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The solutions of a class of operator equations based on different inequality

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Abstract

In this paper, by using random fixed point index theory, some new boundary conditions based on strictly convex or strictly concave functions are established and some new theorems for the solutions of a class of random semi-closed 1-set-contractive operator equations $A(\omega, x) = \mu x$ are obtained, which extend and generalize some corresponding results of Wang [S. Wang, Fixed Point Theory Appl., **2011** (2011), 7 pages]. Finally, an application to a class of random nonlinear integral equations is given to illustrate the usability of the obtained results.

Keywords: Real Banach space, random semi-closed 1-set-contractive operator, random topological degree.
2010 MSC: 47H10, 60H25.

1. Introduction and preliminaries

Random nonlinear analysis is an important mathematical branch, which is mainly concerned with the study of random nonlinear operators and their properties and is widely applied in studying various classes of random differential equations and random integral equations ([1, 6]). In [2], Li introduced the random fixed point index theory of random semi-closed 1-set-contractive operators. Since then, the problem of random nonlinear operators are extensively studied by using such theory (see Refs. [3, 4, 5, 7, 8, 9, 10, 11]). In this paper, by using random fixed point index theory, some new boundary conditions based on strictly convex or strictly concave functions are established, and some new theorems for the solutions of a class of random semi-closed 1-set-contractive operator equations $A(\omega, x) = \mu x$ are obtained in real separable Banach spaces, which extend and generalize some corresponding results in previous literatures [7].

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In this paper, let (Ω, U, γ) be a complete probability measure space, $\gamma(\Omega) = 1$. Let E be a separable real Banach space, (E, B) be a measurable space, where B denotes the σ-algebra generated by all the subsets of E, and D be a bounded open subset in E, ∂D the boundary of D in E.

Definition 1.1 ([2]). A random continuous and bounded operator $A : \Omega \times \overline{D} \to E$ is said to be a E-valued random semi-closed 1-set-contractive operator, if for almost all $\omega \in \Omega$, $A(\omega, \cdot)$ is a semi-closed 1-set-contractive operator in $\overline{D}$, (if $I - A$ is a closed operator and for every $\omega \in \Omega, \alpha(A(\omega, D)) \leq \alpha(D)$, see [5]), and for every $x \in \overline{D}$, $A(\cdot, x) : \Omega \to E$ is an E-valued random operator.

Let D be a nonempty subset of R. If $\varphi : D \to R$ is a real function such that

$$\varphi[tx + (1-t)y] < t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in D$, $x \neq y$, $t \in (0, 1)$, then φ is called a strictly convex function on$D$. If $\varphi : D \to R$ is a real function such that

$$\varphi[tx + (1-t)y] > t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in D$, $x \neq y$, $t \in (0, 1)$, then φ is called a strictly concave function on D.

Lemma 1.2 ([2]). *Let E be a separable real Banach space, X be a closed convex subset of E, D be a bounded open subset in X, and $\theta \in D$. Suppose that $A : \Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator, such that*

$$A(\omega, x) \neq \alpha x$$

for all $(\omega, x) \in \Omega \times \partial D$, and $\alpha > \mu$, where $\mu \geq 1$. Then random operator equations $A(\omega, x) = \mu x$ has a random solution in D.

2. Main results

Theorem 2.1. *Let E be a separable real Banach space, X be a closed convex subset of E, D be a bounded open subset in X and $\theta \in D$. Suppose that $A : \Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator and $\theta \notin A(\omega, \partial D), \forall \omega \in \Omega$. Moreover, if there exist a strictly convex function $\varphi : R^+ \longrightarrow R^+$ with $\varphi(0) = 0$ and a real function $\phi : R^+ \longrightarrow R^+$ with $\phi(t) \geq 1$ for all $t > 1$, such that*

$$\lambda_1\varphi(\| A(\omega, x) - \mu x\|) \geq \lambda_2\varphi(\|A(\omega, x)\|)\phi(\|A(\omega, x) + \mu x\| \cdot \|\mu x\|^{-1}) - \lambda_1\varphi(\|\mu x\|) \tag{2.1}$$

for all $(\omega, x) \in \Omega \times \partial D$, where $0 < \lambda_1 \leq \lambda_2$. Then random operator equations $A(\omega, x) = \mu x (\mu \geq 1)$ has a random solution in D.

