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A novel double integral transform and its applications

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Abstract

We introduce a new double integral equation and prove some related theorems. We then present some useful tools for the new integral operator, and use this operator to solve partial differential equations with singularities of a given type.

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2010 MSC: 31A10, 31A25, 31B20.

1. Introduction and Preliminaries

Partial differential equations with singularities are used to describe many real world problems arising in all the fields of applied science. For instance, the migration of groundwater via the geological formation called aquifer is described by a parabolic equation with singularity [20]. The usefulness of these equations has attracted the attention of many scholars throughout the history of applied sciences. One of the problems with these equations is that they are very difficult to solve analytically, or even numerically. Many papers devoted to underpinning a possible solvability of these equations have been published. One of the well-known methods for dealing with this type of equations is the so-called Frobenius method, which enables us to create a power series solution to such a differential equation, under certain conditions on the variable coefficients of the equation [12]. However there are two major disadvantages in its use. Firstly, the algebra,

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even in simple examples, is inclined to be heavy, and there is certainly no suggestion that the solution can be written down at sight. Secondly, the method may break down completely either because the recurrence relationship in the coefficients is not two-term, and so becomes unmanageable, or because the resulting power series is not convergent, and unfortunately the original differential equation gives no indication of whether such a break-down will occur, so that we may find after some heavy algebra that all our work is wasted. Other techniques such as the homotopy decomposition method [4, 5, 16], the homotopy perturbation method [17, 24], the variational iteration method [15, 21], the Bernstein collateral method [13], the homotopy analysis method [19], asymptotic and perturbation methods [2, 6, 7] were also intensively used to obtain some exact and approximate solutions to this class of equations. In the recent decade, many scholars have used some integral transform operator to solve partial differential equations, for instance, the Laplace transform [3, 18], the Fourier transform [8], the Sumudu transform [1, 14, 22, 23], the Mellin transform and many others [10, 11]. However some of these are not very efficient when dealing with partial differential equations with singularities. For instance the Laplace transform of $\frac{1}{x}f(x)$ is $\int_s^\infty \mathcal{L}(f(x))ds$ where $\mathcal{L}$ denotes the Laplace transform operator, and it is obvious that solving a second order partial differential equation involving such an expression will render the transformed equation very complicated. Our aim in this paper is to introduce a relatively new operator that can be used to handle some types of second order partial differential equations with singularities. We will consider the following equation:

$$D_{xt}u(x,t) + \frac{1}{x^n t^m}u(x,t) = f(x,t), \tag{1.1}$$

where D_{xt} can be the derivative with integer order or a fractional derivative, m, n are positive integers and $f(x,t)$ is a known function.

2. Important facts for the new operator

Definition 2.1. Let f be a continuous function such that the Laplace transform of f is n times and m times partially differentiable. We define a new integral transform as

$$K_{n,m}[f(x,t)](s,p) = \int_0^\infty\int_0^\infty x^n t^m e^{-sx}e^{-pt}f(x,t)dxdt = F(s,p). \tag{2.1}$$

Here, s and p are the Laplace variables.

Theorem 2.2. *The inverse transform of $F(s,p)$ is defined as*

$$f(x,t) = K^{-1}_{n,m}[F(s,p)](x,t) = \frac{(-1)^{n+m+1}}{4\pi^2}\int_{\alpha-i\infty}^{\alpha+i\infty}\int_{\beta-i\infty}^{\beta+i\infty} e^{xs+tp}\left(J^{m+n}_{s,p}\left(K_{n,m}[f(x,t)](s,p)\right)\right)dsdp, \tag{2.2}$$

where $J^{m+n}_{s,p}$ is the inverse of the operator $\dfrac{\partial^{n+m}}{\partial s^n \partial p^m}$.

Proof. In fact,

$$K_{n,m}[f(x,t)](s,p) = \int_0^\infty\int_0^\infty x^n t^m e^{-sx}e^{-pt}f(x,t)dxdt = (-1)^{n+m}\frac{\partial^{n+m}}{\partial s^n\partial p^m}[\mathcal{L}_2(f(x,t))] \tag{2.3}$$

with $\mathcal{L}_2$ denoting the double Laplace transform. Thus, it follows form the above equation that

$$K^{-1}_{n,m}[F(s,p)](x,t) = \frac{(-1)^{n+m+1}}{4\pi^2}\int_{\alpha-i\infty}^{\alpha+i\infty}\int_{\beta-i\infty}^{\beta+i\infty} e^{xs+tp} = \left(J^{m+n}_{s,p}\left(\frac{(-1)^{n+m}}{\frac{\partial^{n+m}}{\partial s^n\partial p^m}[\mathcal{L}_2(f(x,t))]}\right)\right)dsdp. \tag{2.4}$$

This produces

$$K_{n,m}^{-1}[F(s,p)](x,t) = \frac{(-1)^{2n+2m+1}}{4\pi^2 i^2} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_{\beta-i\infty}^{\beta+i\infty} e^{xs+tp} \left(\left(\left([\mathcal{L}_2(f(x,t))]\right)\right)\right) dsdp. \tag{2.5}$$

