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Coupled systems of Riemann-Liouville fractional differential equations with Hadamard fractional integral boundary conditions

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Abstract

In this paper we study existence and uniqueness of solutions for coupled systems consisting from fractional differential equations of Riemann-Liouville type subject to coupled and uncoupled Hadamard fractional integral boundary conditions. The existence and uniqueness of solutions is established by Banach's contraction principle, while the existence of solutions is derived by using Leray-Schauder's alternative. Examples illustrating our results are also presented.

Keywords: Riemann-Liouville fractional derivative, Hadamard fractional integral, coupled system, existence, uniqueness, fixed point theorems.
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1. Introduction

In this paper, we concentrate on the study of existence and uniqueness of solutions for a coupled system of nonlinear Riemann-Liouville fractional differential equations with nonlocal Hadamard fractional boundary

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conditions of the form

$$\begin{cases} {}_{RL}D^q x(t) = f(t, x(t), y(t)), \quad t \in [0, T], \quad 1 < q \le 2, \\ {}_{RL}D^p y(t) = g(t, x(t), y(t)), \quad t \in [0, T], \quad 1 < p \le 2, \\ x(0) = 0, \quad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_i {}_H I^{\rho_i} y(\eta_i), \\ y(0) = 0, \quad y(T) = \displaystyle\sum_{j=1}^{m} \beta_j {}_H I^{\gamma_j} x(\theta_j), \end{cases} \tag{1.1}$$

where ${}_{RL}D^q$, ${}_{RL}D^p$ are the standard Riemann-Liouville fractional derivative of orders q, p, ${}_H I^{\rho_i}$, ${}_H I^{\gamma_j}$ are the Hadamard fractional integral of orders $\rho_i, \gamma_j > 0$, $\eta_i, \theta_j \in (0, T)$, $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and $\alpha_i, \beta_j \in \mathbb{R}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ are real constants such that

$$\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{p-1}}{(p-1)^{\rho_i}} \sum_{j=1}^{m} \frac{\beta_j \theta_j^{q-1}}{(q-1)^{\gamma_j}} \ne T^{q+p-2}.$$

Several interesting and important results concerning existence and uniqueness of solutions, stability properties of solutions, analytic and numerical methods of solutions for fractional differential equations can be found in the recent literature on the topic and the serge for investigating more and more results is in progress. Fractional-order operators are nonlocal in nature and take care of the hereditary properties of many phenomena and processes. Fractional calculus has also emerged as a powerful modeling tool for many real world problems. For examples and recent development of the topic, see ([1, 2, 3, 4, 5, 6, 7, 14, 16, 17, 18, 19, 20]). However, it has been observed that most of the work on the topic involves either Riemann-Liouville or Caputo type fractional derivatve. Besides these derivatives, Hadamard fractional derivative is another kind of fractional derivatives that was introduced by Hadamard in 1892 [12]. This fractional derivative differs from the other ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. For background material of Hadamard fractional derivative and integral, we refer to the papers [8, 9, 10, 13, 14, 15].

The significance of studying in this paper is that, the system (1.1) contains both of Riemann-Liouville and Hadamard calculus which are new theories of boundary value problems. Existence and uniqueness results are obtained by using Banach's contraction principle and Leray-Schauder's alternative. Examples illustrating our results are also presented.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later.

Definition 2.1. The Riemann-Liouville fractional derivative of order $q > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$${}_{RL}D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s) ds, \quad n-1 < q < n,$$

where $n = [q] + 1$, $[q]$ denotes the integer part of a real number q, provided the right-hand side is point-wise defined on $(0, \infty)$, where Γ is the gamma function defined by $\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds$.

Definition 2.2. The Riemann-Liouville fractional integral of order $q > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$${}_{RL}I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.3. The Hadamard derivative of fractional order q for a function $f : (0,\infty) \to \mathbb{R}$ is defined as

$$ {}_HD^q f(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_0^t \left(\log\frac{t}{s}\right)^{n-q-1} \frac{f(s)}{s} ds, \quad n-1 < q < n, \ \ n = [q]+1, $$

where $\log(\cdot) = \log_e(\cdot)$.

Definition 2.4. The Hadamard fractional integral of order $q \in \mathbb{R}^+$ of a function $f(t)$, for all $t > 0$, is defined as

$$ {}_HI^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log\frac{t}{s}\right)^{q-1} f(s)\frac{ds}{s}, $$

provided the integral exists.

Lemma 2.5 ([14], page 113)**.** *Let $q > 0$ and $\beta > 0$. Then the following formulas*

$$ {}_HI^q t^\beta = \beta^{-q} t^\beta \quad \text{and} \quad {}_HD^q t^\beta = \beta^q t^\beta $$

hold.

Lemma 2.6 ([14])**.** *Let $q > 0$ and $x \in C(0,T) \cap L(0,T)$. Then the fractional differential equation*

$$ {}_{RL}D^q x(t) = 0 $$

has a unique solution

$$ x(t) = c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n}, $$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n-1 < q < n$.

Lemma 2.7 ([14])**.** *Let $q > 0$. Then for $x \in C(0,T) \cap L(0,T)$ it holds*

$$ {}_{RL}I^q\,{}_{RL}D^q x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n}, $$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n-1 < q < n$.

Lemma 2.8. *Given $\phi, \psi \in C([0,T],\mathbb{R})$, the unique solution of the problem*

$$ \begin{cases} {}_{RL}D^q x(t) = \phi(t), & t \in [0,T], \quad 1 < q \le 2 \\ {}_{RL}D^p y(t) = \psi(t), & t \in [0,T], \quad 1 < p \le 2, \\ x(0) = 0, \quad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_i\, {}_HI^{\rho_i} y(\eta_i), \\ y(0) = 0, \quad y(T) = \displaystyle\sum_{j=1}^{m} \beta_j\, {}_HI^{\gamma_j} x(\theta_j), \end{cases} \tag{2.1} $$

