist $y_i \in M_{y^*}(x_i)$ and $\xi_i \in \partial_L f_{y^*}(x_i) \subseteq \partial F(x_i, y_i)(y^*)$, $i = 1, 2$ such that

$$\phi(\xi_1, x_1, x_2) \geq 0 \quad \text{and} \quad \phi(\xi_2, x_2, x_1) \geq 0. \quad (4.1)$$

Now, by Remark 3.3, generalized K -pseudoinvexity of F w.r.t. ϕ implies that for any $y^* \in K^+ \setminus \{0\}$, f_{y^*} is generalized pseudoinvex w.r.t. ϕ . Thus, from Lemma 2.8 and Theorem 2.14, we can deduce that $\partial_L f_{y^*}$ is invariant pseudomonotonic. Therefore,

$$\phi(\xi, x_1, x_2) \leq 0 \quad \text{and} \quad \phi(\xi', x_2, x_1) \leq 0, \quad (4.2)$$

for all $\xi \in \partial_L f_{y^*}(x_1)$ and $\xi' \in \partial_L f_{y^*}(x_2)$. Hence, by relations (4.1) and (4.2) and using this fact that $\partial_L f_{y^*}(x_i) \subseteq \partial F(x_i, y_i)(y^*)$, $i = 1, 2$, we can obtain the result. \square

Theorem 4.2. Let X, Y be Asplund spaces and $F : \Omega \subseteq X \rightarrow 2^Y$ be epi-Lipschitz and K -invex w.r.t. η . Let η be continuous in the second variable such that Condition C w.r.t. ϕ holds. Suppose that for each $y^* \in K^+ \setminus \{0\}$, f_{y^*} satisfies the mean-value condition for limiting subdifferential $\partial_L f_{y^*}$ w.r.t. ϕ , and that for each $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous in the $w^* \times \tau$ -topology. If F is generalized K -pseudoinvex w.r.t. ϕ and x_1, x_2 are solutions of the SOP, then, for any $y^* \in K^+ \setminus \{0\}$, there exist $y_i \in M_{y^*}(x_i)$ and $\xi_i \in \partial F(x_i, y_i)(y^*)$, $i = 1, 2$, such that

$$\phi(\xi_1, x_1, x_2) = \phi(\xi_2, x_2, x_1) = 0.$$

Proof. We first show that for any $y^* \in K^+ \setminus \{0\}$, f_{y^*} is pre-quasiinvex w.r.t. η . As a matter of fact, since F is K -invex w.r.t. η , we know that for any $x, y \in \Omega$ and $t \in [0, 1]$,

$$(1-t)F(x) + tF(y) \subseteq F(x + t\eta(y, x)) + K.$$

So, it follows that for any $y^* \in K^+ \setminus \{0\}$,

$$\begin{aligned} f_{y^*}(x + t\eta(y, x)) &= \inf\{y^*(w) : w \in F(x + t\eta(y, x))\} \\ &= \inf\{y^*(w) : w \in F(x + t\eta(y, x)) + K\} \\ &\leq \inf\{y^*(w) : w \in (1-t)F(x) + tF(y)\} \\ &\leq (1-t)f_{y^*}(x) + tf_{y^*}(y) \\ &\leq \max\{f_{y^*}(x), f_{y^*}(y)\}. \end{aligned}$$

This means that f_{y^*} is pre-quasiinvex w.r.t. η . Since x_1, x_2 are solutions of the SOP, by the proof of Theorem 3.7, for any $y^* \in K^+ \setminus \{0\}$, there exist $y_i \in M_{y^*}(x_i)$ and $\xi_i \in \partial_L f_{y^*}(x_i) \subseteq \partial F(x_i, y_i)(y^*)$, $i = 1, 2$ such that (4.1) in the proof of Theorem 4.1 holds. Since the rest of the proof is the same as in the proof of Theorem 4.1, we omit it. \square

Theorem 4.3. Let X, Y be Asplund spaces and $F : \Omega \subseteq X \rightarrow 2^Y$ be epi-Lipschitz. Let $\eta : \Omega \times \Omega \rightarrow X$ be continuous in the second variable such that Condition C w.r.t. ϕ holds. Suppose that for each $y^* \in K^+ \setminus \{0\}$, f_{y^*} is pre-quasiinvex w.r.t. η and satisfies the mean-value condition for limiting subdifferential $\partial_L f_{y^*}$ w.r.t. ϕ and $\partial_L f_{y^*}(\cdot) : \Omega \rightarrow 2^{X^*}$ is \mathcal{H} -hemicontinuous with compact values, and that for each $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous. If F is generalized K -pseudoinvex w.r.t. ϕ and $\bar{x} \in \bar{S}$, then $\bar{S} = S_1 = S_2$, where

$$\begin{aligned} S_1 &= \{x \in \Omega : \forall y^* \in K^+ \setminus \{0\} \exists y \in M_{y^*}(x) \text{ and } \xi \in \partial F(x, y)(y^*) : \phi(\xi, x, \bar{x}) = 0\}, \\ S_2 &= \{x \in \Omega : \forall y^* \in K^+ \setminus \{0\} \exists y \in M_{y^*}(x) \text{ and } \xi \in \partial F(x, y)(y^*) : \phi(\xi, x, \bar{x}) \geq 0\}. \end{aligned}$$

Proof. If $x \in \bar{S}$, then from $\bar{x} \in \bar{S}$ and Theorem 4.1, it follows that, for any $y^* \in K^+ \setminus \{0\}$, there exist $y \in M_{y^*}(x)$ and $\xi \in \partial F(x, y)(y^*)$ such that

$$\phi(\xi, x, \bar{x}) = 0.$$

That is, $x \in S_1$. Hence $\bar{S} \subseteq S_1$.

