Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 350–356
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

A study of some properties of an n-order functional inclusion

Tania Angelica Lazăr[a,*], Vasile Lucian Lazăr[b]

[a] Technical University of Cluj-Napoca, Department of Mathematics, Memorandumului St.28, 400114, Cluj-Napoca, Romania.
[b] "Vasile Goldiş" Western University of Arad, The Faculty of Economics, Mihai Eminescu St.15, 310086, Arad, Romania.

Communicated by Adrian Petruşel

Abstract

The purpose of this paper is to study the solution set of the functional inclusion of n-th order of the following form:

$$x(t) \in G(t,\ x(f_1(t)), \dots, x(f_n(t))),\ t \in X, \tag{1}$$

where the function $G : X \times Y^n \longrightarrow P_{cl,cv}(Y)$ and $f_1, f_2, ..., f_n : X \longrightarrow X$ are given. The approach is based on some fixed point theorems for multivalued operators, satisfying the nonlinear contraction condition, see [V. L. Lazăr, Fixed Point Theory Appl., **2011** (2011), 12 pages].

Keywords: Functional inclusion, multivalued weakly Picard operator, fixed point, φ-contraction, data dependence, well-posedness, Ulam-Hyers stability.
2010 MSC: 47H10, 54H25, 54C60.

1. Introduction

Let X be an arbitrary compact and Hausdorff topological space and $(Y, \|\cdot\|)$ be a Banach space. Let $f_1, \dots, f_n : X \to X$ be continuous mappings and $G : X \times Y^n \longrightarrow P_{cl,cv}(Y)$ be a multivalued operator.

The purpose of this paper is to study existence, uniqueness, data dependence, well-posedness, Ulam-Hyers stability for the solutions of the following n-order functional inclusion

$$x(t) \in G(t,\ x(f_1(t)), \dots, x(f_n(t))),\ t \in X. \tag{1.1}$$

The approach is based on some recent results (see Lazăr [1]) concerning the fixed point problem for multivalued operators, given in terms of multivalued φ-contractions. For related results concerning multivalued nonlinear contractions, see [2], [5], [6] and [7].

*Corresponding author
Email addresses: tanialazar@mail.utcluj.ro (Tania Angelica Lazăr), vasilazar@yahoo.com (Vasile Lucian Lazăr)

Received 2014-10-06

2. Preliminaries

Throughout this section we will recall some of the classical notations and notions in nonlinear analysis, see, for example, [3], [4], and [7].

We consider next the following families of subsets of a metric space (X, d):

$P(X) := \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\}$; $P_b(X) := \{Y \in P(X) |\ Y$ is bounded $\}$;

$P_{cp}(X) := \{Y \in P(X) |\ Y$ is compact$\}$; $P_{cl}(X) := \{Y \in P(X) |\ Y$ is closed$\}$;

$P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$.

Let us define the following generalized functionals:

1. $D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$,

$$D(A,B) = \begin{cases} inf\{d(a,b), a \in A, b \in B\}, & \text{if} \quad A \neq \emptyset \neq B \\ 0, & \text{if} \quad A = \emptyset = B \\ +\infty, & \text{if} \quad A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

D is called the gap functional between A and B. In particular, for $x_0 \in X$, we denote by $D(x_0, B) = D(\{x_0\}, B)$ the distance from the point x_0 to the set B.

2. $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$,

$$\delta(A,B) = \begin{cases} sup\{d(a,b), a \in A, b \in B\}, & \text{if} \quad A \neq \emptyset \neq B \\ 0, & \text{otherwise}. \end{cases}$$

3. $\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$,

$$\rho(A,B) = \begin{cases} sup\{D(a,B), a \in A\}, & \text{if} \quad A \neq \emptyset \neq B \\ 0, & \text{if} \quad A = \emptyset \\ +\infty, & \text{if} \quad A \neq \emptyset = B. \end{cases}$$

ρ is called the excess functional of A over B.

4. $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$,

$$H(A,B) = \begin{cases} max\{\rho(A,B), \rho(B,A)\}, & \text{if} \quad A \neq \emptyset \neq B \\ 0, & \text{if} \quad A = \emptyset \\ +\infty, & \text{if} \quad A \neq \emptyset = B. \end{cases}$$

H is called the generalized Pompeiu-Hausdorff functional of A and B and it is well known that the pair $(P_{b,cl}(X), H)$ is a metric space.

Lemma 2.1. $D(b, A) = 0$ *if and only if* $b \in \bar{A}$.

