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Chaos in nonautonomous discrete fuzzy dynamical systems

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Abstract

This paper is devoted to a study of relations between chaotic properties of nonautonomous dynamical system and its induced fuzzy system. More specially, we study transitivity, periodic density and sensitivity in an original nonautonomous system and its connections with the same ones in its fuzzified system.

Keywords: Discrete dynamical system, nonautonomous, fuzzy, chaos.
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1. Introduction and Preliminaries

A discrete dynamical system uniquely induces its fuzzified system which on the space of fuzzy sets. It is natural to ask the following question: What is the relation between dynamical properties of the original and fuzzified systems? In the present paper we study the relations between some chaotic properties of the nonautonomous discrete dynamical system and its fuzzified system. Let (X, d) be a compact metric space. A nonautonomous discrete system (**NADS**) is the following:

$$x_{n+1} = f_n(x_n), \quad n \geq 0, \tag{1.1}$$

where $\{f_n\}_{n=0}^{\infty}$ is a sequence of continuous maps and each $f_n : X \to X$. Note that the autonomous dynamical system (**ADS**) is a special case of system (1.1) when $f_n = f$ for all $n \geq 0$. For other notions and notations mentioned in this section, we refer to Section 2. **ADSs** have been extensively studied and many elegant

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results have been obtained. The study of dynamics of **NADSs** is more complicated and there are many publications on this area [2, 8, 11, 16], but it seems there are only a few results about chaotic properties of **NADSs**. Chaotic dynamics has been hailed as the third great scientific revolution of the 20th century, along with relativity and quantum mechanics. But there is not a generally accepted definition of chaos yet. The word "chaos" has been introduced into mathematics by Li and Yorke [21], and then different definitions of chaos have been designed to meet different purposes and they are based on very different backgrounds and levels of mathematical sophistication. Among various definitions of chaos, Devaney's chaos [7] is one of the most commonly used. In [1], AlSharawi and his coauthors present an extension of Sharkovsky's theorem and its converse to periodic system. Tian and Chen [28] introduce several new concepts of chaos in the sense of Devaney and prove that two uniformly topologically conjugate **NADS** share the same chaotic properties. However, similar conclusions do not hold for two topological conjugate **NADS** [26], which means it is diffcult to study chaotic behavior of **NADS**. In [26], Shi and Chen also establish a criterion of chaos in the sense of Li-Yorke and discuss chaos of **NADS** in the sense of Devaney, Wiggins, respectively. Until very recently the study of **NADSs**' chaotic behavior has become actively [3, 10, 14, 15, 25].

On the other hand, as the complexity of research subjects increased, an accurate description for systems becomes more and more difficult, the situation would become more complicated when the systems are affected by the uncertainty. In this case, the fuzzy system should be considered. As we mentioned at the first place, it is necessary to study the relations between dynamical properties of the original and fuzzified systems. Actually, there are quite a few elegant results have been obtained [5, 6, 17, 18, 19, 20, 23, 24, 29].

In this paper, we focus on relations between Devaney's chaotic properties of the original and its fuzzified nonautonomous dynamical systems. Below, Section 2 gives basic notions and definitions. Section 3, Section 4 and Section 5 discuss the relation between Devaney's chaotic properties of the original and fuzzified systems, respectively. Finally, a brief conclusion concludes the paper.

2. Preliminaries

In this section, some basic concepts and notations are introduced.

*2.1. Basic concepts of **NADSs***

Let (X, d) be a compact metric space and $\{f_n\}_{n=0}^{\infty}$ be a sequence of continuous maps, where $f_n : X \to X$. An orbit of a point $x_0 \in X$, denoted by $\{x_n\}_{n=0}^{\infty}$, is defined as follows:

$$x_n = f_n(x_{n-1}), \quad n = 1, 2, \cdots .$$

Denote $F_n : X \to X$ by

$$F_n(x) = f_n \circ f_{n-1} \cdots \circ f_2 \circ f_1(x).$$

A point x is ***periodic*** if $F_n(x) = x$ for some $n \geq 1$.

We say that $\{f_n\}_{n=0}^{\infty}$ is ***transitive*** if for any pair of non-empty open sets U and V there exists $n \geq 1$ such that $F_n(U) \cap V \neq \emptyset$.