is

$$ \begin{aligned} x(t) = \ & {}_{RL}I^q \phi(t) - \frac{t^{q-1}}{\Omega} \Bigg[\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{p-1}}{(p-1)^{\rho_i}} \left(\sum_{j=1}^{m} \beta_j\, {}_HI^{\gamma_j}\, {}_{RL}I^q \phi(\theta_j) - {}_{RL}I^p \psi(T) \right) \\ & + T^{p-1} \left(\sum_{i=1}^{n} \alpha_i\, {}_HI^{\rho_i}\, {}_{RL}I^p \psi(\eta_i) - {}_{RL}I^q \phi(T) \right) \Bigg] \end{aligned} \tag{2.2} $$

and

$$ \begin{aligned} y(t) = \ & {}_{RL}I^p \psi(t) - \frac{t^{p-1}}{\Omega} \Bigg[\sum_{j=1}^{m} \frac{\beta_j \theta_j^{q-1}}{(q-1)^{\gamma_j}} \left(\sum_{i=1}^{n} \alpha_i\, {}_HI^{\rho_i}\, {}_{RL}I^p \psi(\eta_i) - {}_{RL}I^q \phi(T) \right) \\ & + T^{q-1} \left(\sum_{j=1}^{m} \beta_j\, {}_HI^{\gamma_j}\, {}_{RL}I^q \phi(\theta_j) - {}_{RL}I^p \psi(T) \right) \Bigg], \end{aligned} \tag{2.3} $$

where

$$ \Omega := \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{p-1}}{(p-1)^{\rho_i}} \sum_{j=1}^{m} \frac{\beta_j \theta_j^{q-1}}{(q-1)^{\gamma_j}} - T^{q+p-2} \neq 0. \tag{2.4} $$

Proof. Using Lemmas 2.6-2.7, the equations in (2.1) can be expressed as equivalent integral equations

$$x(t) = {}_{RL}I^q\phi(t) - c_1t^{q-1} - c_2t^{q-2}, \tag{2.5}$$

$$y(t) = {}_{RL}I^p\psi(t) - d_1t^{p-1} - d_2t^{p-2}, \tag{2.6}$$

for $c_1, c_2, d_1, d_2 \in \mathbb{R}$. The conditions $x(0) = 0, y(0) = 0$ imply that $c_2 = 0, d_2 = 0$. Taking the Hadamard fractional integral of order $\rho_i > 0$ for (2.5) and $\gamma_j > 0$ for (2.6) and using the property of the Hadamard fractional integral given in Lemma 2.5 we get the system

$$ {}_{RL}I^q\phi(T) - c_1T^{q-1} = \sum_{i=1}^{n} \alpha_i {}_HI^{\rho_i} {}_{RL}I^p\psi(\eta_i) - d_1 \sum_{i=1}^{n} \frac{\alpha_i\eta_i^{p-1}}{(p-1)^{\rho_i}},$$

$$ {}_{RL}I^p\psi(T) - d_1T^{p-1} = \sum_{j=1}^{m} \beta_j {}_HI^{\gamma_j} {}_{RL}I^q\phi(\theta_j) - c_1 \sum_{j=1}^{m} \frac{\beta_j\theta_j^{q-1}}{(q-1)^{\gamma_j}},$$

from which we have

$$\begin{aligned} c_1 = \frac{1}{\Omega}\Bigg[&\sum_{i=1}^{n} \frac{\alpha_i\eta_i^{p-1}}{(p-1)^{\rho_i}} \left(\sum_{j=1}^{m} \beta_j {}_HI^{\gamma_j} {}_{RL}I^q\phi(\theta_j) - {}_{RL}I^p\psi(T) \right) \\ &+ T^{p-1}\left(\sum_{i=1}^{n} \alpha_i {}_HI^{\rho_i} {}_{RL}I^p\psi(\eta_i) - {}_{RL}I^q\phi(T) \right)\Bigg] \end{aligned}$$

and

$$\begin{aligned} d_1 = \frac{1}{\Omega}\Bigg[&\sum_{j=1}^{m} \frac{\beta_j\theta_j^{q-1}}{(q-1)^{\gamma_j}} \left(\sum_{i=1}^{n} \alpha_i {}_HI^{\rho_i} {}_{RL}I^p\psi(\eta_i) - {}_{RL}I^q\phi(T) \right) \\ &+ T^{q-1}\left(\sum_{j=1}^{m} \beta_j {}_HI^{\gamma_j} {}_{RL}I^q\phi(\theta_j) - {}_{RL}I^p\psi(T) \right)\Bigg]. \end{aligned}$$

Substituting the values of c_1, c_2, d_1 and d_2 in (2.5) and (2.6), we obtain the solutions (2.2) and (2.3). □

3. Main Results

Throughout this paper, for convenience, we use the following expressions

$$ {}_{RL}I^w h(s, x(s), y(s))(v) = \frac{1}{\Gamma(w)} \int_0^v (v-s)^{w-1} h(s, x(s), y(s))ds$$

and

$$\begin{aligned} &{}_HI^u {}_{RL}I^w h(s, x(s), y(s))(v) \\ &\quad = \frac{1}{\Gamma(u)\Gamma(w)} \int_0^v \int_0^t \left(\log\frac{v}{t}\right)^{u-1} (t-s)^{w-1} h(s, x(s), y(s))ds\frac{dt}{t}, \end{aligned}$$

where $u \in \{\rho_i, \gamma_j\}$, $v \in \{t, T, \eta_i, \theta_j\}$, $w = \{p, q\}$ and $h = \{f, g\}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Let $\mathcal{C} = C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$. Let us introduce the space $X = \{x(t)|x(t) \in C^1([0, T])\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0, T]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{y(t)|y(t) \in C^1([0, T])\}$ be endowed with the norm $\|y\| = \sup\{|y(t)|, t \in [0, T]\}$. Obviously the product space $(X \times Y, \|(x, y)\|)$ is a Banach space with norm $\|(x, y)\| = \|x\| + \|y\|$. In view of Lemma 2.8, we define an operator $\mathcal{T} : X \times Y \to X \times Y$ by

$$\mathcal{T}(x,y)(t)=\begin{pmatrix}\mathcal{T}_1(x,y)(t)\\ \mathcal{T}_2(x,y)(t)\end{pmatrix},$$