Lemma 2.2. *Let (X, d) be a metric space. Then we have:*

(i) Let $Y, Z \in P(X)$ and $q > 1$. Then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq qH(Y, Z)$.

(ii) If $Y, Z \in P_{cp}(X)$ then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq H(Y, Z)$.

(iii) Let $Y, Z \in P_{cl}(X)$. Suppose that there exists $\eta > 0$ such that [for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq \eta$] and [for each $z \in Z$ there exists $y \in Y$ such that $d(y, z) \leq \eta$]. Then, $H(Y, Z) \leq \eta$.

(iv) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $P_{cl}(X)$. Then $A_n \overset{H}{\to} A^* \in P_{cl}(X)$ as $n \to \infty$ if and only if $H(A_n, A^*) \to 0$ as $n \to \infty$.

Theorem 2.3. *If (X, d) is a complete metric space, then $(P_{b,cl}(X), H)$ is a complete metric space.*

Definition 2.4. Let X, Y be Hausdorff topological spaces and $T : X \to P(Y)$ a multivalued operator. T is said to be upper semi-continuous in $x_0 \in X$ (briefly u.s.c.) if and only if for each open subset U of Y with $T(x_0) \subset U$ there exists an open neighborhood V of x_0 such that for all $x \in V$ we have $T(x) \subset U$.

T is u.s.c. on X if it is u.s.c in each $x_0 \in X$.

We present next the definition of lower semi-continuity.

Definition 2.5. Let X, Y be Hausdorff topological spaces and $T : X \to P(Y)$ a multivalued operator. Then T is said to be lower semi-continuous in $x_0 \in X$ (briefly l.s.c.) if and only if for each open subset $U \subset Y$ with $T(x_0) \cap U \neq \emptyset$ there exists an open neighborhood V of x_0 such that for all $x \in V$ we have $T(x) \cap U \neq \emptyset$.

T is l.s.c. on X if it is l.s.c in each $x_0 \in X$.

T is said to be continuous in $x_0 \in X$ if and only if it is l.s.c and u.s.c. in $x_0 \in X$.

Definition 2.6. Let X, Y be two metric spaces and $T : X \to P(Y)$ a multivalued operator. Then T is called H-continuous in $x_0 \in X$ (briefly H-c.) if and only if for all it is H-l.s.c. and H-u.s.c. in $x_0 \in X$.

Definition 2.7. Let (X, d) be a metric space. Then $T : X \to P(X)$ is a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Theorem 2.8 ([4])**.** *Let (X, d) be a complete b-metric space. Suppose that $T : X \to P_{cl}(X)$ is a a-Lipschitz with $a \in [0, b^{-1}[$. Then T is a MWP operator.*

Theorem 2.9 ([4], [5])**.** *Let (X, d) be a complete generalized metric space in Perov' sense (i.e. $d(x, y) \in \mathbb{R}^m_+$) and $T : X \to P_{cl}(X)$ be a multivalued A-contraction, i.e. there exists a matrix $A \in \mathcal{M}_{m,m}(\mathbb{R})$ such that $A^n \to 0$, $n \to \infty$ and for each $x, y \in X$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $d(u, v) \leq Ad(x, y)$.*

Then T is a MWP operator.

3. Some known results concerning multivalued φ-contractions

Further we present some results that will be used in the main section.

Theorem 3.1 ([1])**.** *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued φ-contraction. Then, we have:*

(i) (existence and approximation of the fixed point) T is a MWP operator (see Węgrzyk [8]);
(ii) If additionally $\varphi(qt) \leq q\varphi(t)$ for every $t \in \mathbb{R}_+$ (where $q > 1$), then T is a ψ-MWP operator, with $\psi(t) := t + s(t)$, for each $t \in \mathbb{R}_+$ (where $s(t) := \sum_{n=1}^{\infty} \varphi^n(t)$);
(iii) (Data dependence of the fixed point set) Let $S : X \to P_{cl}(X)$ be a multivalued φ-contraction and $\eta > 0$ be such that $H(S(x), T(x)) \leq \eta$, for each $x \in X$. Suppose that $\varphi(qt) \leq q\varphi(t)$ for every $t \in \mathbb{R}_+$ (where $q > 1$) and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$. Then, $H(F_S, F_T) \leq \psi(\eta)$;
(iv) (sequence of operators) Let $T, T_n : X \to P_{cl}(X), n \in \mathbb{N}$ be multivalued φ-contractions such that $T_n(x) \overset{H}{\to} T(x)$ as $n \to +\infty$, uniformly with respect to each $x \in X$. Then, $F_{T_n} \overset{H}{\to} F_T$ as $n \to +\infty$.