We say that $\{f_n\}_{n=0}^{\infty}$ has ***sensitive dependence on initial conditions*** if there is a constant $\delta > 0$ such that for every point x and every neighborhood U about x, there is a $y \in U$ and a $k \geq 1$ such that $d(F_k(x), F_k(y)) \geq \delta$.

A map that is transitive, has a dense set of periodic points and has sensitive dependence on initial conditions is called *Devaney chaotic*. It is well known that sensitive dependence on initial conditions is a consequence of transitivity together with a dense set of periodic points [4, 27]. More precisely, sensitivity is redundant in the definition if the state space X is infinite. This fact reveals the topological, rather than metric, nature of chaos. However, the situation is complicated when we consider a nonautonomous system. It is not clear that whether transitivity together with periodic density still imply sensitivity in the nonautonomous dynamical systems. Consequently, in this research, we say that $\{f_n\}_{n=0}^{\infty}$ is ***Devaney chaotic***, if it is transitive, sensitive and has dense set of periodic points.

2.2. Zadeh's extension of NADSs

For a given **NADS** $(X, \{\hat{f}_n\}_{n=0}^{\infty})$, its *Zadeh's extension* (or *fuzzification*) is a sequence of maps $\hat{f}_n : \mathbb{F}(X) \to \mathbb{F}(X)$ defined by

$$[\hat{f}_n(u)](x) = sup_{y \in f_n^{-1}(x)}\{u(y)\},$$

for any $u \in \mathbb{F}(X)$ and $x \in X$.

Denote $\mathbb{F}^1(X)$ the space of all *normal fuzzy sets* on X by

$$\mathbb{F}^1(X) = \{u \in \mathbb{F}(X) \mid u(x) = 1 \ for \ some \ x \in X\}$$

Proposition 2.1 ([18, 23]). *Let A, B be two subsets of X. Define $e(A) = \{u \in \mathbb{F}(X) : [u]_0 \subseteq A\}$, then*
(1) If A is an open set, then $e(A)$ is an open subset of $\mathbb{F}(X)$;
(2) $e(A) \neq \emptyset$ if and only if $A \neq \emptyset$;
(3) $e(A \cap B) = e(A) \cap e(B)$.

2.3. Metric space of fuzzy sets

Let $\mathbb{K}(X)$ be the class of all non-empty and compact subset of X. If $A \in \mathbb{K}(X)$ we define the $\varepsilon-$neighborhood of A as the set

$$N(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\},$$

where $d(x, A) = \inf_{a \in A} \|x - a\|$.

The Hausdorff separation $\rho(A, B)$ of $A, B \in \mathbb{K}(X)$ is defined by

$$\rho(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq N(B, \varepsilon)\},$$

The Hausdorff metric on $\mathbb{K}(X)$ is defined by letting

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

Define $\mathbb{F}(X)$ as the class of all upper semicontinuous fuzzy sets $u : X \to [0, 1]$ such that $[u]_\alpha \in \mathbb{K}(X)$, where *$\alpha$-cuts* and the *support* of u are defined by

$$[u]_\alpha = \{x \in X | u(x) \geq \alpha\}, \alpha \in [0, 1] \quad \text{and} \quad supp(u) = \overline{\{x \in X | u(x) > 0\}},$$

respectively.

Moreover, let $\mathbb{F}^1(X)$ denote the space of all *normal fuzzy sets* on X and $\emptyset_X$ denote the *empty fuzzy set* ($\emptyset_X(x) = 0$ for all $x \in X$).

A *levelwise metric* d_∞ on $\mathcal{F}(X)$ is defined by

$$d_\infty(u, v) = \sup_{\alpha \in [0,1]} H([u]_\alpha, [v]_\alpha)$$

for all $u, v \in \mathcal{F}(X)$. It is well known that if (X, d) is complete, then $(\mathcal{F}(X), d_\infty)$ is also complete but is not compact and is not separable (see [9, 13, 17]).

3. Transitivity

In this section, the relations between transitivity of nonautonomous system and its induced fuzzy system has been discussed.

Proposition 3.1. $[\hat{F}_n(u)]_\alpha = F_n([u]_\alpha)$.

