Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 342–349
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Existence of viscosity solutions with asymptotic behavior of exterior problems for Hessian equations

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Abstract

The Perron method is used to establish the existence of viscosity solutions of exterior problems for a class of Hessian type equations with prescribed behavior at infinity.

Keywords: Hessian equation, viscosity solution, asymptotic behavior, exterior problem coincidence.
2010 MSC: 35J40, 35J60.

1. Introduction

In this paper, we study the Hessian equation

$$F(\lambda(D^2u)) = \sigma > 0 \qquad x \in \mathbf{R}^n \setminus \partial\Omega, \tag{1.1}$$

$$u = -\beta \qquad x \in \partial\Omega, \tag{1.2}$$

where σ is a constant, $\Omega \subset \mathbf{R}^n$ $(n \geq 3)$ is a bounded domain, β is a constant, $\lambda(D^2u) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are eigenvalues of the Hessian matrix D^2u. F is assumed to be defined in the symmetric open convex cone Γ, with vertex at the origin, containing

$$\Gamma^+ = \{\lambda \in \mathbf{R}^n : \text{ each component of } \lambda,\ \lambda_i > 0, \quad i = 1, 2, \cdots, n\},$$

and satisfies the fundamental structure conditions:

$$F_i(\lambda) = \frac{\partial F}{\partial \lambda_i} > 0 \quad \text{in} \quad \Gamma, \qquad 1 \leq i \leq n \tag{1.3}$$

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Received 2015-02-26

and F is a continuous concave function. In addition, F will be assumed to satisfy some more technical assumptions, such as

$$F > 0 \quad \text{in } \Gamma, \quad F = 0 \quad \text{on} \quad \partial\Gamma \tag{1.4}$$

and for any $r \geq 1,\ R > 0$

$$F(R(\frac{1}{r^{n-1}}, r, \cdots, r)) \geq F(R(1, 1, \cdots, 1)). \tag{1.5}$$

For every $C > 0$ and every compact set K in Γ there is $\Lambda = \Lambda(C, K)$ such that

$$F(\Lambda\lambda) \geq C \ \text{ for all } \ \lambda \in K. \tag{1.6}$$

There exists a number Λ sufficiently large such that at every point $x \in \partial\Omega$, if $x_1, \cdots, x_{n-1}$ represent the principal curvatures of $\partial\Omega$, then

$$(x_1, \cdots, x_{n-1}, \Lambda) \in \Gamma. \tag{1.7}$$

It is easy to verify that $\Gamma \subset \{\lambda \in \mathbf{R}^n : \sum_{i=1}^{n} \lambda_i > 0\}$.

Equation (1.5) is satisfied by each kth root of elementary symmetric function ($1 \leq k \leq n$) and the $k-l$th root of each quotient of kth elementary symmetric function and lth elementary symmetric function ($1 \leq l < k \leq n$).

The Hessian equation (1.1) is an important class of fully nonlinear elliptic equations. There exist many excellent results in the case of bounded domains, see for examples [2, 3, 7, 13, 16] and the references therein. Caffarelli, Nirenberg and Spruck [2, 3] and Trudinger [15] established the classical solvability of the Dirichlet problems under various hypothesis. In [11] Ivochkina, Trudinger and Wang provided a simple approach the estimation of second derivatives of solutions. In [7] Guan studied the Dirichlet problems in bounded domains of Riemanian manifolds. Other boundary value problems have also been considered. In [13], Trudinger treated the Dirichlet and Neumann problems in balls for the degenerate case and in [16] Urbas studied nonlinear oblique boundary value problems in two dimensions. But for unbounded domains there are few results in this directions.

The study on this kind of fully nonlinear elliptic equations is close to the investigation on prescribed curvature equations and hypersurfaces of constant curvature with boundary, see for example [8, 9, 14] and the references therein.

