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A Time-varying repairable system with repairman vacation and warning device

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Abstract

In this paper, a new kind of repairable system with repairman vacation and warning device is discussed, in which the delayed vacation rate and failure rates are functions related to system working time. The system model is established by using probability analysis method, which then is translated into a initial value problem of a class of abstract semi-linear evolution equation in a suitable Banach space for further study. The conditions of the existence and uniqueness of the system solution as well as system stability is analyzed by using C_0-semigroup theory. Some steady-state reliability indexes are studied by using Laplace transformation. In the end, numerical examples are presented to compare some indexes of the systems with and without warning device.

Keywords: Repairable system, delayed-multiple vacations, semi-linear evolution system, C_0-semigroup theory, well-posedness, stability, sensitivity analysis.
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1. Introduction

Repairable systems can be found in a variety of areas, such as aviation, aerospace, defense, finance and network communications. The well understanding of this type of systems is of both theoretical significance and real applications. From the perspective of rational use of human resources, the models with repairman's vacation makes repairable systems more realistic and flexible. The concept of repairman vacation originally occurred in queueing systems, which has been well studied in the past three decades and successfully applied in many areas such as manufacturing/service and computer/communication network systems. For

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excellent works, one can refer to [2, 7, 16, 17]. In the early twentieth century, inspired by the vacation queueing models, researchers, such as Jain, Rakhee & Singh [5], Ke & Wang [6], Liu, Tang & Luo [13], *et al.*, introduced the concept of repairman vacation into repairable systems, including delayed vacation, single vacation and multiple vacations. However, to the authors' best knowledge, most of available references about repairable systems with repairman vacation are interested in the system steady-state behaviors. Moreover, the failure rates in all available references are either constants or at most, are related to the age of a system (for detailed information, on can refer to [11, 12]), but not related to the working time of a system. But in practice, the failure rate of a unit is generally dependent on the working time of the system. For example, the failure rate can be increased with the system running. Then it will be decreased with the measures such as preventive maintenance, periodic detection and periodic maintenance. For this reason, we are dedicated to studying a repairable system with repairman vacation in which the failure rate of all the units are related to the working time of the system.

As Mobley [14] pointed out, one third of all maintenance costs were wasted as the result of unnecessary or improper maintenance activities. Today, the role of maintenance tends to be a "profit contributor". Much more profit probably be produced when different repairs are considered according to the extent of damages of a system. In practice, in the light of maintenance content, technical requirements and workload size, equipment repair work can be divided into three categories, namely, overhaul, moderate repair and minor repair. Overhaul is the largest planned maintenance work. To achieve the goal of complete elimination of defects prior to repair as well as restoration of specified function and precision, during the process of overhaul, all or most of the equipment components are disassembled and all defective parts are either repaired or replaced. Moderate repair is applied to the equipment when the states of the components of equipment are deteriorated to not able to reach the production process requirements. During the process of moderate repair, a portion of equipment components are generally disassembled and inspected. Then, failed parts are, if necessary, either repaired or replaced to restore their precision and performance. Moderate repair has characteristics of flexible arrangements, strong pertinence, short downtime, low maintenance costs, meeting production needs in a timely manner, and avoiding excess maintenance. For large equipment or single key equipment, moderate repair can be arranged in production gap time (holidays) to ensure normal production based on the issues found in the daily inspection and/or monitoring. Minor repair is the minimal scheduled maintenance. The content for minor repairs is to adjust, replace or repair of failed parts to restore the normal function of the equipment according to the problems found in routine inspection, periodic inspection and condition monitoring diagnostic problems. For the periodic maintenance of the equipment, the main content of minor repair is to replace or repair the parts which are going to be out of work in the maintenance interval period to ensure the normal function of the equipment based on the mastered law of wearing.

Warning systems emerged in the background of repairable systems are stepping into the times of requiring of both advanced warning and real-time fault detection. The so-called warning system is able to send emergency signals and report dangerous situations prior to disasters, catastrophes and/or other dangers need to watch out based on previous experiences and or observed possible omens. Real-time warning systems play an important role in fault management in banking, telecommunications, securities, electric power and other industries. If the warning prompts during system operation, operating staff can choose whether to shut down the system, operate carefully, or repair the system. Warning systems can help users to achieve the 24-hour uninterrupted real-time monitoring and alerting during running of various types of network infrastructure sand application services. Accordingly, the study of repairable systems with warning device is important both in theory and in practice. However, repairable systems with warning device are seldom reported in the current literatures.

To this end, this paper considers a simple repairable system with a warning device which can send an alarm when the system is not in good condition and a repairman who follows delayed-multiple vacations policy and carries out overhaul or minor repair for the system according to the defects. In this paper, we are devoted to studying the transient and asymptotic behavior of the system by semigroup theory, and make comparisons of indices (such as reliability, availability, and the probability of the repairman' vacation) and profit of the two systems with and without warning device.

The rest of the paper is arranged as follows. In the following section, the system model is established by using probability analysis method and then is translated into a initial value problem of a class of abstract semi-linear evolution equation in a Banach space. In Section 3, some properties of the system operator are discussed, thereby the existence and uniqueness, and the continuous dependence of the system solution for the initial value are derived by using C_0-semigroup theory. In Section 4, we derive some of reliability indexes, such as steady-state availability, steady-state failure frequency, and steady-state probability of repairman on vacation by using Laplace transform method. Section 5 does numerical analysis. And a brief conclusion is presented in Section 6 at the end of the paper.

2. System formulation

The system model considered in this paper is an one-unit repairable system with a repairman and a warning device. The system is described specifically as follows: at the initial time $t = 0$, the unit is new, the system begins to work and the repairman starts to prepare for vacation. That is the repairman will not leave for a vacation immediately if there is no component failed. However, there is a stochastic vacation-preparing period in which if a failed component appears he will stop the vacation preparing and serve it immediately; otherwise he will take a rest on the end of the vacation-preparing period. The warning device can send an alarm once the system fails. And we assume that the warning device is sensitive enough to send only true alarms which need to repair. If the warning device sends an alarm in the delayed-vacation period, the repairman will stay in the system until the unit fails. Whenever the repairman returns from a vacation, he either prepares for the next vacation if the unit is working or deals with the failed unit immediately, or stays in the system if the warning device has sent an alarm. That is the repairman follows delayed-multiple vacation policy. And we assume that the time repairman returning to the system can not be late than the time warning device sending next alarm. The system may go for minor repair or overhaul from its warning state with probability $\lambda_1(t)$ or $\lambda_2(t)$ respectively. The repair facility is neither failed nor deteriorated. The unit is repaired as good as new.

Set all possible states of the system at time t as follows.

- 0 the system is working and the repairman is preparing for the vacation.;
- 1 the system is working and the repairman is on vacation;
- 2 the system is warning and the repairman is in the system;
- 21 the unit is minor repaired by the repairman;
- 22 the unit is overhauled by the repairman;
- 3 the system is warning and the repairman is on vacation;
- 31 the unit needs to be minor repaired while the repairman is on vacation;
- 32 the unit needs to be overhauled while the repairman is on vacation.

Then with the probability analysis method, the system model can be described as follows.

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + \varepsilon(t) + \alpha_0\right] P_0(t) = \int_0^\infty r(x)P_1(t,x)\mathrm{d}x + \sum_{i=1}^{2}\int_0^\infty \mu_i(y)P_{2i}(t,y)\mathrm{d}y \tag{2.1}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \alpha_0 + r(x)\right] P_1(t,x) = 0 \tag{2.2}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + \lambda_1(t) + \lambda_2(t)\right] P_2(t) = \alpha_0 P_0(t) + \int_0^\infty r(x)P_3(t,x)\mathrm{d}x \tag{2.3}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1(t) + \lambda_2(t) + r(x)\right] P_3(t,x) = \alpha_0 P_1(t,x) \tag{2.4}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_i(y)\right] P_{2i}(t,y) = 0, \quad i = 1,2 \tag{2.5}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + r(x)\right] P_{3i}(t,x) = \lambda_i(t)P_3(t,x), \quad i = 1,2 \tag{2.6}$$

The boundary conditions are

$$P_1(t,0) = \varepsilon(t)P_0(t); \tag{2.7}$$

$$P_3(t,0) = P_{3i}(t,0) = 0, \quad i = 1,2; \tag{2.8}$$

$$P_{2i}(t,0) = \lambda_i(t)P_2(t) + \int_0^\infty r(x)P_{3i}(t,x)\mathrm{d}x, \quad i = 1,2. \tag{2.9}$$

The initial conditions are

$$P_0(0) = 1, \text{ the others equal to } 0. \tag{2.10}$$

Here $P_i(t)$ represents the probability that the system is in state i at time t, $i = 0,2$. $P_j(t,x)\mathrm{d}x$ represents the probability that the system is in state j with elapsed vacation time lying in $[x,\ x+\mathrm{d}x)$ at time t, $j = 1,3,31,32$. $P_k(t,y)\mathrm{d}y$ represents the probability that the system is in state k with elapsed repair time lying in $[y,\ y+\mathrm{d}y)$ at time t, $k = 21,22$. $\varepsilon(t)$ denotes the delayed vacation rate function, $r(x)$ denotes the vacation rate function, $\mu_1(y)$ and $\mu_2(y)$ denote the minor and overhaul repair rate functions, respectively.