where

$$\begin{aligned}\mathcal{T}_1(x,y)(t)&={}_{RL}I^q f(s,x(s),y(s))(t)\\&-\frac{t^{q-1}}{\Omega}\Bigg[\sum_{i=1}^{n}\frac{\alpha_i\eta_i^{p-1}}{(p-1)^{\rho_i}}\left(\sum_{j=1}^{m}\beta_{j\,H}I^{\gamma_j}{}_{RL}I^q f(s,x(s),y(s))(\theta_j)-{}_{RL}I^p g(s,x(s),y(s))(T)\right)\\&+T^{p-1}\left(\sum_{i=1}^{n}\alpha_{i\,H}I^{\rho_i}{}_{RL}I^p g(s,x(s),y(s))(\eta_i)-{}_{RL}I^q f(s,x(s),y(s))(T)\right)\Bigg]\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_2(x,y)(t)&={}_{RL}I^p g(s,x(s),y(s))(t)\\&-\frac{t^{p-1}}{\Omega}\Bigg[\sum_{j=1}^{m}\frac{\beta_j\theta_j^{q-1}}{(q-1)^{\gamma_j}}\left(\sum_{i=1}^{n}\alpha_{i\,H}I^{\rho_i}{}_{RL}I^p g(s,x(s),y(s))(\eta_i)-{}_{RL}I^q f(s,x(s),y(s))(T)\right)\\&+T^{q-1}\left(\sum_{j=1}^{m}\beta_{j\,H}I^{\gamma_j}{}_{RL}I^q f(s,x(s),y(s))(\theta_j)-{}_{RL}I^p g(s,x(s),y(s))(T)\right)\Bigg].\end{aligned}$$

For the sake of convenience, we set

$$M_1=\frac{T^q}{\Gamma(q+1)}+\frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^q}{q^{\gamma_j}}+\frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)},\tag{3.1}$$

$$M_2=\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}+\frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^p}{p^{\rho_i}},\tag{3.2}$$

$$M_3=\frac{T^{q+p-1}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}+\frac{T^{q+p-2}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^q}{q^{\gamma_j}},\tag{3.3}$$

$$M_4=\frac{T^p}{\Gamma(p+1)}+\frac{T^{p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^p}{p^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}+\frac{T^{q+2p-2}}{|\Omega|\Gamma(p+1)}\tag{3.4}$$

and

$$M_0=\min\{1-(M_1+M_3)k_1-(M_2+M_4)\lambda_1,\ 1-(M_1+M_3)k_2-(M_2+M_4)\lambda_2\},\tag{3.5}$$

$k_i,\ \lambda_i\ge 0\ (i=1,2)$.

The first result is concerned with the existence and uniqueness of solutions for the problem (1.1) and is based on Banach's contraction mapping principle.

Theorem 3.1. *Assume that $f,g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions and there exist constants $m_i,n_i,i=1,2$ such that for all $t\in[0,T]$ and $x_i,y_i\in\mathbb{R},i=1,2,$*

$$|f(t,x_1,x_2)-f(t,y_1,y_2)|\le m_1|x_1-x_2|+m_2|y_1-y_2|$$

and

$$|g(t,x_1,x_2)-g(t,y_1,y_2)|\le n_1|x_1-x_2|+n_2|y_1-y_2|.$$

In addition, assume that

$$(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)<1,$$

where $M_i,i=1,2,3,4$ are given by (3.1)-(3.4). *Then the boundary value problem* (1.1) *has a unique solution.*

Proof. Define $\sup_{t\in[0,1]} f(t,0,0) = N_1 < \infty$ and $\sup_{t\in[0,1]} g(t,0,0) = N_2 < \infty$ such that

$$r \geq \max\left\{\frac{M_4N_2 + M_3N_1}{1-(M_4n_1 + M_3m_1 + M_4n_2 + M_3m_2)}, \frac{M_1N_1 + M_2N_2}{1-(M_2n_1 + M_1m_1 + M_2n_2 + M_1m_2)}\right\}.$$

We show that $\mathcal{T}B_r \subset B_r$, where $B_r = \{(x,y) \in X \times Y : \|(x,y)\| \leq r\}$.

For $(x,y) \in B_r$, we have

$$\begin{aligned}
|\mathcal{T}_1(x,y)(t)| &= \sup_{t\in[0,T]}\Bigg\{ {}_{RL}I^q f(s,x(s),y(s))(t) - \frac{t^{q-1}}{\Omega}\Bigg[\sum_{i=1}^{n}\frac{\alpha_i\eta_i^{p-1}}{(p-1)^{\rho_i}} \\
&\quad \times \left(\sum_{j=1}^{m}\beta_j {}_HI^{\gamma_j}{}_{RL}I^q f(s,x(s),y(s))(\theta_j) - {}_{RL}I^p g(s,x(s),y(s))(T)\right) \\
&\quad + T^{p-1}\left(\sum_{i=1}^{n}\alpha_i {}_HI^{\rho_i}{}_{RL}I^p g(s,x(s),y(s))(\eta_i) - {}_{RL}I^q f(s,x(s),y(s))(T)\right)\Bigg]\Bigg\} \\
&\leq {}_{RL}I^q(|f(s,x(s),y(s)) - f(s,0,0)| + |f(s,0,0)|)(T) \\
&\quad + \frac{T^{q-1}}{|\Omega|}\Bigg[\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\Bigg(\sum_{j=1}^{m}|\beta_j| {}_HI^{\gamma_j}{}_{RL}I^q(|f(s,x(s),y(s)) - f(s,0,0)| \\
&\quad + |f(s,0,0)|)(\theta_j) + {}_{RL}I^p(|g(s,x(s),y(s)) - g(s,0,0)| + |g(s,0,0)|)(T)\Bigg) \\
&\quad + T^{p-1}\Bigg(\sum_{i=1}^{n}|\alpha_i| {}_HI^{\rho_i}{}_{RL}I^p(|g(s,x(s),y(s)) - g(s,0,0)| + |g(s,0,0)|)(\eta_i) \\
&\quad + {}_{RL}I^q(|f(s,x(s),y(s)) - f(s,0,0)| + |f(s,0,0)|)(T)\Bigg)\Bigg] \\
&\leq {}_{RL}I^q(m_1\|x\| + m_2\|y\| + N_1)(T) + \frac{T^{q-1}}{|\Omega|}\Bigg[\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}} \\
&\quad \times \left(\sum_{j=1}^{m}|\beta_j| {}_HI^{\gamma_j}{}_{RL}I^q(m_1\|x\| + m_2\|y\| + N_1)(\theta_j) + {}_{RL}I^p(n_1\|x\| + n_2\|y\| + N_2)(T)\right) \\
&\quad + T^{p-1}\Bigg(\sum_{i=1}^{n}|\alpha_i| {}_HI^{\rho_i}{}_{RL}I^p(n_1\|x\| + n_2\|y\| + N_2)(\eta_i) \\
&\quad + {}_{RL}I^q(m_1\|x\| + m_2\|y\| + N_1)(T)\Bigg)\Bigg] \\
&= (m_1\|x\| + m_2\|y\| + N_1)\Bigg[{}_{RL}I^q(1)(T) + \frac{T^{q-1}}{|\Omega|}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m}|\beta_j| {}_HI^{\gamma_j}{}_{RL}I^q(1)(\theta_j) \\
&\quad + \frac{T^{q+p-2}}{|\Omega|}{}_{RL}I^q(1)(T)\Bigg] + (n_1\|x\| + n_2\|y\| + N_2)\Bigg[\frac{T^{q-1}}{|\Omega|}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}{}_{RL}I^p(1)(T) \\
&\quad + \frac{T^{q+p-2}}{|\Omega|}\sum_{i=1}^{n}|\alpha_i| {}_HI^{\rho_i}{}_{RL}I^p(1)(\eta_i)\Bigg] \\
&= (m_1\|x\| + m_2\|y\| + N_1)\Bigg[\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^q}{q^{\gamma_j}}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}\Bigg] + (n_1\|x\| + n_2\|y\| + N_2)\Bigg[\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}} \\
&+ \frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\Bigg] \\
&= M_1(m_1\|x\| + m_2\|y\| + N_1) + M_2(n_1\|x\| + n_2\|y\| + N_2) \\
&= (M_1m_1 + M_2n_1)\|x\| + (M_1m_2 + M_2n_2)\|y\| + M_1N_1 + M_2N_2 \\
&\le (M_1m_1 + M_2n_1 + M_1m_2 + M_2n_2)r + M_1N_1 + M_2N_2 \le r.
\end{aligned}
$$