If, moreover $T(x) \in P_{cp}(X)$, for each $x \in X$, then we additionally have:

(v) (generalized Ulam–Hyers stability of the inclusion $x \in T(x)$) Let $\epsilon > 0$ and $x \in X$ be such that $D(x, T(x)) \leq \epsilon$. Then there exists $x^* \in F_T$ such that $d(x, x^*) \leq \psi(\epsilon)$;
(vi) T is upper semi-continuous, $\hat{T} : (P_{cp}(X), H) \to (P_{cp}(X), H), \hat{T}(Y) := \bigcup_{x\in Y} T(x)$ is a set-to-set φ-contraction and (thus) $F_{\hat{T}} = \{A^*_T\}$;
(vii) $T^n(x) \overset{H}{\to} A^*_T$ as $n \to +\infty$, for each $x \in X$;
(viii) $F_T \subset A^*_T$ and F_T is compact;

(ix) $A_T^* = \bigcup_{n\in\mathbb{N}^*} T^n(x)$, *for each* $x \in F_T$.

Theorem 3.2 ([1]). *Let* (X, d) *be a complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued* φ*-contraction with* $SF_T \neq \emptyset$. *Then, the following assertions hold:*

(x) $F_T = SF_T = \{x^*\}$;

(xi) If, additionally $T(x)$ *is compact for each* $x \in X$, *then* $F_{T^n} = SF_{T^n} = \{x^*\}$ *for* $n \in \mathbb{N}^*$;

(xii) If, additionally $T(x)$ *is compact for each* $x \in X$, *then* $T^n(x) \overset{H}{\to} \{x^*\}$ *as* $n \to +\infty$, *for each* $x \in X$;

(xiii) Let $S : X \to P_{cl}(X)$ *be a multivalued operator and* $\eta > 0$ *such that* $F_S \neq \emptyset$ *and* $H(S(x), T(x)) \leq \eta$, *for each* $x \in X$. *Then,* $H(F_S, F_T) \leq \beta(\eta)$, *where* $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *is given by* $\beta(\eta) := \sup\{t \in \mathbb{R}_+ |\ t - \varphi(t) \leq \eta\}$;

(xiv) Let $T_n : X \to P_{cl}(X), n \in \mathbb{N}$ *be a sequence of multivalued operators such that* $F_{T_n} \neq \emptyset$ *for each* $n \in \mathbb{N}$ *and* $T_n(x) \overset{H}{\to} T(x)$ *as* $n \to +\infty$, *uniformly with respect to* $x \in X$. *Then,* $F_{T_n} \overset{H}{\to} F_T$ *as* $n \to +\infty$.

(xv) (Well-posedness of the fixed point problem with respect to D) If $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $D(x_n, T(x_n)) \to 0$ *as* $n \to \infty$, *then* $x_n \overset{d}{\to} x^*$ *as* $n \to \infty$;

(xvi) (Well-posedness of the fixed point problem with respect to H) If $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $H(x_n, T(x_n)) \to 0$ *as* $n \to \infty$, *then* $x_n \overset{d}{\to} x^*$ *as* $n \to \infty$;

(xvii) (Limit shadowing property of the multivalued operator) Suppose additionally that φ *is a sub-additive function. If* $(y_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $D(y_{n+1}, T(y_n)) \to 0$ *as* $n \to \infty$, *then there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *of successive approximations for* T, *such that* $d(x_n, y_n) \to 0$ *as* $n \to \infty$.

Theorem 3.3 ([8]). *If* X *is a paracompact Hausdorff topological space and* Y *is a closed and complete metrizable subset of a complete locally convex space over the fields of real or complex numbers, then:*

(i) any lower semi-continuous multivalued function $F : X \to P_{cl,cv}(Y)$ *admits a continuous selection.*

(ii) moreover, if $F|_A$ *is the restriction of a lower semi-continuous multivalued function* $F : X \to P_{cl,cv}(Y)$ *to a closed subset* $A \subset X$ *and* $f : A \to Y$ *is a continuous selection from* $F|_A$ *defined on* A, *then* f *can always be extended to a continuous selection of* F *defined on the whole set* X.