Now, using the properties of the Laplace transform operator, we obtain

$$K_{n,m}^{-1}[F(s,p)](x,t) = \mathcal{L}_2^{-1}[\mathcal{L}_2(f(x,t))] = f(x,t). \tag{2.6}$$

This completers the proof. □

Our first concern is to deal with the existence and uniqueness of the transform. We will start with the existence. Let $f(x,t)$ be a continuous function on the interval $[0,\infty)$ which is of exponential order, that is, for some $\alpha,\beta \in \mathbb{R}$,

$$\sup_{t>\alpha, x>\beta} \frac{|x^n t^m f(x,t)|}{e^{\alpha x + t\beta}} < \infty. \tag{2.7}$$

Under the above condition, the transform of

$$K_{n,m}[f(x,t)](s,p) = \int_0^\infty \int_0^\infty x^n t^m e^{-sx} e^{-pt} f(x,t) dxdt = F(s,p)$$

exists for all $s > \alpha$, $p > \beta$. It is important to point out that all functions in this study are assumed to be of exponential order. We prove the uniqueness of this operator in the next theorem.

Theorem 2.3. *Let $h(x,t)$ and $l(x,t)$ be continuous functions defined for $x,t \geq 0$ and having Laplace transforms $H(s,p)$ and $L(s,p)$, respectively. If $H(s,p) = L(s,p)$, then $h(x,t) = l(x,t)$.*

Proof. If we assume a and b to be sufficiently large then, since

$$f(x,t) = K_{n,m}^{-1}[F(s,p)](x,t) = \frac{(-1)^{n+m+1}}{4\pi^2} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_{\beta-i\infty}^{\beta+i\infty} e^{xs+tp} \left(\left(J_{s,p}^{m+n}\left(K_{n,m}[f(x,t)](s,p)\right)\right)\right) dsdp,$$

we deduce that

$$\begin{aligned} f(x,t) &= K_{n,m}^{-1}[F(s,p)](x,t) \\ &= \frac{(-1)^{n+m+1}}{4\pi^2} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_{\beta-i\infty}^{\beta+i\infty} e^{xs+tp} \left(\left(J_{s,p}^{m+n}\left(H(s,p)\right)\right)\right) dsdp \\ &= \frac{(-1)^{n+m+1}}{4\pi^2} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_{\beta-i\infty}^{\beta+i\infty} e^{xs+tp} \left(\left(J_{s,p}^{m+n}\left(L(s,p)\right)\right)\right) dsdp \\ &= l(x,t) \end{aligned} \tag{2.8}$$

and the theorem is established. □

Theorem 2.4. *The inverse Kanidem transform $K_{(n,m)}[f(x,t)](s,p)$ can be written as*

$$f(x,t) = \lim_{m,n\to\infty} \left(\frac{n}{x}\right)^{n+1} \left(\frac{m}{t}\right)^{m+1} \frac{(-1)^{n+m}}{n!m!} \frac{\partial^{n+m} F(s,p)}{\partial s^n \partial p^m}\Big|_{p=\frac{m}{t}, s=\frac{n}{x}}, \tag{2.9}$$

where $F(s,p)$ is the double Laplace transform of $f(x,t)$.

This result can be proved similarly as in [9].

Theorem 2.5. *Let $g(x,t)$ and $f(x,t)$ be two continuous functions such that their convolution exists and is continuous and their Laplace transforms, $F(s,p)$ and $G(s,p)$ respectively, are $n+m$ times partially differentiable. Then the transform of their convolution is*

$$K_{(n,m)}[(f*g)(x,t)](s,p)=\int_0^\infty\int_0^\infty x^n t^m e^{-sx}e^{-pt}[f*g(x,t)]dxdt. \tag{2.10}$$

Proof. Using the properties of the Laplace transform we obtain that

$$K_{(n,m)}[(f*g)(x,t)](s,p)=\int_0^\infty\int_0^\infty x^n t^m e^{-sx}e^{-pt}[f*g(x,t)]dxdt=(-1)^{n+m}\frac{\partial^{n+m}}{\partial s^n\partial p^m}[\mathcal{L}[f*g(x,t)]]. \tag{2.11}$$

As the Laplace transform of the convolution of two functions is the product of their Laplace transforms, we find that

$$K_{(n,m)}[(f*g)(x,t)](s,p)=(-1)^{n+m}\frac{\partial^{n+m}}{\partial s^n\partial p^m}[F(s,p)G(s,p)].$$