In the same way, we can obtain that

$$
\begin{aligned}
|\mathcal{T}_2(x,y)(t)| &\le (n_1\|x\| + n_2\|y\| + N_2)\Bigg[\frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}} \\
&+ \frac{T^{q+2p-2}}{|\Omega|\Gamma(p+1)}\Bigg] + (m_1\|x\| + m_2\|y\| + N_1)\Bigg[\frac{T^{q+p-1}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}} \\
&+ \frac{T^{q+p-2}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q}}{q^{\gamma_j}}\Bigg] \\
&= (n_1\|x\| + n_2\|y\| + N_2)M_4 + (m_1\|x\| + m_2\|y\| + N_1)M_3 \\
&= (M_4n_1 + M_3m_1)\|x\| + (M_4n_2 + M_3m_2)\|y\| + M_4N_2 + M_3N_1 \\
&\le (M_4n_1 + M_3m_1 + M_4n_2 + M_3m_2)r + M_4N_2 + M_3N_1 \le r.
\end{aligned}
$$

Consequently, $\|\mathcal{T}(x,y)(t)\| \le r$.

Now for $(x_2,y_2),(x_1,y_1) \in X \times Y$, and for any $t \in [0,T]$, we get

$$
\begin{aligned}
&|\mathcal{T}_1(x_2,y_2)(t) - \mathcal{T}_1(x_1,y_1)(t)| \\
&\le {}_{RL}I^q|f(s,x_2(s),y_2(s)) - f(s,x_1(s),y_1(s))|(T) \\
&+ \frac{T^{q-1}}{\Omega}\Bigg[\sum_{i=1}^{n}\frac{\alpha_i\eta_i^{p-1}}{(p-1)^{\rho_i}}\Bigg(\sum_{j=1}^{m}\beta_j\, {}_HI^{\gamma_j}\,{}_{RL}I^q|f(s,x_2(s),y_2(s)) - f(s,x_1(s),y_1(s))|(\theta_j) \\
&+ {}_{RL}I^p|g(s,x_2(s),y_2(s)) - g(s,x_1(s),y_1(s))|(T)\Bigg) \\
&+ T^{p-1}\Bigg(\sum_{i=1}^{n}\alpha_i\ {}_HI^{\rho_i}\,{}_{RL}I^p|g(s,x_2(s),y_2(s)) - g(s,x_1(s),y_1(s))|(\eta_i) \\
&+ {}_{RL}I^q|f(s,x_2(s),y_2(s)) - f(s,x_1(s),y_1(s))|(T)\Bigg)\Bigg] \\
&\le (m_1\|x_2 - x_1\| + m_2\|y_2 - y_1\|)\Bigg[\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q}}{q^{\gamma_j}} \\
&+ \frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}\Bigg] + (n_1\|x_2 - x_1\| + n_2\|y_2 - y_1\|)\Bigg[\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}} \\
&+ \frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\Bigg] \\
&= M_1(m_1\|x_2 - x_1\| + m_2\|y_2 - y_1\|) + M_2(n_1\|x_2 - x_1\| + n_2\|y_2 - y_1\|)
\end{aligned}
$$

$$= (M_1 m_1 + M_2 n_1)\|x_2 - x_1\| + (M_1 m_2 + M_2 n_2)\|y_2 - y_1\|,$$

and consequently we obtain

$$\|\mathcal{T}_1(x_2, y_2)(t) - \mathcal{T}_1(x_1, y_1)\| \le (M_1 m_1 + M_2 n_1 + M_1 m_2 + M_2 n_2)[\|x_2 - x_1\| + \|y_2 - y_1\|]. \tag{3.6}$$

Similarly,

$$\|\mathcal{T}_2(x_2, y_2)(t) - \mathcal{T}_2(x_1, y_1)\| \le (M_3 m_1 + M_4 n_1 + M_3 m_2 + M_4 n_2)[\|x_2 - x_1\| + \|y_2 - y_1\|]. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\|\mathcal{T}(x_2, y_2)(t) - \mathcal{T}(x_1, y_1)(t)\| \le [(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2)](\|x_2 - x_1\| + \|y_2 - y_1\|).$$

Since $(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2) < 1$, therefore, $\mathcal{T}$ is a contraction operator. So, By Banach's fixed point theorem, the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of problem (1.1). This completes the proof. □

In the next result, we prove the existence of solutions for the problem (1.1) by applying Leray-Schauder alternative.