When

$$F(\lambda(D^2u)) = \sigma_k(\lambda(D^2u)),\ \Gamma = \Gamma_k = \{\lambda \in \mathbf{R}^n :\ \sigma_j > 0,\ j = 1, 2, \cdots, k\},$$

where the kth elementary symmetric function

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

for $\lambda = (\lambda_1, \cdots, \lambda_n)$, in [6] Dai obtained the following result:

Theorem 1.1. *Let $k \geq 3$. Then for any $C \in \mathbf{R}$, there exists a constant $\beta_0 \in \mathbf{R}$ such that for any $\beta > \beta_0$ there exists a $k-$convex viscosity solution $u \in C^0(\mathbf{R}^n \setminus \partial\Omega)$ of*

$$\sigma_k(\lambda(D^2u)) = 1, \quad x \in \mathbf{R}^n \setminus \partial\Omega$$

satisfying

$$\limsup_{|x|\to\infty} \left(|x|^{k-2} \left| u(x) - \left(\frac{C_*}{2}|x|^2 + C\right)\right|\right) < \infty,$$
$$u = -\beta \qquad x \in \partial\Omega$$

where $C_* = \left(\frac{1}{C_n^k}\right)^{\frac{1}{k}}$.

The following theorem, which is the main result in this paper, is a generalization of Theorem 1.1 for k-Hessian equations.

Theorem 1.2. *Let $k \geq 3$. Then for any $C \in \mathbf{R}$, there exists a constant $\beta_0 \in \mathbf{R}$ such that for any $\beta > \beta_0$ there exists an admissible viscosity solution $u \in C^0(\mathbf{R}^n \setminus \partial\Omega)$ of (1.1) satisfying*

$$\limsup_{|x|\to\infty} \left(|x|^{n-2} \left| u(x) - \left(\frac{\overline{R}}{2}|x|^2 + C \right) \right| \right) < \infty,$$

$$u = -\beta \qquad x \in \partial\Omega,$$

where $\overline{R}$ is the constant satisfying $F(\overline{R}, \overline{R}, \cdots, \overline{R}) = \sigma$.

This paper is arranged as follows. In section 2, we give some preliminary facts which will be used later. In section 3, we prove the main result of this paper.

2. Preliminaries

The notion of viscosity solutions was introduced by Crandall and Lions [5]. Now viscosity solution is a rather standard concept in partial differential equations. For the completeness of this paper, we first recall the notion of viscosity solutions.

Definition 2.1. A function $u \in C^2(\mathbf{R}^n \setminus \partial\Omega)$ is called admissible if $\lambda(D^2u) \in \overline{\Gamma}$ for every $x \in \mathbf{R}^n \setminus \partial\Omega$.

Definition 2.2. A function $u \in C^0(\mathbf{R}^n \setminus \partial\Omega)$ is called a viscosity subsolution(supersolution) to (1.1), if for any $y \in \mathbf{R}^n \setminus \partial\Omega$ and any admissible function $\xi \in C^2(\mathbf{R}^n \setminus \partial\Omega)$ satisfying

$$u(x) \leq (\geq)\xi(x), \quad x \in \mathbf{R}^n \setminus \partial\Omega, \quad u(y) = \xi(y),$$

we have

$$F(\lambda(D^2\xi(y)) \geq (\leq)\sigma.$$

Definition 2.3. A function $u \in C^0(\mathbf{R}^n \setminus \partial\Omega)$ is called a viscosity solution to (1.1) if it is both a viscosity subsolution and a viscosity supersolution to (1.1).

Definition 2.4. A function $u \in C^0(\mathbf{R}^n \setminus \partial\Omega)$ is called a viscosity subsolution (supersolution, solution) to (1.1)-(1.2), if u is a viscosity subsolution (supersolution, solution) to (1.1) and $u \leq (\geq, =)\varphi(x)$ on $\partial\Omega$.

Definition 2.5. A function $u \in C^0(\mathbf{R}^n \setminus \partial\Omega)$ is called admissible if for any $y \in \mathbf{R}^n \setminus \partial\Omega$ and any function $\xi \in C^2(\mathbf{R}^n \setminus \partial\Omega)$ satisfying $u(x) \leq (\geq)\xi(x), \quad x \in \mathbf{R}^n \setminus \partial\Omega, \quad u(y) = \xi(y)$, we have $\lambda(D^2\xi(y)) \in \overline{\Gamma}$.

It is obvious that if u is a viscosity subsolution, then u is admissible.