4. A theory for an n-order functional inclusion

We will consider now the problem (1.1), i.e. the following problem

$$x(t) \in G(t,\ x(f_1(t)), \ldots, x(f_n(t))),\ t \in X.$$

Throughout this section we will suppose the following settings. Let X be an arbitrary compact and Hausdorff topological space and $(Y, \|\cdot\|)$ be a Banach space. Let $f_1, \ldots, f_n : X \to X$, be continuous mappings and $G : X \times Y^n \longrightarrow P_{cl,cv}(Y)$ be a semi-continuous multivalued operator.

We are looking for solutions of the inclusion (1.1), i.e. continuous mapping $x : X \to Y$ which satisfy (1.1) for each $t \in X$.

We denote the set of all solutions of the inclusion (1.1) by $S_{G;f_1,\ldots,f_n}$, i.e.,

$$S_{G;f_1,\ldots,f_n} := \{x \in C(X, Y)/\ x \text{ satisfies (1.1), for all } t \in X\}.$$

Our first result is an existence theorem for (1.1)

Theorem 4.1. *Consider the n-order functional inclusion* (1.1). *We suppose:*

i) there exists a function $\beta : \mathbb{R}_+^n \to \mathbb{R}_+$ *increasing with respect to each variable such that:*

$$H_{\|\cdot\|}\left(G(t, y_1, \ldots, y_n), G(t, \bar{y}_1, \ldots, \bar{y}_n)\right) \leq \beta\left(\|y_1 - \bar{y}_1\|, \ldots, \|y_n - \bar{y}_n\|\right)$$

for each $t \in X$ *and* $y_i, \bar{y}_i \in Y$ *(when* $i \in \{1, \ldots, n\}$*).*

ii) the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\varphi(t) = \beta(t, \dots, t)$ is a strict comparison function.

Then the inclusion (1.1) *has at least one solution.*

Proof. On $C(X,Y) := \{x : X \to Y/\ x \text{ is continuous}\}$ we consider the supremum norm, i.e.,

$$\|x\|_\infty := \sup_{t\in X} \|x(t)\|.$$

Then $(C(X,Y), \|\cdot\|_\infty)$ is a Banach space. We introduce a multivalued operator $T : C(X,Y) \to P\left(C(X,Y)\right)$ as follows:

$$x \longmapsto Tx,$$

where

$$Tx := \{z \in C(X,Y)/\ z(t) \in G\left(t, x(f_1(t)), \dots, x(f_n(t))\right),\ for\ all\ t \in X\}.$$

Using this notation, a solution for the inclusion (1.1) means a fixed point for T. Hence, it is enough to show that T has at least one fixed point.

Notice first that, by Michael's selection theorem (see Theorem 3.3), the set $Tx \neq \emptyset$, for each $x \in C(X,Y)$.

Indeed, if $x \in C(X,Y)$, since G is lower semi-continuous and f_i $(i \in \{1, \dots, n\})$ are continuous, we get that $G\left(\cdot, x\left(f_1(\cdot)\right), \dots, x\left(f_n(\cdot)\right)\right)$ is lower semi-continuous.

Notice also that $Tx \in P_{cl,cv}(C(X,Y))$ for all $x \in C(X,Y)$.

We will prove now that T is a multivalued φ-contraction on $C(X,Y)$, i.e.,

$$H_{\|\cdot\|_\infty}(Tx_1, Tx_2) \leq \varphi\left(\|x_1 - x_2\|_\infty\right),\ \forall x_1,\ x_2 \in C(X,Y).$$

For this purpose, let $z_1 \in Tx_1$ be arbitrary. It is enough to show that

$$D_{\|\cdot\|_\infty}(z_1, Tx_2) \leq \varphi(\|x_1 - x_2\|).$$

Since

$$D_{\|\cdot\|_\infty}(z_1, Tx_2) = \inf_{z_2 \in Tx_2} \|z_1 - z_2\|_\infty,$$

it is enough to show that for every $q > 1$ there exists $z_2 \in Tx_2$ such that

$$\|z_1 - z_2\|_\infty \leq q\varphi(\|x_1 - x_2\|_\infty).$$

Now, for $z_1 \in Tx_1$ we have

$$\begin{aligned} &D_{\|\cdot\|}\left(z_1(t), G(t, x_2(f_1(t)), \dots, x_2(f_n(t)))\right) \\ &\quad \leq H\left(G(t, x_1(f_1(t)), \dots, x_1(f_n(t))), G(t, x_2(f_1(t)), \dots, x_2(f_n(t)))\right) \\ &\quad \leq \beta\left(\|x_1(f_1(t)) - x_2(f_1(t))\|, \dots, \|x_1(f_n(t)) - x_2(f_n(t))\|\right) \\ &\quad \leq \beta\left(\|x_1 - x_2\|_\infty, \dots, \|x_1 - x_2\|_\infty\right) = \varphi(\|x_1 - x_2\|_\infty). \end{aligned}$$

For a fixed $q > 1$ and for every $t \in X$ (by Lemma 2.2, *ii*)) there exists $u_t \in G(t, x_2(f_1(t)), \dots, x_2(f_n(t)))$ such that $\|z_1(t) - u_t\| \leq q\varphi(\|x_1 - x_2\|_\infty)$.