Lemma 3.2 (Leray-Schauder alternative, [11], page 4)**.** *Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.3. *Assume that there exist real constants $k_i,\ \lambda_i \ge 0$ $(i = 1, 2)$ and $k_0 > 0, \lambda_0 > 0$ such that $\forall x_i \in \mathbb{R},\ (i = 1, 2)$ we have*

$$|f(t, x_1, x_2)| \le k_0 + k_1|x_1| + k_2|x_2|,$$
$$|g(t, x_1, x_2)| \le \lambda_0 + \lambda_1|x_1| + \lambda_2|x_2|.$$

In addition it is assumed that

$$(M_1 + M_3)k_1 + (M_2 + M_4)\lambda_1 < 1 \quad \text{and} \quad (M_1 + M_3)k_2 + (M_2 + M_4)\lambda_2] < 1,$$

where $M_i, i = 1, 2, 3, 4$ are given by (3.1)-(3.4). Then there exists at least one solution for the boundary value problem (1.1).

Proof. First we show that the operator $\mathcal{T} : X \times Y \to X \times Y$ is completely continuous. By continuity of functions f and g, the operator $\mathcal{T}$ is continuous.

Let $\Theta \subset X \times Y$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t, x(t), y(t)| \le L_1, \quad |g(t, x(t), y(t)| \le L_2, \quad \forall (x, y) \in \Theta.$$

Then for any $(x, y) \in \Theta$, we have

$$\begin{aligned}\|\mathcal{T}_1(x, y)(t)\| \le {}& {}_{RL}I^q|f(s, x(s), y(s))|(T) + \frac{T^{q-1}}{|\Omega|}\Bigg[\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}} \\ & \times\left(\sum_{j=1}^{m}|\beta_j|{}_HI^{\gamma_j}{}_{RL}I^q|f(s, x(s), y(s))|(\theta_j) + {}_{RL}I^p|g(s, x(s), y(s))|(T)\right) \\ & + T^{p-1}\left(\sum_{i=1}^{n}|\alpha_i|{}_HI^{\rho_i}{}_{RL}I^p|g(s, x(s), y(s))|(\eta_i) + {}_{RL}I^q|f(s, x(s), y(s))|(T)\right)\Bigg]\end{aligned}$$

$$
\begin{aligned}
&\le \left(\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}+\frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\right)L_2\\
&\quad+\left(\frac{T^{q}}{\Gamma(q+1)}+\frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q}}{q^{\gamma_j}}+\frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}\right)L_1,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\mathcal{T}_1(x,y)\| &\le \left(\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}+\frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\right)L_2\\
&\quad+\left(\frac{T^{q}}{\Gamma(q+1)}+\frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q}}{q^{\gamma_j}}+\frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}\right)L_1\\
&= M_2L_2+M_1L_1.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
\|\mathcal{T}_2(x,y)\| &\le \left(\frac{T^{p}}{\Gamma(p+1)}+\frac{T^{p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}+\frac{T^{q+2p-2}}{|\Omega|\Gamma(p+1)}\right)L_2\\
&\quad+\left(\frac{T^{q+p-1}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}+\frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q}}{q^{\gamma_j}}\right)L_1\\
&= M_4L_2+M_3L_1.
\end{aligned}
$$

Thus, it follows from the above inequalities that the operator $\mathcal{T}$ is uniformly bounded.

Next, we show that $\mathcal{T}$ is equicontinuous. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then we have

$$
\begin{aligned}
&|\mathcal{T}_1(x(t_2),y(t_2))-\mathcal{T}_1(x(t_1),y(t_1))|\\
&\quad\le \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]|f(s,x(s),y(s))|ds\\
&\qquad+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}|f(s,x(s),y(s))|ds+\frac{t_2^{q-1}-t_1^{q-1}}{|\Omega|}\left[\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\right.\\
&\qquad\times\left(\sum_{j=1}^{m}|\beta_j|{}_HI^{\gamma_j}{}_{RL}I^{q}|f(s,x(s),y(s))|(\theta_j)+{}_{RL}I^{p}|g(s,x(s),y(s))|(T)\right)\\
&\qquad\left.+T^{p-1}\left(\sum_{i=1}^{n}|\alpha_i|{}_HI^{\rho_i}{}_{RL}I^{p}|g(s,x(s),y(s))|(\eta_i)+{}_{RL}I^{q}|f(s,x(s),y(s))|(T)\right)\right]\\
&\quad\le \frac{L_1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]ds+\frac{L_1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds\\
&\qquad+\frac{t_2^{q-1}-t_1^{q-1}}{|\Omega|}\left[\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\left(\frac{L_1}{\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q}}{q^{\gamma_j}}+\frac{T^{p}}{\Gamma(p+1)}L_2\right)\right.\\
&\qquad\left.+T^{p-1}\left(\frac{L_2}{\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}+\frac{T^{q}}{\Gamma(q+1)}L_1\right)\right].
\end{aligned}
$$

Analogously, we can obtain

$$
\begin{aligned}
&|\mathcal{T}_2(x(t_2),y(t_2))-\mathcal{T}_2(x(t_1),y(t_1))| \\
&\leq \frac{L_2}{\Gamma(p)}\int_0^{t_1}[(t_2-s)^{p-1}-(t_1-s)^{p-1}]ds+\frac{L_2}{\Gamma(p)}\int_{t_1}^{t_2}(t_2-s)^{p-1}ds \\
&\quad+\frac{t_2^{q-1}-t_1^{q-1}}{|\Omega|}\left[\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}\left(\frac{L_2}{\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^p}{p^{\rho_i}}+\frac{T^q}{\Gamma(q+1)}L_1\right)\right. \\
&\quad\left.+T^{q-1}\left(\frac{L_1}{\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^q}{q^{\gamma_j}}+\frac{T^p}{\Gamma(p+1)}L_2\right)\right].
\end{aligned}
$$