Proof. By Lemma 1.2, we only need to prove that $A(\omega, x) \neq \alpha x, \forall(\omega, x) \in \Omega \times \partial D$, $\alpha > \mu$, where $\mu \geq 1$. Suppose this is not true. Then there exists $(\omega_0, x_0) \in \Omega \times \partial D$ and $\alpha_0 > \mu \geq 1$, such that $A(\omega_0, x_0) = \alpha_0 x_0$, i.e., $x_0 = \alpha_0^{-1} A(\omega_0, x_0)$. From (2.1),we have

$$\begin{aligned}\lambda_1\varphi(\|A(\omega_0, x_0) - \alpha_0^{-1}\mu A(\omega_0, x_0)\|) \geq& \lambda_2\varphi(A(\omega_0, x_0)\|)\phi(\|A(\omega_0, x_0) + \alpha_0^{-1}\mu\|A(\omega_0, x_0)\| \\ &\cdot \|\alpha_0^{-1}\mu A(\omega_0, x_0)\|^{-1}) - \lambda_1\varphi(\|\alpha_0^{-1}\mu A(\omega_0, x_0\|),\end{aligned}$$

which implies that

$$\begin{aligned}&\lambda_1\varphi((1 - \alpha_0^{-1}\mu)\|A(\omega_0, x_0)\|) + \lambda_1\varphi(\|\alpha_0^{-1}\mu A(\omega_0, x_0)\|) \\ &\geq\lambda_2\varphi(\| A(\omega_0, x_0)\|) \cdot \phi[(1 + \alpha_0^{-1}\mu) \cdot \|A(\omega_0, x_0)\| \cdot \alpha_0\mu^{-1}\|A(\omega_0, x_0)\|^{-1}] \\ &=\lambda_2\varphi(\| A(\omega_0, x_0)\|) \cdot \phi(\alpha_0\mu^{-1} + 1).\end{aligned} \tag{2.2}$$

By the strict convexity of φ and $\varphi(0)=0, A(\omega_0,x_0)\neq\theta$, we obtain

$$\begin{aligned}\lambda_1\varphi[(1-\alpha_0^{-1}\mu)\|A(\omega_0,x_0)\|+\alpha_0^{-1}\mu\|\theta\|]+\lambda_1\varphi(\alpha_0^{-1}\mu\|A(\omega_0,x_0)\|)+(1-\alpha_0^{-1}\mu)\|\theta\|)\\ <(1-\alpha_0^{-1}\mu)\lambda_1\varphi(\|A(\omega_0,x_0)\|)+\alpha_0^{-1}\mu\lambda_1\varphi(\|\theta\|)+\alpha_0^{-1}\mu\lambda_1\varphi(\|A(\omega_0,x_0)\|)+(1-\alpha_0^{-1})\mu\lambda_1\varphi(\|\theta\|)\\ =\lambda_1\varphi(\|A(\omega_0,x_0)\|).\end{aligned}\tag{2.3}$$

It is easy to see from (2.2)and(2.3),

$$\lambda_2\varphi(\|A(\omega_0,x_0)\|)\phi[(\alpha_0\mu^{-1}+1)]<\lambda_1\varphi(\|A(\omega_0,x_0)\|).$$