Then, by Theorem 3.3 b)(Wegrzyk) there exists a family of continuous functions $\{z_t \in C(X,Y)/t \in X\}$ such that $z_t(t) = u_t$ and $z_t \in Tx_2$, for $t \in X$.

Then, by the above relations, we get

$$\|z_1(t) - z_t(t)\| = \|z_1(t) - u(t)\| < q\varphi(\|x_1 - x_2\|_\infty),\ \forall\ t \in X.$$

By the continuity of z_1 and z_t there is an open neighborhood U_t of x such that

$$\|z_1(s) - z_t(s)\| < q\varphi(\|x_1 - x_2\|_\infty),\ \forall\ s \in U_t.$$

Since the family $\mathcal{U} := \{\mathcal{U}_t : t \in X\}$ is an open covering of the compact space X and if $p_i : X \to [0,1],\ (i \in \{1,\dots,m\})$ is a finite partition of unity subordinated to $\mathcal{U}$ with $\sup p_i \subset \mathcal{U}_{t_i}$, then the operator

$$z_2(t) := \sum_{i=1}^{m} p_i(t) z_{t_i}(t)$$

is continuous and satisfies the condition

$$\|z_1 - z_2\|_\infty \le q\varphi(\|x_1 - x_2\|_\infty).$$

Moreover $z_2 \in Tx_2$, by the convexity of the values of G.

Hence, we proved that

$$H_{\|\cdot\|_\infty}(Tx_1, Tx_2) \le \varphi(\|x_1 - x_2\|_\infty), \text{ for all } x_1,\ x_2 \in C(X,Y).$$

Since $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function, we can apply Theorem 3.1 i) and thus, the multivalued operator T is a MWP operator, i.e., T has at least one fixed point $x^* \in C(X,Y)$ and there exists a sequence $(x_p)_{p\in\mathbb{N}} \subset C(X,Y)$ with $x_0 \in C(X,Y)$; $x_1 \in C(X,Y)$ and $x_1(t) \in G(t, x_0(f_1(t)), \dots, x_0(f_n(t)))$ for all $t \in X$; $x_{p+1} \in C(X,Y)$ and $x_{p+1} \in G(t, x_p(f_1(t)), \dots, x_p(f_n(t)))$ for all $t \in X$ and $p \in \mathbb{N}$ which converges in $C(X,Y)$ to x^*.

As a consequence, the n-order inclusion (1.1) has at least one solution $x^* \in C(X,Y)$ and the sequence $(x_p)_{p\in\mathbb{N}}$ defined above converges to x^*. □

Another result concerning the properties of the solutions of (1.1) is the following:

Theorem 4.2. *Consider the following two n-order functional inclusions:*

$$x(t) \in G_1(t, x(f_1(t)), \dots, x(f_n(t))),\ t \in X, \tag{4.1}$$

$$y(t) \in G_2(t, y(f_1(t)), \dots, y(f_n(t))),\ t \in X. \tag{4.2}$$

We suppose that $G_1, G_2, f_1, \dots, f_n$ satisfy all the assumptions from Theorem 4.1. *We also suppose:*

(iii) $\varphi(qt) \le q\varphi(t),\ \forall t \in \mathbb{R}_+$ *(where $q > 1$);*

(iv) there exists $\eta > 0$ such that

$$H_{\|\cdot\|}(G_1(t, u_1, \dots, u_n), G_2(t, u_1, \dots, u_n)) \le \eta$$

for all $t \in X$ and $u_1, \dots, u_n \in Y$.

Then

$$H_{\|\cdot\|_\infty}(S_{G_1;f_1,\dots,f_n}, S_{G_2,f_1,\dots,f_n}) \le \psi(\eta),$$

where $\psi(t) = t + s(t)$ and $s(t) := \sum_{n=1}^{\infty} \varphi^n(t)$.