Therefore, the operator $\mathcal{T}(x,y)$ is equicontinuous, and thus the operator $\mathcal{T}(x,y)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E}=\{(x,y)\in X\times Y|(x,y)=\lambda\mathcal{T}(x,y), 0\leq\lambda\leq 1\}$ is bounded. Let $(x,y)\in\mathcal{E}$, then $(x,y)=\lambda\mathcal{T}(x,y)$. For any $t\in[0,T]$, we have

$$x(t)=\lambda\mathcal{T}_1(x,y)(t),\quad y(t)=\lambda\mathcal{T}_2(x,y)(t).$$

Then

$$
\begin{aligned}
|x(t)| &\leq (k_0+k_1\|x\|+k_2\|y\|)\left(\frac{T^q}{\Gamma(q+1)}+\frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^q}{q^{\gamma_j}}\right. \\
&\quad\left.+\frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}\right)+(\lambda_0+\lambda_1\|x\|+\lambda_2\|y\|)\left(\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\right. \\
&\quad\left.+\frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^p}{p^{\rho_i}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
|y(t)| &\leq (\lambda_0+\lambda_1\|x\|+\lambda_2\|y\|)\left(\frac{T^p}{\Gamma(p+1)}+\frac{T^{p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}\right. \\
&\quad\left.+\frac{T^{q+2p-2}}{|\Omega|\Gamma(p+1)}\right)+(k_0+k_1\|x\|+k_2\|y\|)\left(\frac{T^{q+p-1}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}\right. \\
&\quad\left.+\frac{T^{q+p-2}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^q}{q^{\gamma_j}}\right).
\end{aligned}
$$

Hence we have

$$\|x\|\leq(k_0+k_1\|x\|+k_2\|y\|)M_1+(\lambda_0+\lambda_1\|x\|+\lambda_2\|y\|)M_2$$

and

$$\|y\|\leq(k_0+k_1\|x\|+k_2\|y\|)M_3+(\lambda_0+\lambda_1\|x\|+\lambda_2\|y\|)M_4,$$

which imply that

$$
\begin{aligned}
\|x\|+\|y\| = {}&(M_1+M_3)k_0+(M_2+M_4)\lambda_0+[(M_1+M_3)k_1+(M_2+M_4)\lambda_1]\|x\| \\
&+[(M_1+M_3)k_2+(M_2+M_4)\lambda_2]\|y\|.
\end{aligned}
$$

Consequently,

$$\|(x,y)\|\leq\frac{(M_1+M_3)k_0+(M_2+M_4)\lambda_0}{M_0},$$

for any $t\in[0,T]$, where M_0 is defined by (3.5), which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.2, the operator $\mathcal{T}$ has at least one fixed point. Hence the boundary value problem (1.1) has at least one solution. The proof is complete. □

3.1. Examples

Example 1. Consider the following system of coupled Riemann-Liouville fractional differential equations with Hadamard type fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^{3/2}x(t) = \dfrac{e^{t^2}}{(t+7)^2}\dfrac{|x(t)|}{(1+|x(t)|)} + \dfrac{\sin^2(2\pi t)}{(3e^t+1)^2}\dfrac{|y(t)|}{(1+|y(t)|)} + \dfrac{1}{3}, \ t \in [0,2], \\ {}_{RL}D^{5/4}y(t) = \dfrac{1}{25}\cos x(t) + \dfrac{1}{(t+6)^2}\sin y(t) + 1, \quad t \in [0,2], \\ x(0) = 0, \quad x(2) = \dfrac{3}{2}{}_HI^{1/3}y(2/3) + \sqrt{2}{}_HI^{3/7}y(4/3), \\ y(0) = 0, \quad y(2) = \sqrt{3}{}_HI^{1/4}x(1/2) + \dfrac{1}{2}{}_HI^{4/7}x(1) + 2{}_HI^{7/10}x(3/2). \end{cases} \tag{3.8}$$

Here $q = 3/2, p = 5/4, n = 2, m = 3, T = 2, \alpha_1 = 3/2, \alpha_2 = \sqrt{2}, \beta_1 = \sqrt{3}, \beta_2 = 1/2, \beta_3 = 2, \rho_1 = 1/3, \rho_2 = 3/7, \gamma_1 = 1/4, \gamma_2 = 4/7, \gamma_3 = 7/10, \eta_1 = 2/3, \eta_2 = 4/3, \theta_1 = 1/2, \theta_2 = 1, \theta_3 = 3/2$ and $f(t,x,y) = (e^{t^2}|x|)/(((t+7)^2)(1+|x|)) + (\sin^2(2\pi t)|y|)/(((3e^t+1)^2)(1+|y|)) + (1/3)$ and $g(t,x,y) = (\cos x/25) + (\sin y)/((t+6)^2) + 1$. Since $|f(t,x_1,x_2) - f(t,x_2,y_2)| \leq ((1/49)|x_1 - y_1| + (1/16)|x_2 - y_2|)$ and $|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq ((1/25)|x_1 - y_1| + (1/36)|x_2 - y_2|)$ by using the Maple program, we can find

$$\Omega = \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{p-1}}{(p-1)^{\rho_i}} \sum_{j=1}^{m} \frac{\beta_j \theta_j^{q-1}}{(q-1)^{\gamma_j}} - T^{q+p-2} \approx 28.62075873 \neq 0.$$

With the given values, it is found that $m_1 = 1/49, m_2 = 1/16, n_1 = 1/25, n_2 = 1/36, M_1 \simeq 2.930183476, M_2 \simeq 0.6477212729, M_3 \simeq 0.7389741995, M_4 \simeq 2.829885649$, and

$$(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)(n_1 + n_2) \simeq 0.5399075928 < 1.$$

Thus all the conditions of Theorem 3.1 are satisfied. Therefore, by the conclusion of Theorem 3.1, the problem (3.8) has a unique solution on $[0, 2]$.