Lemma 2.6. *Let Ω be a bounded strictly convex domain in $\mathbf{R}^n$, $\partial\Omega \in C^2$, $\varphi \in C^2(\overline{\Omega})$. Then there exists a constant C only dependent on n, φ and Ω such that for any $\xi \in \partial\Omega$, there exists $\overline{x}(\xi) \in \mathbf{R}^n$ such that*

$$|\overline{x}(\xi)| \leq C, \quad w_\xi(x) < \varphi(x) \quad \text{for} \quad x \in \overline{\Omega} \setminus \{\xi\},$$

where $w_\xi(x) = \varphi(\xi) + \frac{\overline{R}}{2}(|x - \overline{x}(\xi)|^2 - |\xi - \overline{x}(\xi)|^2)$ for $x \in \mathbf{R}^n$ and $\overline{R}$ is the constant satisfying $F(\overline{R}, \overline{R}, \cdots, \overline{R}) = \sigma$.

This is a modification of Lemma 5.1 in [1].

Lemma 2.7. *Let Ω be a domain in $\mathbf{R}^n$ and $f \in C^0(\mathbf{R}^n)$ be nonnegative. Assume that the admissible functions $v \in C^0(\overline{\Omega})$, $u \in C^0(\mathbf{R}^n)$ satisfy, respectively,*

$$F(\lambda(D^2 v)) \geq f(x) \quad x \in \Omega,$$

$$F(\lambda(D^2 u)) \geq f(x) \quad x \in \mathbf{R}^n.$$

Moreover

$$\begin{aligned} u \leq v, \quad & x \in \overline{\Omega}; \\ u = v, \quad & x \in \partial\Omega. \end{aligned}$$

Set

$$w(x) = \begin{cases} v(x) & x \in \Omega, \\ u(x) & x \in \mathbf{R}^n \setminus \Omega. \end{cases}$$

Then $w \in C^0(\mathbf{R}^n)$ is an admissible function and satisfies in the viscosity sense

$$F(\lambda(D^2 w(x))) \geq f(x), \quad x \in \mathbf{R}^n.$$

Lemma 2.8. *Let B be a Ball in $\mathbf{R}^n$ and $f \in C^{0,\alpha}(\overline{B})$ be positive. Suppose that $\underline{u} \in C^0(\overline{B})$ satisfies in the viscosity sense*

$$F(\lambda(D^2 \underline{u})) \geq f(x), \quad x \in B.$$

Then the Dirichlet problem

$$F(\lambda(D^2 u)) = f(x) \quad x \in B.$$

$$u = \underline{u}(x) \quad x \in \partial B$$

admits a unique admissible viscosity solution $u \in C^0(\overline{B})$.

We refer to [12] for the proof of Lemma 2.7 and 2.8.

3. Proof of Main Result

We divide the proof of Theorem 1.2 into two steps.

Step 1. By [2], there is an admissible solution $\Phi \in C^\infty(\overline{\Omega})$ of the Dirichlet problem:

$$F(\lambda(D^2\Phi)) = C_0 > \sigma, \quad x \in \Omega,$$

$$\Phi = 0, \qquad x \in \partial\Omega.$$

By the comparison principles in [4], $\Phi \leq 0$ in Ω. Further by Lemma 2.6, for each $\xi \in \partial\Omega$, there exists $\overline{x}(\xi) \in \mathbf{R}^n$ such that

$$W_\xi(x) < \Phi(x), \quad x \in \overline{\Omega} \setminus \{\xi\},$$

where

$$W_\xi(x) = \frac{\overline{R}}{2}\left(|x - \overline{x}(\xi)|^2 - |\xi - \overline{x}(\xi)|^2\right), \quad \xi \in \mathbf{R}^n$$

and $\sup\limits_{\xi \in \partial\Omega} |\overline{x}(\xi)| < \infty$. Therefore

$$W_\xi(\xi) = 0, \quad W_\xi(x) \leq \Phi(x) \leq 0, \quad x \in \overline{\Omega},$$

$$F(\lambda(D^2 W_\xi(x))) = F(\overline{R}, \overline{R}, \cdots, \overline{R}) = \sigma, \quad \xi \in \mathbf{R}^n.$$

Denote

$$W(x) = \sup_{\xi \in \partial\Omega} W_\xi(x).$$

Then

$$W(x) \le \Phi(x), \quad x \in \Omega,$$

and by [10]

$$F(\lambda(D^2 W)) \ge \sigma, \quad x \in \mathbf{R}^n.$$

Define

$$V(x) = \begin{cases} \Phi(x), & x \in \Omega, \\ W(x), & x \in \mathbf{R}^n \setminus \Omega. \end{cases}$$