Proof. The conclusions follows by Theorem 4.1 and Theorem 3.1 $(ii) + (iii)$ □

A result concerning the Ulam-Hyers stability of the n-order functional inclusion (1.1) is the following

Theorem 4.3. *Consider the inclusion* (1.1). *We suppose that $G, f_1, \dots, f_n$ satisfy all the assumptions from* Theorem 4.1. *Moreover, we suppose:*

(v) $G : X \times Y^n \longrightarrow P_{cp,cv}(Y)$

(vi) there exists an increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with the property: for every $\epsilon > 0$ there exists $\delta > 0$ such that if $t, s \in X$, $\|t - s\| < \delta$ implies

$$\delta\left(G(t, x(f_1(t)), \ldots, x(f_n(t))), G(s, x(f_1(s)), \ldots, x(f_n(s)))\right) < \epsilon,$$

for all $x \in C(X, Y)$.

Then, the n-order functional inclusion (1.1) *is generalized Ulam-Hyers stable.*

Proof. From our assumptions, via the Ascoli-Arzela Theorem, it follows that $Tx \in P_{cp}(C(X, Y))$, for every $x \in C(X, Y)$. The conclusion follows now by Theorem 3.1 (v). □

Some other stability results are the following.

Theorem 4.4. *Consider the n-order inclusion* (1.1). *We additionally suppose that there exists $u^* \in C(X, Y)$ such that*

$$G(t, u^*(f_1(t)), \ldots, u^*(f_n(t))) = \{u^*(t)\}.$$

Then:

a) The n-order functional inclusion (1.1) *has u^* as unique solution;*

b) The n-order functional inclusion (1.1) *is well-posed, i.e., if $(x_p)_{p\in\mathbb{N}^*} \subset C(X, Y)$ is a sequence with the property that*

$$D_{\|\cdot\|}(x_p(t), G(t, x_p(f_1(t)), \ldots, x_p(f_n(t)))) \longrightarrow 0 \quad \text{as } p \to \infty$$

then $(x_p)_{p\in\mathbb{N}^*}$ converges to u^* in the supremum norm of $C(X, Y)$;

c) If additional the function φ is subadditive, then the n-order functional inclusion (1.1) *has the limit shadowing property, i.e., if $(y_p)_{p\in\mathbb{N}^*} \subset C(X, Y)$ is such that*

$$D_{\|\cdot\|}(y_{p+1}, G(t, y_p(f_1(t)), \ldots, y_p(f_n(t)))) \longrightarrow 0 \quad \text{as } p \to \infty,$$

then there exists a sequence $(x_p)_{p\in\mathbb{N}} \subset C(X, Y)$ such that

$$x_{p+1}(t) \in G(t, x_p(f_1(t)), \ldots, x_p(f_n(t))), \forall t \in X$$

and $\|x_p - y_p\|_\infty \to 0$ as $p \to \infty$.

Proof. The conclusions follows by Theorems 4.1-4.3 and Theorem 3.2 $(x), (xv), (xvi), (xvii)$. □

References

[1] V. L. Lazăr, *Fixed point theory for multivalued φ-contractions*, Fixed Point Theory Appl., **2011** (2011), 12 pages. 1, 3.1, 3.2

[2] I. R. Petre, *Fixed point theorems in E-b-metric spaces*, J. Nonlinear Sci. Appl., **7** (2014), 264–271. 1

[3] A. Petruşel, *Operatorial Inclusions*, House of the Book of Science, Cluj-Napoca, **2002**. 2

[4] A. Petruşel, *Multivalued weakly Picard operators and applications*, Scientiae Math. Jpn., **59**, (2004), 167–202. 2, 2.8, 2.9

[5] A. Petruşel, I. A. Rus, *Multivalued Picard and weakly Picard operators*, International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, (2004), 207–226. 1, 2.9

[6] A. Petruşel, I. A. Rus, *The theory of a metric fixed point theorem for multivalued operators*, In: L.J. Lin, A. Petruşel, H.K. Xu, Fixed Point Theory and its Applications, Yokohama Publ., (2010), 161–175. 1

[7] A. Petruşel, I. A. Rus, J. C. Yao, *Well-posedness in the generalized sense of the fixed point problems*, Taiwan. J. Math., **11** (2007), 903–914. 1, 2

[8] R. Węgrzyk, *Fixed point theorems for multivalued functions and their applications to functional equations*, Dissertationes Math. (Rozprawy Mat.), **201** (1982), 28 pages. 3.1, 3.3