Concerning the practical background, we can assume that $\varepsilon(t)$, $\lambda_i(t)$, $r(x)$, $\mu_i(y)$ are all nonnegative bounded functions satisfying $\varepsilon(t) \to \varepsilon \geq 0$, $\lambda_i(t) \to \lambda_i \geq 0\,(t \to \infty)$, $r(x)$, $\mu_i(y) \in L[0,T]\,(0 < T < \infty)$ and $\int_0^\infty r(x)\mathrm{d}x = \int_0^\infty \mu_i(y)\mathrm{d}y = \infty$, $i = 1,2$.

For further study, we will translate the system (2.1)-(2.10) into an initial value problem of a class of abstract semi-linear evolution system in a Banach space.

Choose the state space X as below

$$X = \{P = (P_0, P_1, P_2, P_3, P_{21}, P_{22}, P_{31}, P_{32})^{\mathrm{T}} | P_i \in \mathbb{R},\ P_j \in L^1(\mathbb{R}^+),$$
$$\|P\| = \sum_{i=0,2} |P_i| + \sum_{j=1,3,21,22,31,32} \|P_j\| < \infty\},$$

where $\mathbb{R}_+$ represents the set of nonnegative real numbers. Obviously, X is a Banach space.

Define operator $A : D(A) \subset X \to X$ as follows.

$$\begin{aligned} &A(P_0, P_1(x), P_2, P_3(x), P_{21}(y), P_{22}(y), P_{31}(x), P_{32}(x))^{\mathrm{T}} \\ &= \Bigg(\int_0^\infty r(x)P_1(x)\mathrm{d}x + \sum_{i=1}^{2} \int_0^\infty \mu_i(y)P_{2i}(y)\mathrm{d}y, -P_1'(x) - r(x)P_1(x), \\ &\quad \int_0^\infty r(x)P_3(x)\mathrm{d}x, -P_3'(x) - r(x)P_3(x), -P_{21}'(y) - \mu_1(y)P_{21}(y), \\ &\quad - P_{22}'(y) - \mu_2(y)P_{22}(y), -P_{31}'(x) - r(x)P_{31}(x), -P_{32}'(x) - r(x)P_{32}(x) \Bigg)^{\mathrm{T}} \end{aligned}$$

with

$$D(A) = \left(\begin{array}{c} P = (P_0, P_1, P_2, P_3, P_{21}, P_{22}, P_{31}, P_{32})^{\mathrm{T}} \in X | \\ P_j \text{ are differentiable in } \mathbb{R}_+ \text{ and } P_j' \in L^1(\mathbb{R}_+), \\ j = 1,3,21,22,31,32 \end{array} \right).$$

Let $f(t,P) : [0,\infty) \times X \to X$ be

$$\begin{aligned} f(t,P) = (&-[\varepsilon(t) + \alpha_0]P_0(t), -\alpha_0 P_1(t,x), -[\lambda_1(t) + \lambda_2(t)]P_2(t) + \alpha_0 P_0(t), \\ &-[\lambda_1(t) + \lambda_2(t)]P_3(t,x) + \alpha_0 P_1(t,x), 0, 0, \lambda_1(t)P_3(t,x), \lambda_2(t)P_3(t,x))^{\mathrm{T}}. \end{aligned}$$

Then the system (2.1)-(2.10) can be translated into an initial value problem of a class of abstract semi-linear evolution system in Banach space X:

$$\begin{cases} \dfrac{\mathrm{d}P(t,\cdot)}{\mathrm{d}t} = AP(t,\cdot) + f(t, P(t,\cdot)) \quad t \geq 0, \\ P(t,\cdot) = (P_0(t), P_1(t,x), P_2(t), P_3(t,x), P_{21}(t,y), P_{22}(t,y), P_{31}(t,x), P_{32}(t,x))^{\mathrm{T}}, \\ P(0,\cdot) = (1,0,0,0,0,0,0,0)^{\mathrm{T}}. \end{cases} \tag{2.11}$$

3. Existence and uniqueness of system solution

The unique existence of the solution of initial value problem of abstract semi-linear evolution equations, has been studied by few researchers (for detailed information, one can refer to [8, 9, 10, 18]). But in repairable systems, to the authors' best knowledge, this problem has not been discussed. In this section, we mainly study the unique existence of the mild solution of system (2.11) because of its limitation of the physical condition, by using C_0-semigroup theory. To this end, we first present some properties of the system operator.

Lemma 3.1. *The system operator A is densely defined in X.*

Proof. For any $F = (f_0, f_1, f_2, f_3, f_{21}, f_{22}, f_{31}, f_{32})^{\mathrm{T}} \in X$, then $f_j \in L^1(\mathbb{R}_+), j = 1, 3, 2i, 3i,\ i = 1, 2$. Thus for any $\eta > 0$, there exist positive numbers G_j and δ_j such that

$$\int_{G_k}^{\infty} |f_k(x)|\mathrm{d}x < \frac{\eta}{12}, \int_0^{\delta_k} |f_k(x)|\mathrm{d}x < \frac{\eta}{24}, \int_{G_{2i}}^{\infty} |f_{2i}(\xi)|\mathrm{d}\xi < \frac{\eta}{24}, \int_0^{\delta_{2i}} |f_{2i}(\xi)|\mathrm{d}\xi < \frac{\eta}{48}, k = 1, 3, i = 1, 2.$$

Let

$$\delta = \min\left\{\delta_1, \delta_3, \delta_{21}, \delta_{22}, \delta_{31}, \delta_{32}, \frac{2\eta}{12\left\{\varepsilon|f_0| + \sum_{i=1}^{2}[\lambda_i|f_2| + \int_0^{\infty} r(x)|f_{3i}(x)|\mathrm{d}x]\right\} + \eta r}\right\},$$

where $r = \sup_{x\geq 0} r(x)$. Take $P_0 = f_0, P_2 = f_2$ and

$$P_1(x) = \begin{cases} \varepsilon P_0, & 0 \leq x < \delta \\ g_1(x), & \delta \leq x \leq G_1 \\ 0, & G_1 < x < \infty \end{cases} \qquad P_3(x) = \begin{cases} 0, & 0 \leq x < \delta \\ g_3(x), & \delta \leq x \leq G_3 \\ 0, & G_3 < x < \infty \end{cases}$$

$$P_{2i}(y) = \begin{cases} \lambda_i P_2 + \int_0^{\infty} r(x)P_{3i}(x)\mathrm{d}x, & 0 \leq y < \delta \\ g_{2i}(y), & \delta \leq y \leq G_{2i} \\ 0, & G_{2i} < y < \infty \end{cases} \qquad P_{3i}(x) = \begin{cases} 0, & 0 \leq x < \delta \\ g_{3i}(x), & \delta \leq x \leq G_{3i} \\ 0, & G_{3i} < x < \infty. \end{cases}$$

Here, g_j are continuously differentiable functions satisfying $g_j(G_j) = 0, g_1(\delta) = \varepsilon P_0,\ g_3(\delta) = 0,$ $g_{2i}(\delta) = \lambda_i P_2 + \int_0^{\infty} r(x)P_{3i}(x)\mathrm{d}x,\ g_{3i}(\delta) = 0$ and $\int_\delta^{G_k} |f_k(x) - P_k(x)|\mathrm{d}x < \frac{\eta}{12}, \int_\delta^{G_{2i}} |f_{2i}(y) - P_{2i}(y)|\mathrm{d}y < \frac{\eta}{24},$ $k = 1, 3, i = 1, 2$. Then P_j are continuously differentiable functions and $P_j' \in L^1(\mathbb{R}_+)$. Thus $P = (P_0, P_1, P_2, P_3, P_{21}, P_{22}, P_{31}, P_{32})^{\mathrm{T}} \in D(A)$. Furthermore, it can be easily proved that $\|F - P\| < \eta$. Therefore, $D(A)$ is dense in X. □

Lemma 3.2. *$\{\xi|\xi > \varepsilon\} \subset \rho(A)$, where $\rho(A)$ is the resolvent set of system operator A. And there exists a constant $W > 0$, such that for any $\xi > W$,*

$$\|R(\xi; A)\| \leq \frac{1}{\xi - W},$$

where $R(\xi; A) = (\xi I - A)^{-1}$.