Proof. Assume $\hat{f}_i(u_i) = u_{i+1}$.

$$\begin{aligned}[\hat{F}_n(u_1)]_\alpha &= [\hat{f}_n \circ \hat{f}_{n-1} \circ \cdots \circ \hat{f}_1(u_1)]_\alpha) = [\hat{f}_n \circ \hat{f}_{n-1} \circ \cdots \circ \hat{f}_2(u_2)]_\alpha) \\ &= [\hat{f}_n \circ \hat{f}_{n-1} \circ \cdots \circ \hat{f}_3(u_3)]_\alpha) \\ &= \cdots \\ &= [\hat{f}_n(u_n)]_\alpha = f_n([u_n]_\alpha) = f_n \circ f_{n-1}([u_{n-1}]_\alpha) \\ &= \cdots \\ &= f_n \circ f_{n-1} \circ \cdots \circ f_1([u_1]_\alpha) = F_n([u]_\alpha).\end{aligned}$$

□

Proposition 3.2. $\hat{F}_n[e(U)] \subseteq e[F_n(U)]$.

Proof. Suppose $u \in \hat{F}_n[e(U)]$, then there exists $\omega \in e(U)$ such that $u = \hat{F}_n(\omega)$. Therefore, we have

$$[u]_0 = [\hat{F}_n(\omega)]_0 = F_n([\omega]_0).$$

Since $[\omega]_0 \subseteq U$, $[u]_0 = F_n([\omega]_0) \subseteq F_n(U)$, consequently, $u \in e[F_n(U)]$. This completes the proof. □

Theorem 3.3. *If $\{\hat{f}_n\}_{n=1}^\infty$ is transitive, then $\{f_n\}_{n=1}^\infty$ is transitive.*

Proof. Suppose $\{\hat{f}_n\}_{n=1}^\infty$ is transitive. To show that $\{f_n\}_{n=1}^\infty$ is transitive, it suffices to prove for any non-empty open subsets U and V, there is a $k \geq 1$ such that

$$F_k(U) \cap V \neq \emptyset.$$

Due to Proposition 2.1 (1), $e(U)$ and $e(V)$ are open subsets of $\mathbb{F}(X)$ and so, $e(U) \cap e(V)$ is open. Thus, by transitivity of $\hat{F}_n$ and Proposition 3.2, there is a $k \geq 1$ such that

$$\emptyset \neq \hat{F}_k[e(U)] \cap e(V) \subset e[F_k(U)] \cap e(V) = e[F_k(U) \cap V].$$

By Proposition 2.1 (2), we have $F_k(U) \cap V \neq \emptyset$. This completes the proof. □

The following examples show that, in general, the converse of Theorem 3.3 is not true.

Example 3.4. (Irrational rotation of circle)

Consider system (1.1) and let $f_n = f_\lambda : S^1 \to S^1$ defined by $f_\lambda(e^{i\theta}) = e^{i(\theta+2\pi\lambda)}$, where λ is an irrational number. It is well known that for each $z \in S^1$, the orbit of z is dense in S^1 and, consequently, f_λ is transitive. However, $\hat{f}_\lambda$ is not transitive. In fact, assume $u \in \mathbb{F}(S^1)$ and $diam([u]_0) = 1$. Given that $0 < \varepsilon < \frac{1}{2}$, let $\mathcal{U} = B(\hat{1}, \frac{\varepsilon}{2})$ and $\mathcal{V} = B(u, \frac{\varepsilon}{2})$, we obtain

$$\omega \in \mathcal{U} = B(\hat{1}, \frac{\varepsilon}{2}) \Rightarrow diam([\omega]_0) \leq \frac{\varepsilon}{2},$$

$$\nu \in \mathcal{V} = B(u, \frac{\varepsilon}{2}) \Rightarrow diam([\nu]_0) \geq 1 - \varepsilon.$$

Since

$$diam([\hat{f_\lambda}^k(\nu)]_0) = diam(f_\lambda^k[\nu]_0) \geq 1 - \varepsilon$$

for $k \in \mathbb{N}$. Hence, $\mathcal{U} \cap \hat{f_\lambda}^n(\mathcal{V}) = \emptyset$, which means that $\{\hat{f}_n\}_{n=1}^\infty$ is not transitive on $\mathbb{F}(X)$.

In [17, 18] the author proves that no fuzzification can be transitive on the whole $\mathbb{F}(X)$, but there exists a transitive fuzzification on the space of normal fuzzy sets $\mathbb{F}^1(X)$. To finalize this section, we develop a method to prove that $\{f_n\}_{n=0}^{\infty}$ is transitive and it implies $\{\hat{f}_n\}_{n=0}^{\infty}$ is transitive. It should be mentioned that our approach was inspired by the idea presented in [12, 17, 18].