However, since $\alpha_0>\mu\geq 1$, we have $1+\alpha_0\mu^{-1}>1$, and thus $\phi(\alpha_0\mu^{-1}+1)\geq 1$. Noting that $0<\lambda_1\leq\lambda_2$, we have $\lambda_2\varphi(\|A(\omega_0,x_0)\|)\phi[(\alpha_0^{-1}\mu+1)]\geq\lambda_1\varphi(\|A(\omega_0,x_0)\|)$, which is a contradiction. It follows from Lemma 1.2 that random operator equations $A(\omega,x)=\mu x(\mu\geq 1)$ has a random solution in D. □

Remark 2.2. If there exist a convex function $\varphi:R^+\to R^+$, with $\varphi(0)=0$ and a real function $\phi:R^+\to R^+$, with $\phi(t)>1$, for all $t>1$ satisfying (2.1), the conclusion of Theorem2.1 also hold.

Corollary 2.3. *Let E be a separable real Banach space, X be a closed convex subset of E, D be a bounded open subset in X and $\theta\in D$. Suppose that $A:\Omega\times\overline{D}\to X$ is a random semi-closed 1-set-contractive operator and $\theta\notin A(\omega,\partial D),\forall\omega\in\Omega$. Moreover, if there exist $\alpha\in(-\infty,0)\cup(1,+\infty)$ and $\beta\geq 0$, such that*

$$\lambda_1\|A(\omega,x)-\mu x\|^{\alpha}\|\mu x\|^{\beta}\geq\lambda_2\|A(\omega,x)\|^{\alpha}\|A(\omega,x)+\mu x\|^{\beta}-\lambda_1\|\mu x\|^{\alpha+\beta}$$

for all $(\omega,x)\in\Omega\times\partial D$, where $0<\lambda_1\leq\lambda_2$. Then random operator equations $A(\omega,x)=\mu x(\mu\geq 1)$ has a random solution in D.

Proof. Let $\varphi(t)=t^{\alpha}$, $\phi(t)=t^{\beta}$. Then $\varphi(t)$ is a strictly convex function with $\varphi(0)=0$ and $\phi(t)\geq 1$ for all $t>1$. Therefore, by Theorem 2.1, the conclusions follows immediately. □

Remark 2.4. Letting $\beta=0$, $\alpha=2$, $\lambda_1=\lambda_2=1$ and $\mu=1$ in Corollary2.3 , $A(\omega_0,\cdot)=A$ and A is completely continuous operator. Thus Corollary 2.3 becomes the famous *Altman's* theorem. Thus, Corollary 2.3 generalizes *Altman's* theorem.

Theorem 2.5. *Let E be a separable real Banach space, X be a closed convex subset of E, D be a bounded open subset in X, and $\theta\in D$. Suppose that $A:\Omega\times\overline{D}\to X$ is a random semi-closed 1-set-contractive operator and $\theta\notin A(\omega,\partial D),\forall\omega\in\Omega$. Moreover, if there exist a strictly concave function $\varphi:R^+\longrightarrow R^+$ with $\varphi(0)=0$ and a real function $\phi:R^+\longrightarrow R^+$ with $\phi(t)\leq 1$ for all $t>1$, such that*

$$\lambda_1\varphi(\|A(\omega,x)-\mu x\|)\leq\lambda_2\varphi(\|A(\omega,x)\|)\phi(\|A(\omega,x)+\mu x\|\cdot\|\mu x\|^{-1})-\lambda_1\varphi(\|\mu x\|)\tag{2.4}$$

for all $(\omega,x)\in\Omega\times\partial D$, where $\lambda_1\geq\lambda_2>0$. Then random operator equations $A(\omega,x)=\mu x(\mu\geq 1)$ has a random solution in D.