Proof. For any $F = (f_0, f_1, f_2, f_3, f_{21}, f_{22}, f_{31}, f_{32})^{\mathrm{T}} \in X$, consider the operator equation $(\xi I - A)P = F$. That is

$$\xi P_0 - \int_0^{\infty} r(x)P_1(x)\mathrm{d}x - \sum_{i=1}^{2}\int_0^{\infty} \mu_i(y)P_{2i}(y)\mathrm{d}y = f_0 \tag{3.1}$$

$$P_1'(x) + [\xi + r(x)]P_1(x) = f_1(x) \tag{3.2}$$

$$\xi P_2 - \int_0^{\infty} r(x)P_3(x)\mathrm{d}x = f_2 \tag{3.3}$$

$$P_3'(x) + [\xi + r(x)]P_3(x) = f_3(x) \tag{3.4}$$
$$P_{2i}'(y) + [\xi + \mu_i(y)]P_{2i}(y) = f_{2i}(y), \quad i = 1, 2 \tag{3.5}$$
$$P_{3i}'(x) + [\xi + r(x)]P_{3i}(x) = f_{3i}(x), \quad i = 1, 2 \tag{3.6}$$
$$P_1(0) = \varepsilon P_0 \tag{3.7}$$
$$P_3(0) = P_{3i}(0) = 0, \quad i = 1, 2 \tag{3.8}$$
$$P_{2i}(0) = \lambda_i P_2 + \int_0^\infty r(x)P_{3i}(x)\mathrm{d}x, \quad i = 1, 2. \tag{3.9}$$

Solving equations (3.2)-(3.6) with the help of (3.7)-(3.9) derives

$$P_1(x) = \varepsilon P_0 e^{-\int_0^x [\xi + r(s)]\mathrm{d}s} + Y_1(x), \quad P_3(x) = Y_3(x), \quad P_{3i}(x) = Y_{3i}(x) \tag{3.10}$$
$$P_{2i}(y) = \left[\lambda_i P_2 + \int_0^\infty r(x)P_{3i}(x)\mathrm{d}x\right] e^{-\int_0^y [\xi + \mu_i(s)]\mathrm{d}s} + Y_{2i}(y) \tag{3.11}$$
$$P_2 = \frac{1}{\xi}\left[f_2 + \int_0^\infty r(x)Y_3(x)\mathrm{d}x\right], \tag{3.12}$$

where $Y_j(x) = \int_0^x f_j(s)e^{-\int_s^x [\xi + r(\tau)]\mathrm{d}\tau}\mathrm{d}s$, $Y_{2i}(y) = \int_0^y f_{2i}(s)e^{-\int_s^y [\xi + \mu_i(\tau)]\mathrm{d}\tau}\mathrm{d}s$, $j = 1, 3, 3i$; $i = 1, 2$. Substituting (3.10)-(3.12) into (3.1) yields

$$\begin{aligned}(\xi - \varepsilon M)P_0 =& f_0 + \int_0^\infty r(x)Y_1(x)\mathrm{d}x + \sum_{i=1}^2 \int_0^\infty \mu_i(y)Y_{2i}(y)\mathrm{d}y \\ &+ \frac{1}{\xi}\left[f_2 + \int_0^\infty r(x)Y_3(x)\mathrm{d}x\right](\sum_{i=1}^2 \lambda_i N_i) + \sum_{i=1}^2 N_i \int_0^\infty r(x)Y_{3i}(x)\mathrm{d}x,\end{aligned}$$

where $M = \int_0^\infty r(x)e^{-\int_0^x [\xi + r(s)]\mathrm{d}s}\mathrm{d}x$, $N_i = \int_0^\infty \mu_i(y)e^{-\int_0^y [\xi + \mu_i(s)]\mathrm{d}s}\mathrm{d}y$, $i = 1, 2$. It is easy to know $M < 1$ for $\xi > 0$. Then $\xi - \varepsilon M > \xi - \varepsilon > 0$, for $\xi > \varepsilon$. Thus

$$\begin{aligned}P_0 = \frac{1}{\xi - \varepsilon M}\Bigg[& f_0 + \int_0^\infty r(x)Y_1(x)\mathrm{d}x + \sum_{i=1}^2 \int_0^\infty \mu_i(y)Y_{2i}(y)\mathrm{d}y \\ &+ \frac{1}{\xi}\left[f_2 + \int_0^\infty r(x)Y_3(x)\mathrm{d}x\right](\sum_{i=1}^2 \lambda_i N_i) + \sum_{i=1}^2 N_i \int_0^\infty r(x)Y_{3i}(x)\mathrm{d}x\Bigg].\end{aligned} \tag{3.13}$$

So, it is easy to deduce that for any $\xi > \varepsilon$, equations (3.1)-(3.9) have a unique solution $P = (P_0, P_1, P_2, P_3, P_{21}, P_{22}, P_{31}, P_{32})^{\mathrm{T}} \in D(A)$. This means that $(\xi I - A)$ is surjective. Because $(\xi I - A)$ is closed and $D(A)$ is dense in X, then $(\xi I - A)^{-1}$ exists and is bounded by Inverse operator theorem, for any $\xi > \varepsilon$.

Furthermore, from equations (3.10)-(3.13), it is not hard to derive the following estimation. That is there exists a constant $W > 0$, such that

$$\|P\| = \sum_{i=0,2} |P_i| + \sum_{j=3,21,22,31,32} \|P_j\| < \frac{1}{\xi - W}\|F\|.$$

This means that for any $\xi > W$, $(\xi I - A)^{-1}$ exits and $\|(\xi I - A)^{-1}\| < \frac{1}{\xi - W}$. □

According to Hille-Yosida Theorem [15] with lemmas 3.1 and 3.2, the following result is obvious.

Theorem 3.3. *The system operator A generates a C_0 semigroup $T(t)$.*

Lemma 3.4 ([15])**.** *Let $f : [t_0, T] \times X \to X$ be continuous about t on $[t_0, T]$ and uniformly Lipschitz continuous (with constant L) on X, if $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, on X, then for every $u_0 \in X$ the initial value problem*

$$\begin{cases}\dfrac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = f(t, u(t)), & t > t_0 \\ u(t_0) = u_0\end{cases}$$

has a unique mild solution $u \in C([t_0, T] : X)$. Moreover, the mapping $u_0 \to u$ is Lipschitz continuous from X into $C([t_0, T] : X)$.

Theorem 3.5. *For any $T > 0$, assume $\varepsilon(t)$ and $\lambda_i(t)\,(i = 1, 2)$ are continuous on $[0, T]$. Then for any $P_1(t, x), P_3(t, x) \in C([0, T]; L^1(R_+))$, where P_1, P_3 are the second and forth components of P, the semi-linear evolution system (2.11) has a unique mild solution $P \in C([0, T] : X)$. Moreover, the mapping $P_0 \to P$ is Lipschitz continuous from X into $C([0, T] : X)$.*

Proof. For any $T > 0$, with the assumptions of the theorem, it is obvious that $f(t, P)$ is continuous about t on $[0, T]$. Furthermore, for any $t \in [0, T]$ and P, $Q \in X$, it is easy to know that

$$\begin{aligned}\|f(t,P)-f(t,Q)\| =& [\varepsilon(t)+2\alpha_0]|P_0-Q_0| + 2\alpha_0\|P_1-Q_1\| + [\lambda_1(t)\\ &+\lambda_2(t)]|P_2-Q_2| + 2[\lambda_1(t)+\lambda_2(t)]\|P_3-Q_3\|\\ \le& L\|P-Q\|\end{aligned}$$

where P_0, P_1, P_2, P_3 and Q_0, Q_1, Q_2, Q_3 are respectively the first, second, third and forth components of P and Q, and $L = 2\max\left\{\sup\limits_{t\in[0,T]} \varepsilon(t) + \alpha_0,\ \sup\limits_{t\in[0,T]} [\lambda_1(t) + \lambda_2(t)]\right\}$. Therefore, the result of Theorem 3.5 is obvious by using Lemma 3.4 combing Theorem 3.3. □

4. Stability of system solution

In this section, we are dedicated to studying the asymptotic stability of the system solution by substituting the limit values ε and λ_i respectively for the delayed vacation rate $\varepsilon(t)$ and the failure rate $\lambda_i(t)\,(i = 1, 2)$ in system (2.1)-(2.10). We first translate the system equations (2.1)-(2.10) into an abstract Cauchy problem in a suitable Banach space. Then we present some spectrum properties of the system operator and its adjoint operator. Thus the asymptotic stability of the system solution can be derived readily with the preparation.