Example 2. Consider the following system of coupled Riemann-Liouville fractional differential equations with Hadamard type fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^{\sqrt{2}}x(t) = 1 + \dfrac{\sqrt{2}}{81}x(t)\cos y(t) + \dfrac{\sqrt{3}}{36\pi}y(t), \ t \in [0,\pi], \\ {}_{RL}D^{\sqrt{3}}y(t) = \dfrac{3}{2} + \dfrac{\sqrt{3}}{64\pi}\sin x(t) + \dfrac{1}{63\pi}y(t), \ t \in [0,\pi], \\ x(0) = 0, \quad x(\pi) = \dfrac{\sqrt{3}}{2}{}_HI^{1/2}y(\pi/4) + \dfrac{2}{17}{}_HI^{2/3}y(\pi/3) + \dfrac{4}{9}{}_HI^{3/4}y(\pi/2), \\ y(0) = 0, \quad y(\pi) = \dfrac{1}{2}{}_HI^{3/5}x(\pi/6) + \dfrac{\sqrt{5}}{14}{}_HI^{5/6}x(\pi/3). \end{cases} \tag{3.9}$$

Here $q = \sqrt{2}, p = \sqrt{3}, n = 3, m = 2, T = \pi, \alpha_1 = \sqrt{3}/2, \alpha_2 = 2/17, \alpha_3 = 4/9, \beta_1 = 1/2, \beta_2 = \sqrt{5}/14, \rho_1 = 1/2, \rho_2 = 2/3, \rho_3 = 3/4, \gamma_1 = 3/5, \gamma_2 = 5/6, \eta_1 = \pi/4, \eta_2 = \pi/3, \eta_3 = \pi/2, \theta_1 = \pi/6, \theta_2 = \pi/3$, $f(t,x,y) = 1 + (\sqrt{2}x\cos y)/(81) + (\sqrt{3}y)/(36\pi)$ and $g(t,x,y) = (3/2) + (\sqrt{3}\sin x)/(64\pi) + (y)/(63)$. By using the Maple program, we get

$$\Omega = \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{p-1}}{(p-1)^{\rho_i}} \sum_{j=1}^{m} \frac{\beta_j \theta_j^{q-1}}{(q-1)^{\gamma_j}} - T^{q+p-2} \approx -1.955428761 \neq 0.$$

Since $|f(t,x,y)| \leq k_0 + k_1|x| + k_2|y|, |g(t,x,y)| \leq \lambda_0 + \lambda_1|x| + \lambda_2|y|$, where $k_0 = 1, k_1 = \sqrt{2}/81, k_2 = \sqrt{3}/36\pi, \lambda_0 = 3/2, \lambda_1 = \sqrt{3}/64\pi, \lambda_2 = 1/63\pi$, it is found that $M_1 \simeq 12.01088124, M_2 \simeq 8.095664081, M_3 \simeq 5.051706267, M_4 \simeq 14.14407333$. Furthermore,

$$(M_1 + M_3)k_1 + (M_2 + M_4)\lambda_1 \approx 0.6297371340 < 1$$

and

$$(M_1 + M_3)k_2 + (M_2 + M_4)\lambda_2 \approx 0.3736753802 < 1.$$

Thus all the conditions of Theorem 3.3 holds true and consequently the conclusion of Theorem 3.3, the problem (3.9) has at least one solution on $[0, \pi]$.

4. Uncoupled integral boundary conditions case

In this section we consider the following system

$$\begin{cases} {}_{RL}D^q x(t) = f(t, x(t), y(t)), \quad t \in [0, T], \quad 1 < q \le 2, \\ {}_{RL}D^p y(t) = g(t, x(t), y(t)), \quad t \in [0, T], \quad 1 < p \le 2, \\ x(0) = 0, \quad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_{i\,H} I^{\rho_i} x(\eta_i), \\ y(0) = 0, \quad y(T) = \displaystyle\sum_{j=1}^{m} \beta_{j\,H} I^{\gamma_j} y(\theta_j). \end{cases} \tag{4.1}$$

Lemma 4.1 (Auxiliary Lemma). *For $h \in C([0, T], \mathbb{R})$, the unique solution of the problem*

$$\begin{cases} {}_{RL}D^q x(t) = h(t), \quad 1 < q \le 2, \quad t \in [0, T] \\ x(0) = 0, \quad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_{i\,H} I^{\rho_i} x(\eta_i), \end{cases} \tag{4.2}$$

is given by

$$x(t) = {}_{RL}I^q h(t) - \frac{t^{q-1}}{\Lambda} \left({}_{RL}I^q h(T) - \sum_{i=1}^{n} \alpha_i ({}_H I^{\rho_i} {}_{RL} I^q h)(\eta_i) \right), \tag{4.3}$$

where

$$\Lambda := T^{q-1} - \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{q-1}}{(q-1)^{\rho_i}} \neq 0. \tag{4.4}$$

4.1. Existence results for uncoupled case

In view of Lemma 4.1, we define an operator $\mathfrak{T} : X \times Y \to X \times Y$ by

$$\mathfrak{T}(x, y)(t) = \begin{pmatrix} \mathfrak{T}_1(x, y)(t) \\ \mathfrak{T}_2(x, y)(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathfrak{T}_1(x, y)(t) = {} & {}_{RL}I^q f(s, x(s), y(s))(t) \\ & - \frac{t^{q-1}}{\Lambda} \left({}_{RL}I^q f(s, x(s), y(s))(T) - \sum_{i=1}^{n} \alpha_{i\,H} I^{\rho_i} {}_{RL} I^q f(s, x(s), y(s))(\eta_i) \right) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{T}_2(x, y)(t) = {} & {}_{RL}I^p g(s, x(s), y(s))(t) \\ & - \frac{t^{p-1}}{\Phi} \left({}_{RL}I^p g(s, x(s), y(s))(T) - \sum_{j=1}^{m} \beta_{j\,H} I^{\gamma_j} {}_{RL} I^p g(s, x(s), y(s))(\theta_j) \right), \end{aligned}$$

where

$$\Phi := T^{p-1} - \sum_{j=1}^{m} \frac{\beta_j \theta_j^{p-1}}{(p-1)^{\gamma_j}} \neq 0.$$

In the sequel, we set

$$\delta_1 = \frac{T^q}{\Gamma(q+1)} + \frac{T^{2q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^q}{q^{\rho_i}}. \tag{4.5}$$

$$\delta_2 = \frac{T^p}{\Gamma(p+1)} + \frac{T^{2p-1}}{|\Phi|\Gamma(p+1)} + \frac{T^{p-1}}{|\Phi|\Gamma(p+1)} \sum_{j=1}^{m} \frac{|\beta_j|\theta_j^p}{p^{\gamma_j}}. \tag{4.6}$$

Now we present the existence and uniqueness result for the problem (4.1). We do not provide the proof of this result as it is similar to the one for Theorem 3.1.