Proof. By Lemma 1.2, we only need to prove that $A(\omega,x)\neq\alpha x$, $\forall(\omega,x)\in\Omega\times\partial D$, $\alpha>\mu$, where $\mu\geq 1$. Suppose this is not true. Then there exist $(\omega_0,x_0)\in\Omega\times\partial D$ and $\alpha_0>\mu\geq 1$, such that $A(\omega_0,x_0)=\alpha_0x_0$, i.e., $x_0=\alpha_0^{-1}A(\omega_0,x_0)$. From (2.4), we have

$$\begin{aligned}\lambda_1\varphi(\|A(\omega_0,x_0)-\alpha_0^{-1}\mu A(\omega_0,x_0)\|)\leq&\lambda_2\varphi(\|A(\omega_0,x_0)\|)\phi(\|A(\omega_0,x_0)+\alpha_0^{-1}\mu A(\omega_0,x_0)\|\\ &\cdot\|\alpha_0^{-1}\mu A(\omega_0,x_0)\|^{-1})-\lambda_1\varphi(\|\alpha_0^{-1}\mu A(\omega_0,x_0\|),\end{aligned}$$

which implies that

$$\begin{aligned}\lambda_1\varphi[(1-\alpha_0^{-1}\mu)\|A(\omega_0,x_0)\|]+\lambda_1\varphi[\|\alpha_0^{-1}\mu A(\omega_0,x_0)\|]\\ \leq\lambda_2\varphi(\|A(\omega_0,x_0)\|)\cdot\phi[(1+\alpha_0^{-1}\mu)\cdot\|A(\omega_0,x_0)\|\cdot\alpha_0\mu^{-1}\|A(\omega_0,x_0)\|^{-1}]\\ =\lambda_2\varphi(\|A(\omega_0,x_0)\|)\cdot\phi(\alpha_0\mu^{-1}+1).\end{aligned}\tag{2.5}$$

By the strict concavity of φ and $\varphi(0) = 0$, we obtain

$$\begin{aligned}
&\lambda_1\varphi[(1-\alpha_0^{-1}\mu)\|A(\omega_0,x_0)\| + \alpha_0^{-1}\mu\|\theta\|] + \lambda_1\varphi[\alpha_0^{-1}\mu\|A(\omega_0,x_0)\|) + (1-\alpha_0^{-1}\mu)\|\theta\|] \\
&\quad > (1-\alpha_0^{-1}\mu)\lambda_1\varphi(\|A(\omega_0,x_0)\|) + \alpha_0^{-1}\mu\lambda_1\varphi(\|\theta\|) + \alpha_0^{-1}\mu\lambda_1\varphi(\|A(\omega_0,x_0)\|) + (1-\alpha_0^{-1}\mu)\lambda_1\varphi(\|\theta\|) \\
&\quad = \lambda_1\varphi(\|A(\omega_0,x_0)\|).
\end{aligned} \tag{2.6}$$

It is easy to see from (2.5) and (2.6) that

$$\lambda_2\varphi(\|A(\omega_0,x_0)\|)\phi[(\alpha_0\mu^{-1}+1)] > \lambda_1\varphi(\|A(\omega_0,x_0)\|).$$

However, since $\alpha_0 > \mu \geq 1$, we have $\alpha_0\mu^{-1}+1 > 1$ and thus $\phi(\alpha_0\mu^{-1}+1) \leq 1$. Noting that $\lambda_1 \geq \lambda_2 > 0$, we have $\lambda_2\varphi(\|A(\omega_0,x_0)\|)\phi[(\alpha_0^{-1}\mu+1)] \leq \lambda_1\varphi(\|A(\omega_0,x_0)\|)$, which is a contradiction. It follows from Lemma 1.2 that random operator equations $A(\omega,x) = \mu x(\mu \geq 1)$ has a random solution in D. □

Remark 2.6. If there exist a concave function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ and a real function $\phi : R^+ \to R^+$, with $\phi(t) < 1$ for all $t > 1$ satisfying (2.4), the conclusion of Theorem 2.5 also hold.