We define system operator B in state space X which has been defined in Section 3 as below.

$$BP = \begin{pmatrix} -(\varepsilon+\alpha_0)P_0 + \int_0^\infty r(x)P_1(x)\mathrm{d}x + \sum\limits_{i=1}^{2}\int_0^\infty \mu_i(y)P_{2i}(y)\mathrm{d}y \\ -P_1'(x) - [\alpha_0 + r(x)]P_1(x) \\ \alpha_0 P_0 - (\lambda_1+\lambda_2)P_2 + \int_0^\infty r(x)P_3(x)\mathrm{d}x \\ \alpha_0 P_1(x) - P_3'(x) - [\lambda_1+\lambda_2+r(x)]P_3(x) \\ -P_{21}'(y) - \mu_1(y)P_{21}(y) \\ -P_{22}'(y) - \mu_2(y)P_{22}(y) \\ \lambda_1 P_3(x) - P_{31}'(x) - r(x)P_{31}(x) \\ \lambda_2 P_3(x) - P_{32}'(x) - r(x)P_{32}(x) \end{pmatrix}$$

with domain

$$\begin{aligned}D(B) = \Big\{&P = (P_0, P_1, P_2, P_{21}, P_{22}, P_3, P_{31}, P_{32})^{\mathrm{T}} \in X:\ P_j' \in L^1(\mathbb{R}^+) \text{ are absolutely}\\ &\text{continuous functions satisfying } P_1(0) = \varepsilon P_0, P_{2i}(0) = \lambda_i P_2 + \int_0^\infty r(x)P_{3i}(x)\mathrm{d}x,\\ &P_3(0) = P_{3i}(0) = 0,\ j = 1, 3, 2i, 3i,\ i = 1, 2\Big\}.\end{aligned}$$

Then the system equations (2.1)-(2.10) can be rewritten as an abstract Cauchy problem in Banach space X.

$$\begin{cases} \dfrac{\mathrm{d}P(t,\cdot)}{\mathrm{d}t} = BP(t,\cdot), \quad t \ge 0 \\ P(t,\cdot) = (P_0(t), P_1(t,x), P_2(t), P_3(t,x), P_{21}(t,y), P_{22}(t,y), P_{31}(t,x), P_{32}(t,x))^{\mathrm{T}} \\ P(0,\cdot) \triangleq P_0 = (1, 0, 0, \ldots, 0)^{\mathrm{T}}_{1\times 8}. \end{cases} \tag{4.1}$$

In the following, we present some properties of system operator B including its spectrum distribution.

Lemma 4.1. *The system operator B is a densely closed dissipative operator.*

Proof. Firstly, with the method in the proof of Lemma 3.1, we can obtain that $D(B)$, the domain of operator B is dense in X.

Next, we prove that B is a closed operator. Choose $P^n = (P_0^n, P_1^n, P_2^n, P_3^n, P_{21}^n, P_{22}^n, P_{31}^n, P_{32}^n)^{\mathrm{T}} \in D(B)$. Let $P^n \to P = (P_0, P_1, P_2, P_3, P_{21}, P_{22}, P_{31}, P_{32})^{\mathrm{T}}, BP^n \to Q = (Q_0, Q_1, Q_2, Q_3, Q_{21}, Q_{22}, Q_{31}, Q_{32})^{\mathrm{T}},\ n \to \infty$. According to Proposition 1 ([4], II.2.10), the differential operator $\mathcal{D}$ is the infinitesimal generator of a left translation semigroup $\{T_l(t)\}_{t\geq 0}$ defined on

$$D(\mathcal{D}) = \left\{ f \in L^1(\mathbb{R}^+) \middle| f \text{ is absolutely continuous satisfying } f^{'} \in L^1(\mathbb{R}^+) \right\}.$$

Then $P_j \in D(\mathcal{D})$ due to $D(\mathcal{D})$ is closed and $P_j^n \in D(\mathcal{D})$, which is equivalent to $P_j^{'} \in L^1(\mathcal{R}^+)$ are absolutely continuous, $j = 1, 3, 21, 22, 31, 32$. Furthermore, $P_1^n(0) = \varepsilon P_0^n \to \varepsilon P_0 = P_1(0)$, $P_{2i}^n(0) = \lambda_i P_2^n + \int_0^\infty r(x) P_{3i}^n(x)\mathrm{d}x \to \lambda_i P_2 + \int_0^\infty r(x) P_{3i}(x)\mathrm{d}x = P_{2i}(0),\ n \to \infty, i = 1, 2$. Thus $P \in D(B)$. Noting the bounded measurable of $r(x), \mu_i(y), i = 1, 2$, it is not hard to deduce that $BP = Q$. This implies that B is a closed operator.

Finally, we prove that B is a dissipative operator. For any $P = (P_0, P_1, P_2, P_3, P_{21}, P_{22}, P_{31}, P_{32})^{\mathrm{T}} \in D(B)$, set $Q_k = \|P\|\mathrm{sgn}(P_k),\ k = 0, 1, 2, 3, 21, 22, 31, 32$ and take $Q = (Q_0, Q_1, Q_2, Q_3, Q_{21}, Q_{22}, Q_{31}, Q_{32})^{\mathrm{T}}$. Clearly, $Q \in X^* = \mathbb{R} \times L^\infty(\mathbb{R}^+) \times \mathbb{R} \times (L^\infty(\mathbb{R}^+))^5$, the duel space of X. Moreover, it is easy to know that $\langle P, Q\rangle = \|P\|^2 = \|Q\|^2$ and $\langle BP, Q\rangle \leq 0$. This manifests that B is a dissipative operator. □

Lemma 4.2. *If γ is a complex number γ with positive real part or a pure imaginary number, then γ is a regular point of the system operator B.*

Proof. For any $G \in X$, it is not very hard to prove that the operator equation $(\gamma I - B)P = G$ has a unique solution $P \in D(B)$ if γ is a complex number satisfying conditions of the lemma. This implies that $(\gamma I - B)^{-1}$ exists and is bounded by using Banach Inverse Operator Theorem. Thus the proof is completed. □

Lemma 4.3. *0 is an eigenvalue of the system operator B with algebraic multiplicity one.*

Proof. Repeating the proof process of Lemma 4.2 with $\gamma = 0$ and $G = 0$, it can be yielded readily that 0 is an eigenvalue of the system operator B with geometric multiplicity one. Then by recalling Ref.[3], it only needs to prove that the algebraic index of eigenvalue 0 is one, which can be easily obtained by using the reduction to absurdity. □

In the following, we will present some properties of B^*, the adjoint operator of system operator B, including its spectrum distribution.