Let $\mathcal{U}$ be a subset of $\mathbb{F}^1(X)$. Set

$$r(\mathcal{U}) = \{A \in \mathbb{K}(X) \mid \exists u \in \mathcal{U} \ s.t. \ A \subseteq [u]_0\}.$$

Proposition 3.5. *Let $\mathcal{U}$ and $\mathcal{V}$ be subsets of $\mathbb{F}^1(X)$.*

(1) *$r(\mathcal{U}) \neq \emptyset$ if and only if $\mathcal{U} \neq \emptyset_X$, where $\emptyset_X$ is the empty fuzzy set ($\emptyset_X = 0$ for each $x \in X$);*

(2) *Suppose that $u \neq v$ implies $[u]_0 \cap [v]_0 = \emptyset$, then $r(\mathcal{U} \cap \mathcal{V}) = r(\mathcal{U}) \cap r(\mathcal{V})$;*

(3) *$F_n[r(\mathcal{U})] \subseteq r[\hat{F}_n(\mathcal{U})]$;*

(4) *If $\mathcal{U}$ is a non-empty open subset of $\mathbb{F}^1(X)$, then $r(\mathcal{U})$ is a non-empty open subset of X.*

Proof. (1) follows directly from the definitions.

(2) If $A \in r(\mathcal{U} \cap \mathcal{V})$, then there exists $\omega \in \mathcal{U} \cap \mathcal{V}$ such that $A \in [\omega]_0$. Then $A \in r(\mathcal{U})$ and $A \in r(\mathcal{V})$. Therefore, the inclusion $r(\mathcal{U} \cap \mathcal{V}) \subseteq r(\mathcal{U}) \cap r(\mathcal{V})$ follows. Conversely, let $A \in r(\mathcal{U}) \cap r(\mathcal{V})$. Then there exist $u \in \mathcal{U}$ and $v \in \mathcal{V}$ such that $A \subseteq [u]_0$ and $A \subseteq [v]_0$, respectively. Hence, by hypothesis, $A \subseteq [u]_0 \cap [v]_0$ which means that $[u]_0 \cap [v]_0 \neq \emptyset$ and so, $u = v$. Consequently, there is $u \in \mathcal{U} \cap \mathcal{V}$ such that $A \in r(\mathcal{U} \cap \mathcal{V})$ and the inclusion $r(\mathcal{U} \cap \mathcal{V}) \supseteq r(\mathcal{U}) \cap r(\mathcal{V})$ is true.

(3) If $y \in F_n[r(\mathcal{U})]$, then there exists $x \in A \subseteq [u]_0$ such that $y = F_n(x)$. Thus, by Proposition 3.1, we have $y = F_n(x) \in F_n([u]_0) = [\hat{F}_n(u)]_0$, consequently, $y \in r[\hat{F}_n(\mathcal{U})]$, which follows that $F_n[r(\mathcal{U})] \subseteq r[\hat{F}_n(\mathcal{U})]$.

(4) Suppose that $r(\mathcal{U})$ is not open. For any $A \in r(\mathcal{U}) \backslash int(r(\mathcal{U}))$ and $\varepsilon > 0$, there exists open $\varepsilon-$neighborhood N of A such that $N \cap r(\mathcal{U}) \neq \emptyset$ and $N \nsubseteq r(\mathcal{U})$. Consider a fuzzy set $\chi_{\{A\}}$. Since $\chi_{\{A\}} \in \mathcal{U}$ and

$$D(\chi_N, \chi_{\{A\}}) = \sup_{\alpha \in [0,1]} H([\chi_N]_\alpha, [\chi_{\{A\}}]_\alpha) \leq \varepsilon,$$

we obtain $\chi_N \in B(\chi_{\{A\}}, \varepsilon)$, where $B(\chi_{\{A\}}, \varepsilon)$ is an open ball in $\mathbb{F}^1(X)$. However, $\chi_N \notin \mathcal{U}$, and consequently, $B(\chi_{\{A\}}, \varepsilon) \nsubseteq \mathcal{U}$. That is to say, no $\varepsilon-$neighborhood of $\chi_{\{A\}}$ contains in $\mathcal{U}$, this contradicts the fact that $\mathcal{U}$ is open in $\mathbb{F}^1(X)$. □