Corollary 2.7. *Let E be a separable real Banach space,X be a closed convex subset of E, D be a bounded open subset in X and $\theta \in D$. Suppose that $A : \Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator and $\theta \notin A(\omega,\partial D)$, for all $\omega \in \Omega$, where $\lambda_1 \geq \lambda_2 > 0$. Moreover, if there exist $\alpha \in (0,1)$ and $\beta \leq 0$, such that*

$$\lambda_1\|A(\omega,x)-\mu x\|^\alpha\|\mu x\|^\beta \leq \lambda_2\|A(\omega,x)\|^\alpha\|A(\omega,x)+\mu x\|^\beta - \lambda_1\|\mu x\|^{\alpha+\beta}$$

for all $(\omega,x) \in \Omega \times \partial D$, $\lambda_1 \geq \lambda_2 > 0$. Then random operator equations $A(\omega,x) = \mu x(\mu \geq 1)$ has a random solution in D.

Proof. Let $\varphi(t) = t^\alpha$, $\phi(t) = t^\beta$. Then $\varphi(t)$ is a strictly convex function with $\phi(t) \geq 1$ for all $t > 1$ and $\varphi(0) = 0$. Therefore, by Theorem 2.5, the conclusion follows immediately. □

Theorem 2.8. *Let E be a separable real Banach space, X be a closed convex subset of E, D be a bounded open subset in X and $\theta \in D$. Suppose that $A : \Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator and $\theta \notin A(\omega,\partial D)$, $\forall\omega \in \Omega$. Moreover, if there exist a strictly convex function $\varphi : R^+ \longrightarrow R^+$ with $\varphi(0) = 0$ and a real function $\phi : R^+ \longrightarrow R^+$ with $\phi(t) \geq 1$ for all $t > 1$, such that*

$$\lambda_1\varphi(\| A(\omega,x)-\mu x\|) \geq \lambda_2\varphi(\|A(\omega,x)\|)\phi(\|A(\omega,x)\|\cdot\|\mu x\|^{-1}) - \lambda_1\varphi(\|\mu x\|) \tag{2.7}$$

for all $(\omega,x) \in \Omega \times \partial D$, where $\lambda_2 \geq \lambda_1 > 0$. Then random operator equations $A(\omega,x) = \mu x(\mu \geq 1)$ has a random solution in D.

Proof. By Lemma 1.2, we only need to prove that $A(\omega,x) \neq \alpha x$, $\forall(\omega,x) \in \Omega \times \partial D$, $\alpha > \mu \geq 1$. Suppose this is not true. Then there exists $(\omega_0,x_0) \in \Omega \times \partial D$ and $\alpha_0 > \mu \geq 1$, such that $A(\omega_0,x_0) = \alpha_0 x_0$, i.e., $x_0 = \alpha_0^{-1}A(\omega_0,x_0)$. From (2.7), we have

$$\begin{aligned}
\lambda_1\varphi(\|A(\omega_0,x_0)-\alpha_0^{-1}\mu A(\omega_0,x_0)\|) \geq{}& \lambda_2\varphi(\|A(\omega_0,x_0)\|)\phi(\|A(\omega_0,x_0)\|\cdot\|\alpha_0^{-1}\mu A(\omega_0,x_0)\|^{-1}) \\
&- \lambda_1\varphi(\|\alpha_0^{-1}\mu A(\omega_0,x_0)\|),
\end{aligned}$$

which implies that

$$\lambda_1\varphi[(1-\alpha_0^{-1}\mu)\|A(\omega_0,x_0)\|] + \lambda_1\varphi[\|\alpha_0^{-1}\mu A(\omega_0,x_0)\|] \geq \lambda_2\varphi(A(\omega_0,x_0)\|)\cdot\phi(\mu^{-1}\alpha_0). \tag{2.8}$$