First, the dual space of X is: $X^* = \mathbb{R} \times L^\infty(\mathbb{R}^+) \times \mathbb{R} \times (L^\infty(\mathbb{R}^+))^5$, with norm $\|Q\| = \sup\{|Q_i|,\ \|Q_j\|_{L^\infty(\mathbb{R}^+)},$ $i = 0, 2,\ j = 1, 3, 21, 22, 31, 32\}$, for $Q \in X^*$. Then the adjoint operator B^* can be obtained as presented below with the equality $\langle BP,\ Q\rangle = \langle P,\ B^*Q\rangle$ where $P \in D(B)$ and $Q \in X^*$.

$$B^*Q = \begin{pmatrix} -(\varepsilon + \alpha_0)Q_0 + \varepsilon Q_1(0) + \alpha_0 Q_2 \\ Q_1^{'}(x) - [\alpha_0 + r(x)]Q_1(x) + r(x)Q_0 + \alpha_0 Q_3(x) \\ -(\lambda_1 + \lambda_2)Q_2 + \lambda_1 Q_{21}(0) + \lambda_2 Q_{22}(0) \\ Q_3^{'}(x) - (\lambda_1 + \lambda_2)Q_3(x) + \lambda_1 Q_{31}(x) + \lambda_2 Q_{32}(x) + r(x)[Q_2 - Q_3(x)] \\ Q_{21}^{'}(y) - \mu_1(y)Q_{21}(y) + \mu_1(y)Q_0 \\ Q_{22}^{'}(y) - \mu_2(y)Q_{22}(y) + \mu_2(y)Q_0 \\ Q_{31}^{'}(x) - r(x)Q_{31}(x) + r(x)Q_{21}(0) \\ Q_{32}^{'}(x) - r(x)Q_{32}(x) + r(x)Q_{22}(0) \end{pmatrix} \triangleq (C + D)Q,$$

$$D(B^*) = \Big\{ Q = (Q_0, Q_1, Q_2, Q_3, Q_{21}, Q_{22}, Q_{31}, Q_{32})^{\mathrm{T}} \in X^* : Q_j^{'} \in L^\infty(\mathbb{R}^+) \text{ is an absolutely}$$

continuous function satisfying $Q_j(\infty) < \infty,\ j = 1, 3, 21, 22, 31, 32\Big\}$.

Here

$$C = \mathrm{diag}\Big(-(\varepsilon+\alpha_0), \frac{\mathrm{d}}{\mathrm{d}x} - [\alpha_0 + r(x)], -(\lambda_1+\lambda_2), \frac{\mathrm{d}}{\mathrm{d}x} - [\lambda_1+\lambda_2+r(x)], \frac{\mathrm{d}}{\mathrm{d}y} - \mu_1(y), \frac{\mathrm{d}}{\mathrm{d}y} - \mu_2(y), \frac{\mathrm{d}}{\mathrm{d}x} - r(x), \frac{\mathrm{d}}{\mathrm{d}x} - r(x)\Big)$$

$$D = \begin{pmatrix} 0 & \varepsilon\theta_1(\cdot) & \alpha_0 & 0 & 0 & 0 & 0 & 0 \\ r(x) & 0 & 0 & \alpha_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1\theta_{21}(\cdot) & \lambda_2\theta_{22}(\cdot) & 0 & 0 \\ 0 & 0 & r(x) & 0 & 0 & 0 & \lambda_1 & \lambda_2 \\ \mu_1(y) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_2(y) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r(x)\theta_{21}(\cdot) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r(x)\theta_{22}(\cdot) & 0 \end{pmatrix}$$

and $\theta_k(\cdot):\ L^\infty(\mathbb{R}^+) \to \mathbb{C}$ satisfying $\theta_k(f) = f(0), k = 1, 21, 22$.

Lemma 4.4. *If γ is a complex number satisfying*

$$\sup\left\{ \frac{\varepsilon+\alpha_0}{|\gamma+\varepsilon+\alpha_0|}, \frac{\alpha_0+M}{Re\gamma+\alpha_0+M}, \frac{\lambda_1+\lambda_2}{|\gamma+\lambda_1+\lambda_2|}, \frac{\lambda_1+\lambda_2+M}{Re\gamma+\lambda_1+\lambda_2+M}, \frac{M}{Re\gamma+M} \right\} < 1, \tag{4.2}$$

then γ is in the resolvent set of B^*. Here $M = \sup\{r, \mu_1, \mu_2\}$ and $\mu_i = \sup_{y\geq 0}\mu_i(y)$, $i = 1, 2$.

Proof. For any $W \in X^*$, it is not hard to prove that the operator equation $(\gamma I - C)Q = DW$ has a unique solution Q for a complex number γ satisfying (4.2), and $\|Q\| < \|W\|$. This implies that $\|(\gamma I - C)^{-1}D\| < 1$. Then $[I - (\gamma I - C)^{-1}D]$ is invertible. Therefore $\gamma I - B^*$ is invertible and $(\gamma I - B^*)^{-1} = [\gamma I - (C + D)]^{-1} = [I - (\gamma I - C)^{-1}D]^{-1}(\gamma I - C)^{-1}$. □

The following result of eigenvalue 0 of B^* can be obtained with the same method of Lemma 4.3.

Lemma 4.5. *0 is an eigenvalue of operator B^* with algebraic multiplicity one.*

With the above preparation and strongly continuous semigroup theory, we can deduce that the system operator B generates a positive C_0 semigroup of contraction $\widehat{T}(t)$. Then the system (4.1) has a unique nonnegative time-dependent solution $P(t,\cdot)$ which can be expressed by $P(t,\cdot) = \widehat{T}(t)P_0\,(t \in [0,\infty))$,where P_0 is the initial value of the system (4.1). Moreover, the asymptotic stability of the system can be obtained as follows.

Theorem 4.6. *Let $\widehat{P}$ be the nonnegative eigenfunction corresponding to eigenvalue 0 of the system operator B satisfying $\|\ \widehat{P}\ \| = 1$ and $Q^* = (1,1,1,1,1,1,1,1)^T \in X^*$, then the time-dependent solution $P(t,\cdot)$ of system (4.1) converges to the nonnegative steady-state solution $\widehat{P}$. That is $\lim\limits_{t\to\infty} P(t,\cdot) = \langle P_0,\ Q^*\rangle \widehat{P} = \widehat{P}$.*

5. Reliability indexes

In Section 4, we have obtained the unique existence and stability of the solution of system (2.1)-(2.10). So in this section we can study steady-state reliability indexes of the system with the method of Laplace transformation because the premise of Laplace transformation needs the condition that the system solution is unique existed and stable.

Applying the Laplace transformation to equations (2.1)-(2.9), we can obtain the following equations.

$$(s+\varepsilon+\alpha_0)P_0^*(s) = 1 + \int_0^\infty r(x)P_1^*(s,x)\mathrm{d}x + \sum_{i=1}^{2}\int_0^\infty \mu_i(y)P_{2i}^*(s,y)\mathrm{d}y \tag{5.1}$$

$$\frac{\mathrm{d}P_1^*(s,x)}{\mathrm{d}x} + [s+\alpha_0+r(x)]P_1^*(s,x) = 0 \tag{5.2}$$

$$(s+\lambda_1+\lambda_2)P_2^*(s) = \alpha_0 P_0^*(s) + \int_0^\infty r(x)P_3^*(s,x)\mathrm{d}x \tag{5.3}$$

$$\frac{\mathrm{d}P_3^*(s,x)}{\mathrm{d}x} + [s+\lambda_1+\lambda_2+r(x)]P_3^*(s,x) = \alpha_0 P_1^*(s,x) \tag{5.4}$$

$$\frac{\mathrm{d}P_{2i}^*(s,y)}{\mathrm{d}y} + [s+\mu_i(y)]P_{2i}^*(s,y) = 0 \tag{5.5}$$

$$\frac{\mathrm{d}P_{3i}^*(s,x)}{\mathrm{d}x} + [s+r(x)]P_{3i}^*(s,x) = \lambda_i P_3^*(s,x) \tag{5.6}$$

$$P_1^*(s,0) = \varepsilon P_0^*(s) \tag{5.7}$$

$$P_{2i}^*(s,0) = \lambda_i P_2^*(s) + \int_0^\infty r(x)P_{3i}^*(s,x)\mathrm{d}x \tag{5.8}$$

$$P_3^*(s,0) = P_{3i}^*(s,0) = 0, \tag{5.9}$$

where $i = 1,2$. Solving equations (5.1)-(5.6) with the help of (5.7)-(5.9) follows

$$P_0^*(s) = \frac{(s+\lambda_1+\lambda_2)(\alpha_0-\lambda_1-\lambda_2)(\lambda_1+\lambda_2)}{sN(s)}$$

$$P_1^*(s,x) = \varepsilon P_0^*(s)e^{-\int_0^x[s+\alpha_0+r(\tau)]\mathrm{d}\tau}$$