Theorem 3.6. *Let $\{\hat{f}_n\}_{n=1}^{\infty}$ be a sequence of maps on normal fuzzy sets $\mathbb{F}^1(X)$. If $\{f_n\}_{n=0}^{\infty}$ is transitive then $\{\hat{f}_n\}_{n=0}^{\infty}$ is transitive.*

Proof. Suppose $\{f_n\}_{n=0}^{\infty}$ is transitive. To prove $\{\hat{f}_n\}_{n=0}^{\infty}$ is transitive, it suffices to show that for any non-empty open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathbb{F}^1(X)$, there is a $k \geq 1$ such that

$$\hat{F}^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset.$$

Since $\mathcal{U}$ and $\mathcal{V}$ are open, by Proposition 3.5 (4), $r(\mathcal{U})$ and $r(\mathcal{V})$ are also open sets. Due to $\{f_n\}_{n=0}^{\infty}$ is transitive, there is a $k \geq 1$ such that

$$F_k(r(\mathcal{U})) \cap r(\mathcal{V}) \neq \emptyset.$$

By Propositions 3.5 (3) and 3.5 (4), we have

$$\emptyset \neq F_k[r(\mathcal{U})] \cap r(\mathcal{V}) \subseteq r[\hat{F}_k(\mathcal{U})] \cap r(\mathcal{V}) = r[\hat{F}_k(\mathcal{U}) \cap \mathcal{V}].$$

Thus, using Proposition 3.5 (1), it follows that

$$\hat{F}^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset_X.$$

This completes the proof. □

4. Periodic Density

It has been proven that $f \in C(X)$ has periodic density then both $\bar{f}$ and $\hat{f}$ share the same property, but the converse are not true [18, 23]. In this section, we first show that the periodic density of $\{f_n\}_{n=0}^{\infty}$ implies the periodic density of $\{\hat{f}_n\}_{n=0}^{\infty}$, and then some conditions are discussed, under which the converse implication is true.

Theorem 4.1. *If $\{f_n\}_{n=0}^{\infty}$ has periodic density, then $\{\hat{f}_n\}_{n=0}^{\infty}$ has periodic density.*

Proof. The proof, with slighter modifications, is similar to Theorem 5 in [23]. □

In the converse direction of Theorem 4.1, we discuss sufficient conditions on $\{\hat{f}_n\}_{n=0}^{\infty}$ for the periodic density of $\{f_n\}_{n=0}^{\infty}$ as follows. Before passing to the next theorem, we give some preliminary notations.

Let $\mathbb{M}$ be a subspace of $\mathbb{F}(X)$. Notice that $\hat{f}_{\mathbb{M}}(u) = \hat{f}(u)$ for all $u \in \mathbb{M}$. We say that a topological space X has the fixed point property (in short, *f.p.p.*) if every continuous map $f_n : X \to X$ has a fixed point. We will denote the family of all non-empty compact subsets of X which have the *f.p.p.* by $\mathbb{K}_p(X)$. Define $\mathbb{F}_p(X) = \{u \in \mathbb{F}(X) : [u]_\alpha \in \mathbb{K}_p(X)\}$. The next theorem shows that when $\{\hat{f}_n\}_{n=0}^{\infty}$ has periodic density will imply $\{f_n\}_{n=0}^{\infty}$ has periodic density.

Remark 4.2. *Let U be a subset of X and let $e_M(U) = \{u \in \mathbb{M} : [u]_0 \subseteq U\}$. We can conclude that if U is an open subset of X, then $e_M(U)$ is an open subset of $\mathbb{F}(X)$.*

Theorem 4.3. *Let $\mathbb{M}$ be a subspace of $\mathbb{F}(X)$. If $\mathbb{M} \subseteq \mathbb{F}_p(X)$, then $\{\hat{f}_n\}_{n=0}^{\infty}$ has periodic density implies $\{f_n\}_{n=0}^{\infty}$ has periodic density.*

Proof. Let $x \in X$ and $\chi_{\{x\}} \in \mathbb{M}$, then by periodic density of $\{\hat{f}_n\}_{n=0}^{\infty}$, for any $\varepsilon > 0$, there exist $\nu \in \mathbb{M}$ and $n \in \mathbb{N}$ such that
(a) $d_\infty(\chi_{\{x\}}, \nu) < \varepsilon$;
(b) $\hat{F}_n(\nu) = \nu$.