By the strict convexity of φ and $\varphi(0) = 0$, we obtain

$$\begin{aligned}
&\lambda_1\varphi[(1-\alpha_0^{-1}\mu)\|A(\omega_0,x_0)\| + \alpha_0^{-1}\mu\|\theta\|] + \lambda_1\varphi[\alpha_0^{-1}\mu\|A(\omega_0,x_0)\|] + (1-\alpha_0^{-1}\mu)\|\theta\|) \\
&\quad < \lambda_1(1-\alpha_0^{-1}\mu)\varphi(\|A(\omega_0,x_0)\|) + \lambda_1\alpha_0^{-1}\mu\varphi(\|\theta\|) + \lambda_1\alpha_0^{-1}\mu\varphi(\|A(\omega_0,x_0)\|) + \lambda_1(1-\alpha_0^{-1}\mu)\varphi(\|\theta\|) \\
&\quad = \lambda_1\varphi(\|A(\omega_0,x_0)\|).
\end{aligned} \tag{2.9}$$

It is easy to see from (2.8) and (2.9) that

$$\lambda_2\varphi(\|A(\omega_0,x_0)\|)\phi(\alpha_0\mu^{-1}) < \lambda_1\varphi(\|A(\omega_0,x_0)\|).$$

However, since $\alpha_0 > \mu \geq 1$, we have $\alpha_0\mu^{-1} \geq 1$, and thus $\phi(\alpha_0\mu^{-1}) > 1$. Noting that $\lambda_2 \geq \lambda_1 > 0$, we have $\lambda_2\varphi(\|A(\omega_0,x_0)\|)\phi(\mu^{-1}\alpha_0) \geq \lambda_1\varphi(\|A(\omega_0,x_0)\|)$, which is a contradiction. Therefore, it follows from Lemma 1.2 that random operator equations $A(\omega,x) = \mu x(\mu \geq 1)$ has a random solution in D. □

Remark 2.9. If there exist a convex function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ and a real function $\phi : R^+ \to R^+$ with $\phi(t) > 1$ for all $t > 1$ satisfying (2.7), the conclusion of Theorem 2.8 also hold.

Corollary 2.10. *Let E be a separable real Banach space, X be a closed convex subset of E, D be a bounded open subset in X and $\theta \in D$. Suppose that $A : \Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator and $\theta \notin A(\omega,\partial D), \forall\omega \in \Omega$. Moreover, if there exist $\alpha \in (-\infty,0] \cup [1,+\infty)$ and $\beta > 0$, such that*

$$\lambda_1\|A(\omega,x) - \mu x\|^{\alpha}\|\mu x\|^{\beta} \geq \lambda_2\|A(\omega,x)\|^{\alpha+\beta} - \lambda_1\|\mu x\|^{\alpha+\beta}$$

for all $(\omega,x) \in \Omega \times \partial D, \lambda_2 \geq \lambda_1 > 0$. Then random operator equations $A(\omega,x) = \mu x(\mu \geq 1)$ has a random solution in D.

Proof. Let $\varphi(t) = t^{\alpha}$, $\phi(t) = t^{\beta}$. Then $\varphi(t)$ is a convex function with $\varphi(0) = 0$ and $\phi(t) > 1$ for all $t > 1$. Therefore, by theorem 2.8, the conclusion follows immediately. □

Remark 2.11. Letting $\mu = 1$ and $\lambda_1 = \lambda_2 = 1$ in Corollary 2.10, $A(\omega,\cdot) = A$, we can obtain Theorem 2.17 of [7].

Theorem 2.12. *Let E be a separable real Banach space, X be a closed convex subset of E, D be a bounded open subset in X, and $\theta \in D$. Suppose that $A : \Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator, and $\theta \notin A(\omega,\partial D), \forall\omega \in \Omega$. Moreover, if there exist a strictly concave function $\varphi : R^+ \longrightarrow R^+$ with $\varphi(0) = 0$ and a real function $\phi : R^+ \longrightarrow R^+$ with $\phi(t) \leq 1$ for all $t > 1$, such that*

$$\lambda_1\varphi(\| A(\omega,x) - \mu x\|) \leq \lambda_2\varphi(\|A(\omega,x)\|)\phi(\|A(\omega,x)\| \cdot \|\mu x\|^{-1}) - \lambda_1\varphi(\|\mu x\|) \tag{2.10}$$

for all $(\omega,x) \in \Omega \times \partial D$, where $0 < \lambda_2 \leq \lambda_1$. Then random operator equations $A(\omega,x) = \mu x$ has a random solution in D.