Theorem 4.2. *Assume that $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist constants $\bar{m}_i, \bar{n}_i, i = 1,2$ such that for all $t \in [0,T]$ and $x_i, y_i \in \mathbb{R}, i = 1,2$,*

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le \bar{m}_1|x_1 - y_1| + \bar{m}_2|x_2 - y_2|$$

and

$$|g(t,x_1,x_2) - g(t,y_1,y_2)| \le \bar{n}_1|x_1 - y_1| + \bar{n}_2|x_2 - y_2|.$$

In addition, assume that

$$\delta_1(\bar{m}_1 + \bar{m}_2) + \delta_2(\bar{n}_1 + \bar{n}_2) < 1,$$

where δ_1 and δ_2 are given by (4.5) and (4.6) respectively. Then the boundary value problem (4.1) has a unique solution.

Example 3. Consider the following system of coupled Riemann-Liouville fractional differential equations with uncoupled Hadamard type fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^{7/6}x(t) = \dfrac{e^{-t}}{(5+t)^2}\dfrac{|x(t)|}{|x(t)|+1} + \dfrac{1}{(e^t+3)^2}\dfrac{|y(t)|}{|y(t)|+1} + \dfrac{\pi}{2}, \quad t \in [0,3], \\ {}_{RL}D^{\sqrt{5}/2}y(t) = \dfrac{4|x(t)|}{33(t+1)^2} + \dfrac{2\sin y(t)}{17(e^t+1)} + \sqrt{3}, \quad t \in [0,3], \\ x(0) = 0, \quad x(3) = \dfrac{1}{6}{}_HI^{\sqrt{2}}x(1/2) - \dfrac{1}{5}{}_HI^{3/4}x(1) + \dfrac{2}{9}{}_HI^{\sqrt{5}}x(3/2), \\ y(0) = 0, \quad y(3) = \dfrac{3}{4}{}_HI^{2/3}y(1/2) + \dfrac{1}{2}{}_HI^{\sqrt{3}}y(3/2) + \dfrac{\pi}{2}{}_HI^{5/4}y(5/3). \end{cases} \tag{4.7}$$

Here $q = 7/6, p = \sqrt{5}/2, n = 3, m = 3, T = 3, \alpha_1 = 1/6, \alpha_2 = -1/5, \alpha_3 = 2/9, \beta_1 = 3/4, \beta_2 = 1/2, \beta_3 = \pi/2, \rho_1 = \sqrt{2}, \rho_2 = 3/4, \rho_3 = \sqrt{5}, \gamma_1 = 2/3, \gamma_2 = \sqrt{3}, \gamma_3 = 5/4, \eta_1 = 1/2, \eta_2 = 1, \eta_3 = 3/2, \theta_1 = 1/2, \theta_2 = 3/2, \theta_3 = 5/3, f(t,x,y) = (e^{-t}|x|)/(((5+t)^2)(|x|+1)) + (|y|)/(((e^t+3)^2)(|y|+1)) + (\pi/2)$ and $g(t,x,y) = (4|x|)/(33((5+t)^2)) + (2\sin y(t))/(17(e^t+1)) + \sqrt{3}$. Since $|f(t,x_1,y_1) - f(t,x_2,y_2)| \le ((1/25)|x_1 - x_2| + (1/16)|y_1 - y_2|)$ and $|g(t,x_1,y_1) - g(t,x_2,y_2)| \le ((4/33)|x_1 - x_2| + (1/17)|y_1 - y_2|)$. By using the Maple program, we can find

$$\Lambda := T^{q-1} - \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{q-1}}{(q-1)^{\rho_i}} \approx -12.96942934 \ne 0$$

and

$$\Phi := T^{p-1} - \sum_{j=1}^{m} \frac{\beta_j \theta_j^{p-1}}{(p-1)^{\gamma_j}} \approx -47.08574657 \ne 0.$$

With the given values, it is found that $\bar{m}_1 = 1/25, \bar{m}_2 = 1/16, \bar{n}_1 = 4/33, \bar{n}_2 = 1/17, \delta_1 \simeq 3.678923396, \delta_2 \simeq 3.402792438$. In consequence,

$$\delta_1(\bar{m}_1 + \bar{m}_2) + \delta_2(\bar{n}_1 + \bar{n}_2) \approx 0.9897135986 < 1.$$

Thus all the conditions of Theorem 4.2 are satisfied. Therefore, there exists a unique solution for the problem (4.7) on $[0, 3]$.

The second result dealing withe the existence of solutions for the problem (4.1) is analogous to Theorem 4.3 and is given below.

Theorem 4.3. *Assume that there exist real constants* κ_i, $\nu_i \geq 0$ $(i = 1, 2)$ *and* $\kappa_0 > 0, \nu_0 > 0$ *such that* $\forall x_i \in \mathbb{R}$, $(i = 1, 2)$ *we have*

$$|f(t, x_1, x_2)| \leq \kappa_0 + \kappa_1|x_1| + \kappa_2|x_2|,$$
$$|g(t, x_1, x_2)| \leq \nu_0 + \nu_1|x_1| + \nu_2|x_2|.$$

In addition it is assumed that

$$\delta_1\kappa_1 + \delta_2\nu_1 < 1 \quad \text{and} \quad \delta_1\kappa_2 + \delta_2\nu_2 < 1,$$

where δ_1 *and* δ_2 *are given by* (4.5) *and* (4.6) *respectively. Then the boundary value problem* (4.1) *has at least one solution.*

Proof. Setting

$$\delta_0 = \min\{1 - (\delta_1\kappa_1 + \delta_2\nu_1),\ 1 - (\delta_1\kappa_2 + \delta_2\nu_2)\}, \quad \kappa_i,\ \nu_i \geq 0\ (i = 1, 2),$$

the proof is similar to that of Theorem 4.3. So we omit it. □

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