On one hand, by Proposition 3.1, we have $F_n([\nu]_\alpha) = [\nu]_\alpha$. Thus, combing (a) and (b), we have

$$d(x, F_n(y)) < \varepsilon \tag{4.1}$$

for all $y \in [\nu]_\alpha$.

On the other hand, the map $g : [\nu]_\alpha \to [\nu]_\alpha$ given by $g(y) = F_n(y)$ for every $y \in [\nu]_\alpha$ is a continuous map. Since $[\nu]_\alpha$ has the *f.p.p.* (recall that $\mathbb{M} \subseteq \mathbb{F}_p(X)$), it follows that g has a fixed point y_p such that $g(y_p) = F_n(y_p) = y_p$, that is to say, y_p is a periodic point of $\{f_n\}_{n=0}^{\infty}$ contained in $[\nu]_\alpha$. Thus, due to (4.1), we obtain $d(x, y_p) < \varepsilon$ for all $x \in X$. Consequently, $\{f_n\}_{n=0}^{\infty}$ has periodic density on X. This completes the proof. □

5. Sensitivity

In this section, we study the relations between sensitivity of nonautonomous dynamical system and its fuzzified system. An counterexample has been given to show that, in general, sensitivity of $\{f_n\}_{n=0}^{\infty}$ does not imply sensitivity of $\{\hat{f}_n\}_{n=0}^{\infty}$.

Theorem 5.1. *If $\{\hat{f}_n\}_{n=0}^{\infty}$ is sensitive, then $\{f_n\}_{n=0}^{\infty}$ is sensitive.*

Proof. Let $u_0 \in \mathbb{F}(X)$. Since $\{\hat{f}_n\}_{n=0}^{\infty}$ is sensitive, there exists a constant δ such that for every $\epsilon > 0$ we can find $\nu \in \mathbb{F}(X)$ and $k \in \mathbb{N}$ satisfying $\nu \in B(u_0, \varepsilon)$ and

$$\begin{aligned} d_\infty(\hat{F}_k(u_0), \hat{F}_k(\nu)) &= \sup_{\alpha \in [0,1]} H([\hat{F}_k(u_0)]_\alpha, [\hat{F}_k(\nu)]_\alpha) \\ &= \sup_{\alpha \in [0,1]} H(F_k[u_0]_\alpha, F_k[\nu]_\alpha) > \delta. \end{aligned}$$

Thus, there exist $x_0 \in [u_0]_\alpha$ and $y_0 \in [\nu]_\alpha$ such that $d(F_k(x_0), F_k(y_0)) > \delta$. On the other hand, since $\nu \in B(u_0, \varepsilon)$, we have $d(x_0, y_0) < \varepsilon$. Therefore, $\{f_n\}_{n=0}^\infty$ is sensitive. □

The following example shows that, in general, the converse of Theorem 5.1 is not true.

Example 5.2. We first need some previous notations and results. In [7], the author perform "surgery" on the circle S^1 to construct a Denjoy homeomorphism, more specifically, take any point $x_0 \in S^1$, we cut out each point $R_\lambda^n(x_0)$ on the orbit of x_0 and replace it with a small interval I_n, where $R_\lambda : S^1 \to S^1$ is the irrational rotation of the circle S^1. Consequently, a new circle S^* has been constructed. The Denioy map $D_\lambda : S^* \to S^*$ is an orientation preserving homeomorphism of S^*. There exists a Cantor set $C_\lambda \subset S^*$ on which D_λ acts minimally. It is known that there exists a continuous surjection $h_\lambda : S^* \to S^1$ that semi-conjugates D_λ with R_λ. In [22], the authors show that the system $(\mathbb{K}(C_\lambda), D_\lambda)$ is not sensitive.

Now turning to our problem. Let $f_n = D_\lambda$, $n = 1, 2, \cdots$. Define $i_\lambda : \mathbb{K}(C_\lambda) \to \mathbb{F}(C_\lambda)$ by $i_\lambda(K) = \lambda\chi_K$ for any $K \in \mathbb{K}(C_\lambda)$ and any $\lambda \in (0, 1]$, where χ_K is the characteristic function of K. Hence, $i_\lambda \circ \overline{D}_\lambda = \widehat{D_\lambda} \circ i_\lambda$. Note that i_λ is continuous. We show that the sensitivity of D_λ cannot be inherited by $\widehat{D_\lambda}$ as follows.