Proof. By Lemma 1.2, we only need to prove that $A(\omega,x) \neq \alpha x$, $\forall(\omega,x) \in \Omega \times \partial D$, $\alpha > \mu \geq 1$. Suppose this is not true. Then there exist $(\omega_0,x_0) \in \Omega \times \partial D$ and $\alpha_0 > \mu \geq 1$ such that $A(\omega_0,x_0) = \alpha_0 x_0$, i.e., $x_0 = \alpha_0^{-1}A(\omega_0,x_0)$. From (2.10), we have

$$\begin{aligned}\lambda_1\varphi(\|A(\omega_0,x_0) - \alpha_0^{-1}\mu A(\omega_0,x_0)\|) \leq &\lambda_2\varphi(\|A(\omega_0,x_0)\|)\phi(\|A(\omega_0,x_0)\| \cdot \|\alpha_0^{-1}\mu A(\omega_0,x_0)\|^{-1})\\ &- \lambda_1\varphi(\|\alpha_0^{-1}\mu A(\omega_0,x_0)\|),\end{aligned}$$

which implies that

$$\lambda_1\varphi[(1-\alpha_0^{-1}\mu)\|A(\omega_0,x_0)\|] + \lambda_1\varphi[\|\alpha_0^{-1}\mu A(\omega_0,x_0)\|] \leq \lambda_2\varphi(A(\omega_0,x_0)\|) \cdot \phi(\mu^{-1}\alpha_0). \tag{2.11}$$

By the strict concavity of φ and $\varphi(0) = 0$, we obtain

$$\begin{aligned}&\lambda_1\varphi[(1-\alpha_0^{-1}\mu)\|A(\omega_0,x_0)\| + \alpha_0^{-1}\mu\|\theta\|] + \lambda_1\varphi[\alpha_0^{-1}\mu\|A(\omega_0,x_0)\|] + (1-\alpha_0^{-1}\mu)\|\theta\|)\\ &\quad>\lambda_1(1-\alpha_0^{-1}\mu)\varphi(\|A(\omega_0,x_0)\|) + \lambda_1\alpha_0^{-1}\mu\varphi(\|\theta\|) + \lambda_1\alpha_0^{-1}\mu\varphi(\|A(\omega_0,x_0)\|) + \lambda_1(1-\alpha_0^{-1}\mu)\varphi(\|\theta\|)\\ &\quad=\lambda_1\varphi(\|A(\omega_0,x_0)\|).\end{aligned} \tag{2.12}$$

It is easy to see from (2.11) and (2.12) that

$$\lambda_2\varphi(\|A(\omega_0,x_0)\|)\phi(\alpha_0\mu^{-1}) > \lambda_1\varphi(\|A(\omega_0,x_0)\|).$$

However, since $\alpha_0 > \mu \geq 1$, we have $\alpha_0\mu^{-1} > 1$, and thus $\phi(\alpha_0\mu^{-1}) \leq 1$. Noting that $\lambda_1 \geq \lambda_2 > 0$, we have

$$\lambda_2\varphi(\|A(\omega_0, x_0)\|)\phi(\mu^{-1}\alpha_0) \leq \lambda_1\varphi(\|A(\omega_0, x_0)\|),$$

which is a contradiction. Therefore, it follows from Lemma 1.2 that random operator equations $A(\omega, x) = \mu x(\mu \geq 1)$ has a random solution in D. □