Then $V \in C^0(\mathbf{R}^n)$ is an admissible viscosity solution of

$$F(\lambda(D^2 V)) \ge \sigma, \quad x \in \mathbf{R}^n.$$

Fix some $R_1 > 0$ such that $\overline{\Omega} \subset B_{R_1}(0)$ where $B_{R_1}(0)$ is the ball centered at the origin with radius R_1. Let $R_2 = 2R_1 \overline{R}^{\frac{1}{2}}$. For $a > 1$, define

$$W_a(x) = \inf_{B_{R_1}} V + \int_{2R_2}^{|\overline{R}^{\frac{1}{2}} x|} (s^n + a)^{\frac{1}{n}} ds, \quad x \in \mathbf{R}^n.$$

Then

$$D_{ij} W_a = (|y|^n + a)^{\frac{1}{n}-1} \left[\left(|y|^{n-1} + \frac{a}{|y|} \right) \overline{R} \delta_{ij} - \frac{a \overline{R}^2 x_i x_j}{|y|^3} \right], \quad |x| > 0,$$

where $y = \overline{R}^{\frac{1}{2}} x$. By rotating the coordinates we may set $x = (r, 0, \cdots 0)$. Therefore

$$D^2 W_a = (R^n + a)^{\frac{1}{n}-1} \overline{R} \mathrm{diag} \left(R^{n-1}, R^{n-1} + \frac{a}{R}, \cdots, R^{n-1} + \frac{a}{R} \right),$$

where $R = |y|$. Consequently $\lambda(D^2 W_a) \in \Gamma$ for $|x| > 0$ and by (1.5)

$$F(\lambda(D^2 W_a)) \ge F(\overline{R}, \overline{R}, \cdots, \overline{R}) = \sigma, \quad |x| > 0.$$

Moreover

$$W_a(x) \le V(x), \quad |x| \le R_1. \tag{3.1}$$

Fix some $R_3 > 3R_2$ satisfying

$$R_3 \overline{R}^{\frac{1}{2}} > 3R_2.$$

We choose $a_1 > 1$ such that for $a \ge a_1$,

$$W_a(x) > \inf_{B_{R_1}} V + \int_{2R_2}^{3R_2} (s^n + a)^{\frac{1}{n}} ds \ge V(x), \quad |x| = R_3.$$

Then by (3.1), $R_3 \ge R_1$. According to the definition of W_a,

$$\begin{aligned} W_a(x) &= \inf_{B_{R_1}} V + \int_{2R_2}^{|\overline{R}^{\frac{1}{2}} x|} s \left(\left(1 + \frac{a}{s^n} \right)^{\frac{1}{n}} - 1 \right) ds + \int_{2R_2}^{|\overline{R}^{\frac{1}{2}} x|} s ds \\ &= \frac{\overline{R}}{2} |x|^2 + C + \inf_{B_{R_1}} V + \int_{2R_2}^{+\infty} s \left(\left(1 + \frac{a}{s^n} \right)^{\frac{1}{n}} - 1 \right) ds - C - \\ &\quad - 2R_2^2 - \int_{|\overline{R}^{\frac{1}{2}} x|}^{+\infty} s \left(\left(1 + \frac{a}{s^n} \right)^{\frac{1}{n}} - 1 \right) ds, \quad x \in \mathbf{R}^n. \end{aligned}$$

Let

$$\mu(a) = \inf_{B_{R_1}} V + \int_{2R_2}^{+\infty} s\left(\left(1+\frac{a}{s^n}\right)^{\frac{1}{n}} - 1\right) ds - C - 2R_2^2.$$

Then $\mu(a)$ is continuous and monotonic increasing for a and when $a \to \infty$, $\mu(a) \to \infty$. Moreover,

$$W_a(x) = \frac{\overline{R}}{2}|x|^2 + C + \mu(a) - O(|x|^{2-n}), \quad \text{when} \quad |x| \to \infty. \tag{3.2}$$

Define, for $a \geq a_1$, set $\beta_0 = \mu(a)$ and define, for any $\beta > \beta_0$,

$$\underline{u}_a(x) = \begin{cases} \max\{V(x), W_a(x)\} - \beta, & |x| \leq R_3, \\ W_a - \beta, & |x| \geq R_3. \end{cases}$$