It is trivial that $S_1 \subseteq S_2$. Now, suppose that $x \in S_2$. Hence, for any $y^* \in K^+ \setminus \{0\}$, there exist $y \in M_{y^*}(x)$ and $\xi \in \partial F(x, y)(y^*)$ such that

$$\phi(\xi, x, \bar{x}) \geq 0.$$

Now, the generalized K -pseudoinvexity of F w.r.t. ϕ implies that

$$y^*(\bar{y}) \geq y^*(y),$$

for all $\bar{y} \in M_{y^*}(\bar{x})$. Since \bar{x} is a solution of the SOP, it shows that x is also a solution of the SOP and therefore is in \bar{S} , which completes the proof. \square

Theorem 4.4. Let X, Y be Asplund spaces and $F : \Omega \subseteq X \rightarrow 2^Y$ be epi-Lipschitz and K -invex w.r.t. η . Let η be continuous in the second variable such that Condition C w.r.t. ϕ holds. Suppose that for each $y^* \in K^+ \setminus \{0\}$, f_{y^*} satisfies the mean-value condition for limiting subdifferential $\partial_L f_{y^*}$ w.r.t. ϕ , and that for each $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous in the $w^* \times \tau$ -topology. If F is generalized K -pseudoinvex w.r.t. ϕ and $\bar{x} \in \bar{S}$, then $\bar{S} = S_1 = S_2$, where

$$S_1 = \{x \in \Omega : \forall y^* \in K^+ \setminus \{0\} \exists y \in M_{y^*}(x) \text{ and } \xi \in \partial F(x, y)(y^*); \phi(\xi, x, \bar{x}) = 0\},$$

$$S_2 = \{x \in \Omega : \forall y^* \in K^+ \setminus \{0\} \exists y \in M_{y^*}(x) \text{ and } \xi \in \partial F(x, y)(y^*); \phi(\xi, x, \bar{x}) \geq 0\}.$$

Proof. Since the proof is similar to that of Theorem 4.3, we omit it. \square

Remark 4.5. Theorems 4.1–4.2 and Theorems 4.3–4.4 extend, improve and develop Theorem 4.1 and Theorem 4.2 in [24] for equilibrium-like function and Condition C for η w.r.t. each ϕ_i , respectively. However, Theorem 4.2 in [24] extends Theorem 1 in [17], Theorem 3.1 in [13] and Theorem 3.1 in [27].

As applications of Theorems 4.3 and 4.4, we obtain the following Corollaries 4.6–4.7 and Corollaries 4.8–4.9, which extend and improve Corollary 4.1 and Corollary 4.2 in [24], respectively. Here it is worth pointing out that Corollaries 4.1 and 4.2 in [24] are the one for K -pseudoconvex set-valued maps and the second one extending partially Theorem 4.1 of [32].

Corollary 4.6. Let X, Y be Asplund spaces, $\Omega \subseteq X$ be convex and $F : \Omega \rightarrow 2^Y$ be epi-Lipschitz. Let $\eta(y, x) = y - x$, for all $x, y \in \Omega$ such that Condition C w.r.t. ϕ holds. Suppose that for each $y^* \in K^+ \setminus \{0\}$, f_{y^*} is pre-quasiconvex w.r.t. η and satisfies the mean-value condition for limiting subdifferential $\partial_L f_{y^*}$ w.r.t. ϕ and $\partial_L f_{y^*}(\cdot) : \Omega \rightarrow 2^{X^*}$ is \mathcal{H} -hemicontinuous with compact values, and that for each $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous. If F is generalized K -pseudoconvex w.r.t. ϕ and $\bar{x} \in \bar{S}$, then

$$\bar{S} = S_1 = S_2.$$

Corollary 4.7. Let X, Y be Asplund spaces, $\Omega \subseteq X$ be convex and $\eta(y, x) = y - x$, for all $x, y \in \Omega$ such that Condition C w.r.t. ϕ holds. Let $F : \Omega \subseteq X \rightarrow 2^Y$ be epi-Lipschitz and K -convex. Suppose that for each $y^* \in K^+ \setminus \{0\}$, f_{y^*} satisfies the mean-value condition for limiting subdifferential $\partial_L f_{y^*}$ w.r.t. ϕ , and that for each $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous in the $w^* \times \tau$ -topology. If F is generalized K -pseudoconvex w.r.t. ϕ and $\bar{x} \in \bar{S}$, then