Corollary 2.13. *Let E be a separable real Banach space, X be a closed convex subset of E, D be a bounded open subset in X and $\theta \in D$. Suppose that $A : \Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator and $\theta \notin A(\omega, \partial D)$, $\forall\omega \in \Omega$. Moreover, if there exist $\alpha \in (0, 1)$ and $\beta \leq 0$, such that*

$$\lambda_1\|A(\omega, x) - \mu x\|^{\alpha}\|\mu x\|^{\beta} \leq \lambda_2\|A(\omega, x)\|^{\alpha+\beta} - \lambda_1\|\mu x\|^{\alpha+\beta}$$

for all $(\omega, x) \in \Omega \times \partial D$, where $0 < \lambda_2 \leq \lambda_1$. Then random operator equations $A(\omega, x) = \mu x(\mu \geq 1)$has a random solution in D.

Proof. Let $\varphi(t) = t^{\alpha}$, $\phi(t) = t^{\beta}$. Then $\varphi(t)$ is a strictly concave function with $\varphi(0) = 0$ and $\phi(t) \leq 1$ for all $t > 1$. Therefore, by Theorem 2.12, the conclusion follows immediately. □

3. An application

In this section, we give an example to demonstrate Corollary 2.10.

Example 3.1. Let us consider the following random nonlinear integral equation

$$\mu\varphi(x) = \int_G k(\omega, x, y, \varphi(y))dy, \tag{3.1}$$

where $\mu \geq 1$, and G is a bounded open subset in $\mathbb{R}^n$.

Suppose that

(i)$k(\omega, x, y, t)$ is random continuous on $\Omega \times G \times G \times \mathbb{R}^1$.

(ii)There exist $a, b > 0$ and $b \cdot mesG < m$, where $m > 0$, such that for any $(\omega, x, y, t) \in \Omega \times G \times G \times \mathbb{R}^1$,

$$| k(\omega, x, y, t) | \leq a + b \mid t \mid .$$

Then Equation (3.1) has a random continuous solution $\varphi(\omega, x)$.

Proof. Let $C(G)$ be the Banach space of all continuous functions defined on G with norm defined by

$$\| f \|_C = \sup_{x\in G} \| f(x) \|, f \in C(G).$$

Imitating the proof of Theorem 2.1 of [4], we know that the operator

$$A(\omega, \varphi) = \int_G k(\omega, x, y, \varphi(y))dy$$

is a random compact continuous operator. Therefore, $A(\omega, \varphi) : \Omega \times C(G) \longrightarrow C(G)$ is a random semi-closed 1-set-contractive operator.

Let $r = \frac{a.mesG}{m-b\cdot mesG} > 0$ and $D = \{\varphi \mid\| \varphi(x) \|_C \leq r\}$ be a closed ball in $C(G)$. From (ii), we have

$$\begin{aligned}| A(\omega, \varphi) | &\leq \int_G | k(\omega, x, y, \varphi(y)) | dy \leq \int_G (a + b|\varphi(y)|)dy = a\int_G dy + b\int_G |\varphi(y)|dy\\ &= a \cdot mesG + b \cdot mesG \| \varphi \|_C, \forall(\omega, \varphi) \in \Omega \times \partial D.\end{aligned}$$

Thus

$$| A(\omega, \varphi) | \leq a \cdot mesG + b \cdot mesG \| \varphi \|_C \leq amesG + br \cdot mesG = mr, \forall(\omega, \varphi) \in \Omega \times \partial D,$$

which implies that

$$\| A(\omega, \varphi) \| \leq \| m\varphi \|, \forall(\omega, \varphi) \in \Omega \times \partial D.$$

Take $m = \frac{2\lambda_1}{\lambda_2}, \alpha = 0, \beta = 1$ in Corollary 2.10. Then all the conditions of Corollary 2.10 are satisfied. Therefore, the random nonlinear integral equation (3.1) has a random continuous solution $\varphi(\omega, x)$. □

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