Since $(\mathbb{K}(C_\lambda), \overline{D}_\lambda)$ is not sensitive, for $\varepsilon > 0, \delta > 0$, there exists a nonempty set $M \in \mathbb{K}(C_\lambda)$ and $B(M, \delta)$ such that for all $N \in B(M, \delta)$,

$$H(\overline{D}_\lambda^n(M), \overline{D}_\lambda^n(N)) < \varepsilon. \tag{5.1}$$

Suppose $u \in e(M)$ (recall that $e(M) = \{u \in \mathbb{F}(C_\lambda) \mid [u]_0 \subseteq M\}$), by continuity of i_λ and (5.1), we have

$$\begin{aligned} H(\overline{D}_\lambda^n([u]_0), \overline{D}_\lambda^n(N)) < \varepsilon \quad & \Rightarrow H(i_\lambda \circ \overline{D}_\lambda^n([u]_0), i_\lambda \circ \overline{D}_\lambda^n(N)) < \varepsilon \\ & \Rightarrow H(\widehat{D_\lambda}^n \circ i_\lambda([u]_0), \widehat{D_\lambda}^n \circ i_\lambda(N)) \\ & = d_\infty(\widehat{D_\lambda}^n(u), \widehat{D_\lambda}^n(\nu)) < \varepsilon, \end{aligned}$$

where $\nu = i_\lambda(N) \in \mathbb{F}(C_\lambda)$.

6. Conclusions and Discussions

In this paper, we discuss relations between some chaotic properties of the nonautonomous discrete dynamical systems and its fuzzified dynamical systems. More specifically, we study transitivity, periodic density and sensitivity, respectively. Several examples are also presented to illustrate the relations between two dynamical systems. We show that the dynamical properties of the original system and its fuzzy extension mutually inherit some global characteristics. The following main results are obtained:

Theorem 3.3. If $\{\hat{f}_n\}_{n=1}^\infty$ is transitive, then $\{f_n\}_{n=1}^\infty$ is transitive.

Theorem 3.6. Let $\{\hat{f}_n\}_{n=1}^\infty$ be a sequence of maps on normal fuzzy sets $\mathbb{F}^1(X)$. If $\{f_n\}_{n=1}^\infty$ is transitive, then $\{\hat{f}_n\}_{n=1}^\infty$ is transitive.

Theorem 4.1. If $\{f_n\}_{n=0}^\infty$ has periodic density, then $\{\hat{f}_n\}_{n=0}^\infty$ has periodic density.

Theorem 4.3. Let $\mathbb{M}$ be a subspace of $\mathbb{F}(X)$. If $\mathbb{M} \subseteq \mathbb{F}_p(X)$, then $\{\hat{f}_n\}_{n=0}^\infty$ has periodic density implies $\{f_n\}_{n=0}^\infty$ has periodic density.

Theorem 5.1. If $\{\hat{f}_n\}_{n=0}^\infty$ is sensitive, then $\{f_n\}_{n=0}^\infty$ is sensitive.

In general, the converse of the Theorem 5.1 is not true, please see Example 5.2.

From the results obtained above, we can conclude that $\{\hat{f}_n\}_{n=0}^\infty$ is Devaney chaotic implies $\{f_n\}_{n=0}^\infty$ is Devaney chaotic, provided that $\mathbb{F}(X)$ has the *f.p.p.* It is worth noting that if the system is autonomous, then sensitive dependence on initial conditions is a consequence of transitivity together with a dense set of periodic points [4, 27]. More precisely, sensitivity is redundant in the definition of Devaney's chaos if the state space X is infinite. However, it is not clear whether similar conclusion still holds for the nonautonomous systems.

On the other hand, the results mentioned above are restricted to the special case that all (X_n, d_n) $(n \geq 0)$ are same space. In the current literature, there are few results about chaotic properties of nonautonomous systems in general case.

According to above analysis, there are some open problems still exist.

Problem 1. In nonautonomous dynamical systems, does transitivity together with periodic density imply sensitivity?

Problem 2. In general case, although it is indeed difficult to study complexity of a nonautonomous system, it would be a challenge to discuss the relations between dynamics of a nonautonomous system and its induced fuzzy system.

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