$$P_2^*(s) = \alpha_0 P_0^*(s)\frac{\alpha_0-\lambda_1-\lambda_2+\varepsilon[(s+\alpha_0)g(s)-(s+\lambda_1+\lambda_2)h(s)]}{(\alpha_0-\lambda_1-\lambda_2)(s+\lambda_1+\lambda_2)}$$

$$P_3^*(s,x) = \frac{\alpha_0\varepsilon P_0^*(s)}{\alpha_0-\lambda_1-\lambda_2}\left[e^{-\int_0^x[s+\lambda_1+\lambda_2+r(\tau)]\mathrm{d}\tau} - e^{-\int_0^x[s+\alpha_0+r(\tau)]\mathrm{d}\tau}\right]$$

$$P_{2i}^*(s,y) = \frac{\alpha_0\lambda_i P_0^*(s)}{\alpha_0-\lambda_1-\lambda_2}\Big[\frac{[1+\varepsilon g(s)](\alpha_0-\lambda_1-\lambda_2)}{s+\lambda_1+\lambda_2} + \varepsilon s\big[\frac{f(s)-g(s)}{\alpha_0} - \frac{f(s)-h(s)}{\lambda_1+\lambda_2}\big]\Big]e^{-\int_0^y[s+\mu_i(\tau)]\mathrm{d}\tau}$$

$$P_{3i}^*(s,x) = \frac{\alpha_0\varepsilon\lambda_i P_0^*(s)}{\alpha_0-\lambda_1-\lambda_2}\Big[\frac{e^{-\alpha_0 x}-1}{\alpha_0} - \frac{e^{-(\lambda_1+\lambda_2)x}-1}{\lambda_1+\lambda_2}\Big]e^{-\int_0^x[s+r(\tau)]\mathrm{d}\tau}.$$

Here

$$f(s) = \int_0^\infty e^{-\int_0^x(s+r(\tau))\mathrm{d}\tau}\mathrm{d}x, \quad g(s) = \int_0^\infty e^{-\int_0^x(s+\alpha_0+r(\tau))\mathrm{d}\tau}\mathrm{d}x$$

$$h(s) = \int_0^\infty e^{-\int_0^x(s+\lambda_1+\lambda_2+r(\tau))\mathrm{d}\tau}\mathrm{d}x, \quad k_i(s) = \int_0^\infty e^{-\int_0^y(s+\mu_i(\tau))\mathrm{d}\tau}\mathrm{d}y, \ i=1,2.$$

$$\begin{aligned}N(s) = &(\alpha_0-\lambda_1-\lambda_2)\Big[(s+\lambda_1+\lambda_2)[(\lambda_1+\lambda_2)(1+\varepsilon g(s)+\varepsilon f(s)) - \varepsilon f(s)s(k_1\lambda_1+k_2\lambda_2)]\\ &+\alpha_0(\lambda_1+\lambda_2)(1+\varepsilon g(s))(1+k_1\lambda_1+k_2\lambda_2)\Big]\\ &+\varepsilon(s+\lambda_1+\lambda_2)[\lambda_1 g(s)+\lambda_2 g(s)-\alpha_0 h(s)](\lambda_1+\lambda_2-sk_1\lambda_1-sk_2\lambda_2).\end{aligned}$$

Then the steady state reliability indexes of the system (2.1)-(2.10) can be derived as follows.

Theorem 5.1. *The stead-state availability of the system is*

$$A_v = \frac{(1+\varepsilon g)(\alpha_0+\lambda_1+\lambda_2)(\alpha_0-\lambda_1-\lambda_2)}{N}. \tag{5.10}$$

Proof. The instantaneous availability of the system at time t is $A_v(t) = P_0(t) + \int_0^\infty P_1(t,x)\mathrm{d}x + P_2(t) + \int_0^\infty P_3(t,x)\mathrm{d}x$. According to Taubert theorem, the stead-state availability of the system can be obtained readily. That is

$$\begin{aligned}A_v &= \lim_{t\to\infty} A_v(t) = \lim_{s\to 0} sA_v^*(s) = \lim_{s\to 0} s\left[P_0^*(s) + \int_0^\infty P_1^*(s,x)\mathrm{d}x + P_2^*(s) + \int_0^\infty P_3^*(s,x)\mathrm{d}x\right]\\ &= \frac{(1+\varepsilon g)(\alpha_0+\lambda_1+\lambda_2)(\alpha_0-\lambda_1-\lambda_2)}{N},\end{aligned}$$

where $N = (\alpha_0 - \lambda_1 - \lambda_2)[(1 + \varepsilon f + \varepsilon g)(\lambda_1 + \lambda_2) + \alpha_0(1 + \varepsilon g)(1 + k_1\lambda_1 + k_2\lambda_2)] + \varepsilon(\lambda_1 + \lambda_2)(\lambda_1 g + \lambda_2 g - \alpha_0 h)$, and $f = \int_0^\infty e^{-\int_0^x r(s)\mathrm{d}s}\mathrm{d}x$, $g = \int_0^\infty e^{-\int_0^x (\alpha_0 + r(s))\mathrm{d}s}\mathrm{d}x$, $h = \int_0^\infty e^{-\int_0^x (\lambda_1+\lambda_2+r(s))\mathrm{d}s}\mathrm{d}x$, $k_i = \int_0^\infty e^{-\int_0^y \mu_i(s)\mathrm{d}s}\mathrm{d}y$, $i = 1, 2$. □

Theorem 5.2. *The stead-state probability of the repairman on vacation is*

$$P_v = \frac{\varepsilon f(\lambda_1 + \lambda_2)(\alpha_0 - \lambda_1 - \lambda_2)}{N}. \tag{5.11}$$

Proof. The probability that the repairman is on vacation at time t is $P_v(t) = \int_0^\infty P_1(t,x)\mathrm{d}x + \int_0^\infty P_3(t,x)\mathrm{d}x + \sum_{i=1}^{2}\int_0^\infty P_{3i}(t,x)\mathrm{d}x$. Then the stead-state probability of the repairman on vacation can be yielded by using the limit theorem as below.

$$\begin{aligned} P_v &= \lim_{t\to\infty} P_v(t) = \lim_{s\to 0} sP_v^*(s) = \lim_{s\to 0} s\left[\int_0^\infty P_1^*(s,x)\mathrm{d}x + \int_0^\infty P_3^*(s,x)\mathrm{d}x + \sum_{i=1}^{2}\int_0^\infty P_{3i}^*(s,x)\mathrm{d}x\right] \\ &= \frac{\varepsilon f(\lambda_1 + \lambda_2)(\alpha_0 - \lambda_1 - \lambda_2)}{N}, \end{aligned}$$

where f and N are defined in Theorem 5.1. □

Theorem 5.3. *The steady-state probability of the system in warning state is*

$$P_w = \frac{\alpha_0(1 + \varepsilon g)(\alpha_0 - \lambda_1 - \lambda_2)}{N}. \tag{5.12}$$

Proof. The probability of the system in warning state at time t is $P_w(t) = P_2(t) + \int_0^\infty P_3(t,x)\mathrm{d}x$. Then the stead-state probability of the repairman on vacation can be yielded by using the limit theorem as below.

$$\begin{aligned} P_w &= \lim_{t\to\infty} P_w(t) = \lim_{s\to 0} sP_w^*(s) = \lim_{s\to 0} s\left[P_2^*(s) + \int_0^\infty P_3^*(s,x)\mathrm{d}x\right] \\ &= \frac{\alpha_0(1 + \varepsilon g)(\alpha_0 - \lambda_1 - \lambda_2)}{N}, \end{aligned}$$

where g and N are defined in Theorem 5.1. □

Theorem 5.4. *The stead-state failure frequency of the system is*

$$W_f = \frac{\alpha_0(\lambda_1 + \lambda_2)(1 + \varepsilon g)(\alpha_0 - \lambda_1 - \lambda_2)}{N}. \tag{5.13}$$

Proof. According to Ref. [1] the instantaneous failure frequency of the system at time t is $W_f(t) = (\lambda_1 + \lambda_2)\left[P_2(t) + \int_0^\infty P_3(t,x)\mathrm{d}x\right]$. Applying the limit theorem and noting the result of Theorem 5.3, the result of Theorem 5.4 can be yielded readily. □

6. Numerical analysis

In this section, we mainly concentrate on that how the warning device will affect the system. We will present some numerical examples to show the transient behavior of the system (2.1)-(2.10) with warning device. And we compare some reliability indexes of systems with and without warning device, and give the corresponding numerical examples. For simplicity, we assume that the times of overhaul and minor repair the vacation time all follow exponential distributions. That is, $r(x) \equiv r$, $\mu_i(y) \equiv \mu_i$, where r and μ_i are nonnegative constants, $i = 1, 2$.