$$\bar{S} = S_1 = S_2.$$

Corollary 4.8. Let X be an Asplund space and $f : \Omega \subseteq X \rightarrow \mathbf{R}$ be locally Lipschitz. Let $\eta : \Omega \times \Omega \rightarrow X$ be continuous in the second variable such that Condition C w.r.t. ϕ holds. Suppose that f is pre-quasinvex w.r.t. η and satisfies the mean-value condition for limiting subdifferential $\partial_L f$ w.r.t. ϕ and $\partial_L f(\cdot) : \Omega \rightarrow 2^{X^*}$ is \mathcal{H} -hemicontinuous with compact values, and that for each $y \in \Omega$, $\phi(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous. If f is generalized pseudoinvex w.r.t. ϕ and $\bar{x} \in \bar{S}$, then $\bar{S} = S_1 = S_2$, where

$$S_1 = \{x \in \Omega : \exists \xi \in \partial_L f(x); \phi(\xi, x, \bar{x}) = 0\},$$

$$S_2 = \{x \in \Omega : \exists \xi \in \partial_L f(x); \phi(\xi, x, \bar{x}) \geq 0\}.$$

Proof. It is enough to apply Theorem 4.3 for $F = f : \Omega \subseteq X \rightarrow \mathbf{R}$ and $K = \mathbf{R}_+$. \square

Corollary 4.9. Let X be an Asplund space and $f : \Omega \subseteq X \rightarrow \mathbf{R}$ be locally Lipschitz and K -invex w.r.t. η . Let η be continuous in the second variable such that Condition C w.r.t. ϕ holds. Suppose that f is pre-quasinvex w.r.t. η and satisfies the mean-value condition for limiting subdifferential $\partial_L f$ w.r.t. ϕ , and $\partial_L f$ is locally bounded and has closed graph, and that for each $y \in \Omega$, $\phi(\cdot, \cdot, y)$ is continuous in the $w^* \times \tau$ -topology. If f is generalized pseudoinvex w.r.t. ϕ and $\bar{x} \in \bar{S}$, then $\bar{S} = S_1 = S_2$, where

$$S_1 = \{x \in \Omega : \exists \xi \in \partial_L f(x); \phi(\xi, x, \bar{x}) = 0\},$$

$$S_2 = \{x \in \Omega : \exists \xi \in \partial_L f(x); \phi(\xi, x, \bar{x}) \geq 0\}.$$

Proof. It is enough to apply Theorem 4.4 for $F = f : \Omega \subseteq X \rightarrow \mathbf{R}$ and $K = \mathbf{R}_+$. \square

For real vector-valued functions, we have the following results.

Theorem 4.10. *Let X be an Asplund space and $\Omega \subseteq X$ be invex w.r.t. η that is continuous in the second variable and satisfies Condition C w.r.t. each $\phi_i, i = 1, \dots, n$, where $\Phi(\xi, x, y) = (\phi_1(\xi_1, x, y), \dots, \phi_n(\xi_n, x, y))$ for all $(\xi, x, y) \in \mathcal{L}(X, \mathbf{R}^n) \times \Omega \times \Omega$ with $\xi = (\xi_1, \dots, \xi_n)$. Let $f = (f_1, \dots, f_n) : \Omega \rightarrow \mathbf{R}^n$ be locally Lipschitz on Ω and suppose for $i = 1, \dots, n$ that*

- (i) *each f_i is pre-quasiinvex w.r.t. η and generalized pseudoinvex w.r.t. ϕ_i ;*
- (ii) *each f_i satisfies the mean-value condition for limiting subdifferential $\partial_L f_i$ w.r.t. ϕ_i ;*
- (iii) *each $\partial_L f_i : \Omega \rightarrow 2^{X^*}$ is \mathcal{H} -hemicontinuous with compact values;*
- (iv) *$\phi_i(\cdot, \cdot, y) : X^* \times \Omega \rightarrow \mathbf{R}$ is continuous for each $y \in \Omega$.*

If x_0, y_0 are weakly efficient solutions to problem (3.1), then one has

$$\Phi(\partial_L f(x_0), x_0, y_0) \not\subseteq -\text{int}K \quad \text{and} \quad \Phi(\partial_L f(x_0), x_0, y_0) \not\subseteq \text{int}K.$$

Proof. From Theorems 2.14 and 3.8 we can get the proof. \square

Theorem 4.11. *Suppose conditions (iii), (iv) in Theorem 4.10 are replaced by the following ones:*

- (iii) *each $\partial_L f_i$ is locally bounded and has closed graph;*
- (iv) *$\phi_i(\cdot, \cdot, y)$ is continuous in the $w^* \times \tau$ -topology for each $y \in \Omega$.*

Assume other conditions in Theorem 4.10 are not changed. If x_0, y_0 are weakly efficient solutions to problem (3.1), then one has

$$\Phi(\partial_L f(x_0), x_0, y_0) \not\subseteq -\text{int}K \quad \text{and} \quad \Phi(\partial_L f(x_0), x_0, y_0) \not\subseteq \text{int}K.$$

Proof. From Theorems 2.14 and 3.9 we can get the proof. \square

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