Then by (3.2),

$$\underline{u}_a(x) = \frac{\overline{R}}{2}|x|^2 + C - O(|x|^{2-n}), \text{ when } |x| \to \infty$$

and by the definition of V,

$$\underline{u}_a(x) = -\beta, \quad x \in \partial\Omega.$$

Choose $a_2 \geq a_1$ large enough such that when $a \geq a_2$,

$$\begin{aligned} V(x) - \beta &\leq V(x) - \beta_0 \\ &= V(x) - \inf_{B_{R_1}} V - \int_{2R_2}^{+\infty} s\left(\left(1+\frac{a}{s^n}\right)^{\frac{1}{n}} - 1\right) ds + C + 2R_2^2 \\ &\leq C \\ &\leq \frac{\overline{R}}{2}|x|^2 + C, \quad |x| \leq R_3. \end{aligned}$$

Therefore

$$\underline{u}_a(x) \leq \frac{\overline{R}}{2}|x|^2 + C, \quad a \geq a_2, \quad x \in \mathbf{R}^n.$$

By Lemma 2.7, $\underline{u}_a \in C^0(\mathbf{R}^n)$ is admissible and satisfies in the viscosity sense

$$F(\lambda(D^2\underline{u}_a)) \geq \sigma, \quad x \in \mathbf{R}^n.$$

Step 2. We define the solution of (1.1) by Perron method.
For $a \geq a_2$, let S_a denote the set of admissible function $V \in C^0(\mathbf{R}^n)$ which satisfies

$$F(\lambda(D^2V)) \geq \sigma, \quad x \in \mathbf{R}^n \setminus \partial\Omega,$$

$$V(x) = -\beta, \quad x \in \partial\Omega,$$

$$V(x) \leq \frac{\overline{R}}{2}|x|^2 + C, \quad x \in \mathbf{R}^n.$$

It is obvious that $\underline{u}_a \in S_a$. Hence $S_a \neq \emptyset$. Define

$$u_a(x) = \sup\{V(x) : \ V \in S_a\}, \quad x \in \mathbf{R}^n.$$

Next we prove that u_a is a viscosity solution of (1.1). From the definition of u_a, it is a viscosity subsolution of (1.1) and satisfies

$$u_a(x) \leq \frac{\overline{R}}{2}|x|^2 + C, \quad x \in \mathbf{R}^n.$$

So we need only to prove that u_a is a viscosity supersolution of (1.1) satisfying (1.2).

For any $x_0 \in \mathbf{R}^n \setminus \partial\Omega$, fix $\varepsilon > 0$ such that $B = B_\varepsilon(x_0) \subset \mathbf{R}^n \setminus \partial\Omega$. Then by Lemma 2.8, there exists an admissible viscosity solution $\widetilde{u} \in C^0(\overline{B})$ to the princelet problem

$$F(\lambda(D^2\widetilde{u})) = \sigma, \quad x \in B,$$

$$\widetilde{u} = u_a, \quad x \in \partial B.$$

By the comparison principle in [4],

$$u_a \leq \widetilde{u}, \quad x \in B. \tag{3.3}$$

Define

$$\psi(x) = \begin{cases} \widetilde{u}(x), & x \in B, \\ u_a(x), & x \in \mathbf{R}^n \setminus \{B \cup \partial\Omega\}. \end{cases}$$

By Lemma 2.7,

$$F(\lambda(D^2\psi(x))) \geq \sigma, \quad x \in \mathbf{R}^n.$$

As

$$F(\lambda(D^2\widetilde{u})) = \sigma = F(\lambda(D^2 g)), \quad x \in B,$$

$$\widetilde{u} = u_a \leq g, \quad x \in \partial B,$$

where $g(x) = \frac{\overline{R}}{2}|x|^2 + C$, we have

$$\widetilde{u} \leq g, \quad x \in \overline{B},$$

by the comparison principle in [4]. Therefore $\psi \in S_a$.

By the definition of u_a, $u_a \geq \psi$ in $\mathbf{R}^n$. Consequently $\widetilde{u} \leq u_a$ in B and further $\widetilde{u} = u_a$, $x \in B$ in view of (3.3). Since x_0 is arbitrary, we conclude that u_a is an admissible viscosity solution of (1.1).

By the definition of u_a,

$$\underline{u}_a \leq u_a \leq g, \quad x \in \mathbf{R}^n,$$

so u_a satisfies (1.2) and we complete the proof of Theorem 1.2.

Acknowledgements:

This work was supported by the National Natural Science Foundation of China (Grant No. 11371110).

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