Choose $\varepsilon(t) = 2$, $\lambda_1 = 1$ and $\lambda_2 = 0.5$ for $t \in [0, 5)$, $\varepsilon(t) = 1$, $\lambda_1 = 0.5$ and $\lambda_2 = 0.2$ for $t \in [5, 10)$, and $\varepsilon(t) = 0.2$, $\lambda_1 = 0.2$ and $\lambda_2 = 0.1$ for $t \in [10, \infty)$. Figures 1-4 present the transient behaviors of the system (2.1)-(2.10) with different α_0. From the figures we can see that the system steady-state behavior will arrive when $\varepsilon(t)$ and $\lambda_i(t)$ reach their limit values, $i = 1, 2$, and the transient availability and vacation probability increase while the transient probabilities of the system in failure state and warning state decrease with the decreasing of α_0.

Figures 1∼4

The steady-state indexes, specifically, the stable reliability $\widetilde{A}_v$, failure frequency $\widetilde{W}_f$ and the probability of repairman on vacation $\widetilde{P}_v$) of the system without warning device corresponding to system (2.1)-(2.10) are presented as follows.

$$\widetilde{A}_v = \frac{1+\varepsilon h}{1+\varepsilon f + (1+\varepsilon h)(\lambda_1 k_1 + \lambda_2 k_2)} \tag{6.1}$$

$$\widetilde{W}_f = \frac{(\lambda_1+\lambda_2)(1+\varepsilon h)}{1+\varepsilon f + (1+\varepsilon h)(\lambda_1 k_1 + \lambda_2 k_2)} \tag{6.2}$$

$$\widetilde{P}_v = \frac{\varepsilon f}{1+\varepsilon f + (1+\varepsilon h)(\lambda_1 k_1 + \lambda_2 k_2)}. \tag{6.3}$$

In the following, we will compare the above three reliability indexes of the two systems with and without warning device.

(1) Set $\varepsilon = \lambda_1 = \mu_1 = 0.5$, $r = 0.2$, $\alpha_0 = 0.1$, $= 0.5$. Figures 5, 7, 9 and Figures 6, 8, 10 present the availabilities, failure frequencies and vacation probabilities of systems with and without warning device with $\mu_2 = 0.2$ and $\lambda_2 = 0.5$, 0.2, 0.05, respectively. From the figures we can deduce the following results.

(i) The availabilities and vacation probabilities of both systems are increasing with the decreasing of λ_2. The failure frequencies of both systems are first decreasing and then increasing with the decreasing of λ_2.

(ii) The availabilities and vacation probabilities of the system with warning device are greater than those of the system without warning device. While failure frequencies are just the opposite.

Figures 5∼10

(2) Set $\varepsilon = \lambda_1 = \mu_1 = 0.5$, $r = 0.2$, $\alpha_0 = 0.1$, $= 0.5$. Figures 11, 13, 15 and Figures 12, 14, 16 present the availabilities, failure frequencies and vacation probabilities of systems with and without warning device with $\lambda_2 = 0.2$ and $\mu_2 = 0.5$, 0.2, 0.08, respectively. From the figures we can deduce the following results.

(i) The availabilities, frequencies and vacation probabilities of both systems are decreasing with the decreasing of μ_2.

(ii) The availabilities and vacation probabilities of the system with warning device are greater than those of the system without warning device. While failure frequencies are just the opposite.

Figures 11∼16

(3) Let c_1, c_2 and c_3 represent the income of the system for working unit per unit time, the loss of the system for failed unit per unit time and the income of the system for the repairman vacation per unit time, respectively. And let I and $\widetilde{I}$ respectively be the total profit of the system with and without warning device in steady state. Then

$$I = c_1 A_v - c_2 W_f + c_3 P_v, \quad \widetilde{I} = c_1 \widetilde{A}_v - c_2 \widetilde{W}_f + c_3 \widetilde{P}_v \tag{6.4}$$

From the results of (1)(ii) and (2)(ii), it is easy to deduce that it can be assured the total profit of the system with warning is greater than that of the system without warning device by adjusting c_1, c_2 and c_3.

Therefore, by the results of (1)-(3), it can be yielded readily that the system with warning device is superior to the system without warning device. Then warning device in a system is important and practical.

7. CONCLUSION

In this paper, a repairable system with warning device and a repairman is considered. The warning device will send an alarm when the system is not working smoothly before it fails. The repairman follows delayed-multiple vacations policy. The system will be overhauled or minor repaired by the repairman after the warning device alarms. It is worth noting that the delayed vacation rate and failure rates of the system are functions of the system working time. We first present the system model with a group of integro-differential equations by using probability analysis method, and translate them into an abstract Cauchy problem of semi-linear evolution system by choosing a suitable Banach space. Then we give the conditions of the unique existence, and even the continuous dependence on initial value of the system solution by using strongly continuous semigroup theory. To study the asymptotic stability of system solution, we replace the delayed vacation rate and failure rates with their limit values. Then by analyzing the properties and spectrum distribution of the system operator, the stability of system solution is obtained. Because the system solution is unique existent and stable, the stability reliability indexes can be obtained by Laplace transformation. In the end, numerical examples of some indexes are presented, from which we can see that the system is stable when the delayed vacation rate and failure rates all reach their limit values. We also try to discuss the differences between the systems with and without warning device. Because too many parameters are concerned, it is very hard to get exact analysis result. Thus we also present several numerical examples to compare the indexes of system with and without warning device with different failure rates and repair rates respectively. From these figures we can see that the availability and vacation probability of system with warning device are greater than those of the system without warning device, while failure frequency is just the opposite. Then the profit of system with warning device can be greater than that of the system without warning device. Thus we can deduce that the system with warning device is superior to the system without warning device. The exact analysis results of the two systems with and without warning device will be our further work.

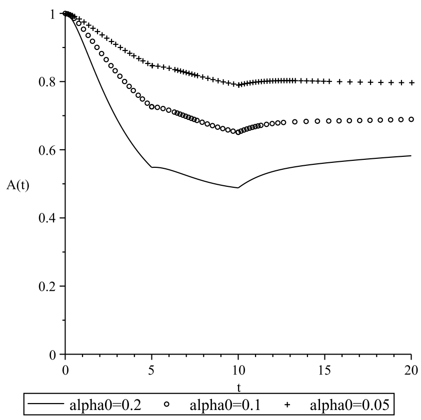

Figure 1:
Transient availabilities with different α_0

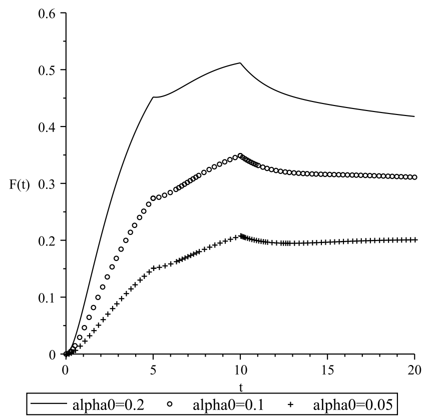

Figure 2:
Transient failure probabilities with different α_0

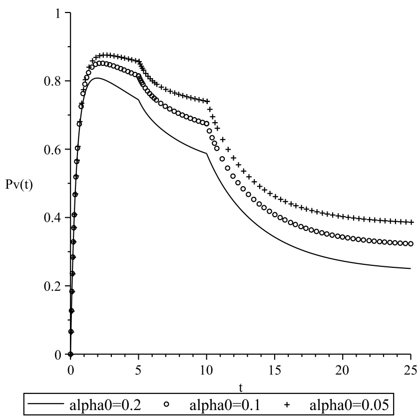

Figure 3:
Transient vacation probabilities with different α_0

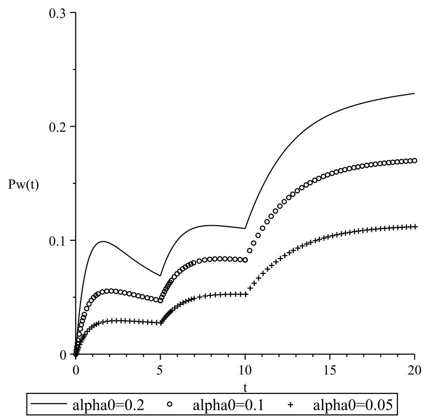

Figure 4:
Transient warning probabilities with different α_0

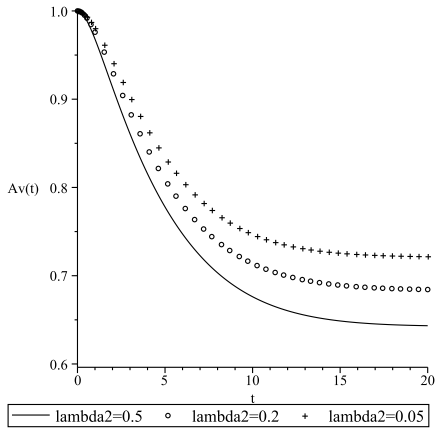

Figure 5:
System availabilities with warning device with different λ_2

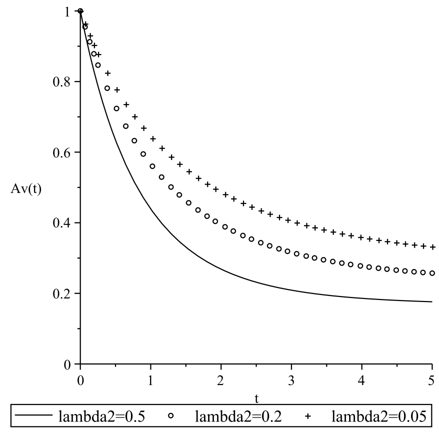

Figure 6:
System availabilities without warning device with different λ_2

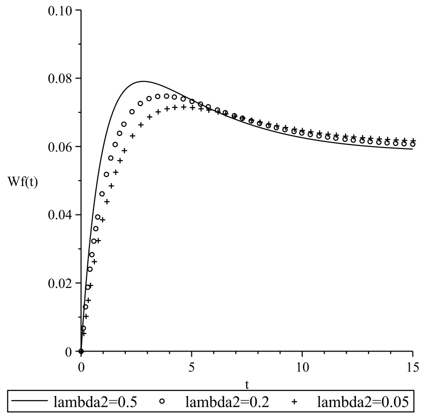

Figure 7:
System failure frequencies with warning device with different λ_2

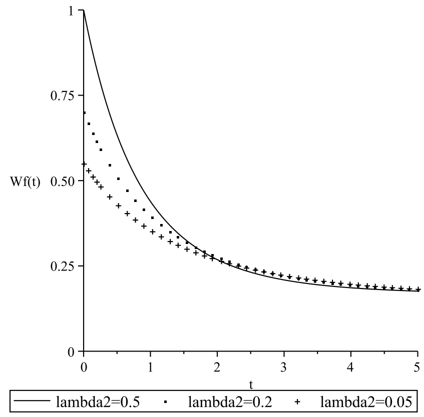

Figure 8:
System failure frequencies without warning device with different λ_2

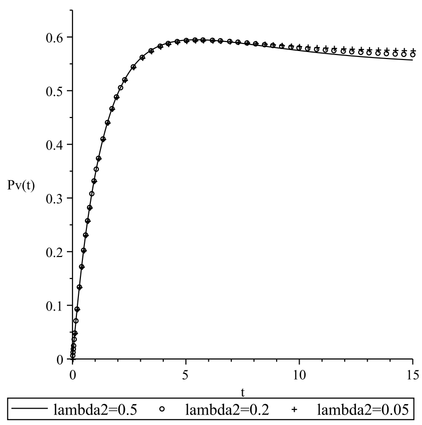

Figure 9:
Vacation probabilities of system with warning device with different λ_2

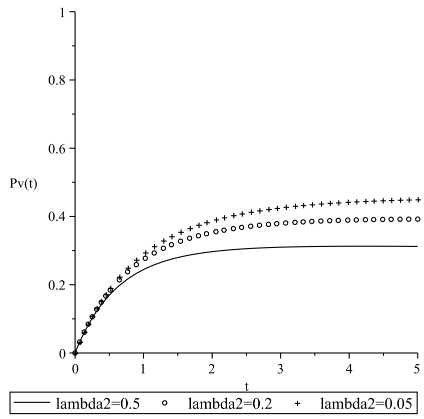

Figure 10:
Vacation probabilities of system without warning device with different λ_2

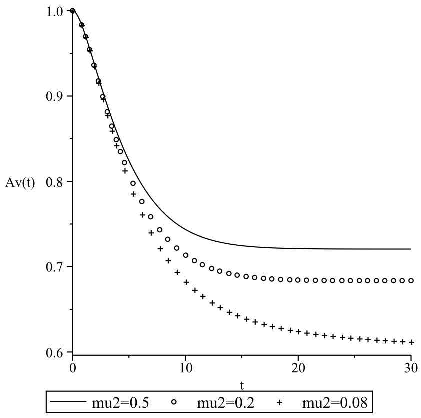

Figure 11:
System availabilities with warning device with different μ_2

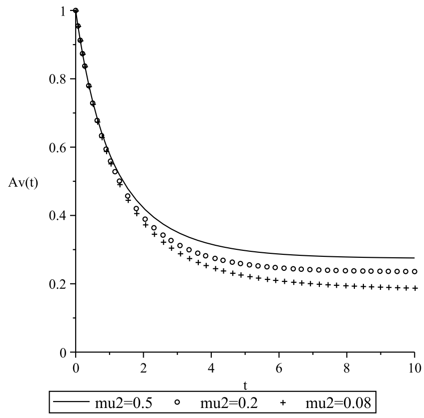

Figure 12:
System availabilities without warning device with different μ_2

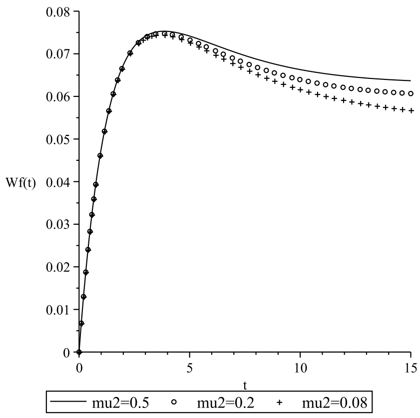

Figure 13:
System failure frequencies with warning device with different μ_2

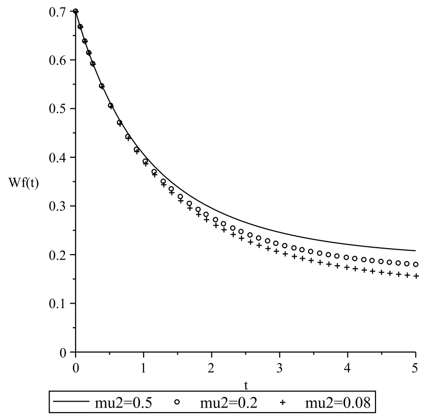

Figure 14:
System failure frequencies without warning device with different μ_2

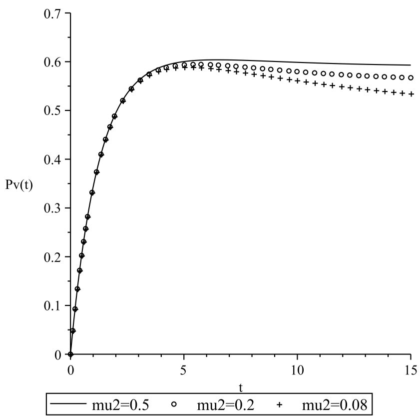

Figure 15:
Vacation probabilities of system with warning device with different μ_2

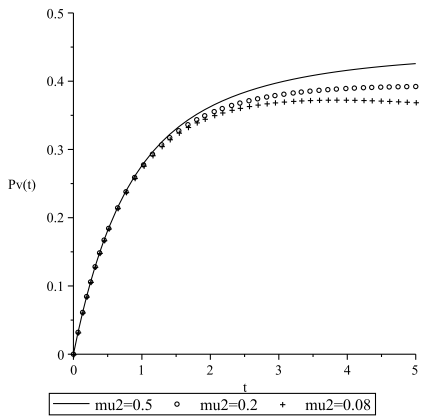

Figure 16:
Vacation probabilities of system without warning device with different μ_2

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