Finally, using the property of the derivative of the product of two functions, we obtain

$$K_{(n,m)}[(f*g)(x,t)](s,p)=(-1)^{n+m}\sum_{i=0}^{n-1}C_n^i\sum_{j=0}^{m-1}C_m^j\frac{\partial^{i+j}F(s,p)}{\partial s^i\partial p^j}\frac{\partial^{n+m-i-j}G(s,p)}{\partial s^{n-i}\partial p^{m-j}} \tag{2.12}$$

and the proof is complete. □

3. Properties

Theorem 3.1. *The operator transform is a linear operator, that is, given two constants a and b and two continuous functions f and g, we have*

$$K_{(n,m)}[(af+bg)(x,t)](s,p)=aK_{(n,m)}[f(x,t)](s,p)+bK_{(n,m)}[f(x,t)](s,p). \tag{3.1}$$

Proof. By definition,

$$\begin{aligned}K_{(n,m)}[(af+bg)(x,t)](s,p)&=\int_0^\infty\int_0^\infty x^n t^m e^{-sx}e^{-pt}[(af+bg)(x,t)]dxdt\\&=a\int_0^\infty\int_0^\infty x^n t^m e^{-sx}e^{-pt}[f(x,t)]dxdt+b\int_0^\infty\int_0^\infty x^n t^m e^{-sx}e^{-pt}[f(x,t)]dxdt\\&=aK_{(n,m)}[f(x,t)](s,p)+bK_{(n,m)}[f(x,t)](s,p).\end{aligned}$$

□

Theorem 3.2. *The first mixed derivative property of the operator transform states*

$$K_{(n,m)}\left[\left(\frac{\partial^2 f(x,t)}{\partial x\partial t}\right)\right](s,p)=(-1)^{n+m}\frac{\partial^{n+m}}{\partial s^n\partial p^m}[spF(s,p)-pF(p,0)-sF(0,s)]. \tag{3.2}$$

Proof. In fact,

$$\begin{aligned}K_{(n,m)}\left[\left(\frac{\partial^2 f(x,t)}{\partial x\partial t}\right)\right](s,p)&=\int_0^\infty\int_0^\infty x^n t^m e^{-sx}e^{-pt}\left[\left(\frac{\partial^2 f(x,t)}{\partial x\partial t}\right)\right]dxdt\\&=(-1)^{m+n}\frac{\partial^{n+m}}{\partial s^n\partial p^m}\left[L_2\left(\frac{\partial^2 f(x,t)}{\partial x\partial t}\right)\right].\end{aligned} \tag{3.3}$$

But the double Laplace transform of $\left(\dfrac{\partial^2 f(x,t)}{\partial x \partial t}\right)$ is

$$L_2\left(\frac{\partial^2 f(x,t)}{\partial x \partial t}\right) = spF(s,p) - pF(p,0) - sF(0,s) - F(0,0). \tag{3.4}$$

Therefore,

$$K_{(n,m)}\left[\left(\frac{\partial^2 f(x,t)}{\partial x \partial t}\right)\right](s,p) = (-1)^{n+m}\frac{\partial^{n+m}}{\partial s^n \partial p^m}[spF(s,p) - pF(p,0) - sF(0,s)].$$

□

Lemma 3.3. *The (n, m)-mixed derivative property of the operator transform states that*

$$\begin{aligned} K_{(n,m)}\left[\left(\frac{\partial^{n+m} f(x,t)}{\partial x^n \partial t^m}\right)\right](s,p) =&(-1)^{n+m}\frac{\partial^{n+m}}{\partial s^n \partial p^m}\left[s^n p^m F(s,p) - \sum_{i=1}^{m} p^{m-i}s^n \frac{\partial^{i-1}F(s,0)}{\partial t^{i-1}}\right. \\ &\left. - \sum_{j=1}^{n} p^m s^{n-i}\frac{\partial^{j-1}F(0,p)}{\partial x^{j-1}} + \sum_{i=1}^{m}\sum_{j=1}^{n} p^{m-i}s^{n-i}\frac{\partial^{i+j-2}f(0,0)}{\partial x^{j-1}\partial t^{i-1}}\right]. \end{aligned} \tag{3.5}$$

The above lemma can be proved via the recursive technique on the positive integers n and m.

Remark 3.4. It is important to point out that, if $n = m = 1$, then the double integral operator introduced here becomes the double Laplace-Carson integral transform. Also if $n = m = 0$, we obtain the double Laplace transform operator.

Theorem 3.5 (Integration theorem)**.** *The following relation holds:*

$$K_{(n,m)}\left[\int_0^x \int_0^t f(v,l)dvdl\right](s,p) = (-1)^{n+m}\frac{\partial^{n+m}}{\partial s^n \partial p^m}\left[\frac{F(s,p)}{sp}\right].$$

Proof. We have

$$\begin{aligned} K_{(n,m)}\left[\int_0^x \int_0^t f(v,l)dvdl\right](s,p) &= \int_0^\infty \int_0^\infty x^n t^m e^{-sx}e^{-pt}\left[\int_0^x \int_0^t f(v,l)dvdl\right]dtdx \\ &= \int_0^\infty x^n e^{-sx}\left[\int_0^x \int_0^\infty e^{-pt}t^m \int_0^t f(v,l)dvdt\right]dldx. \end{aligned}$$

Using the property of the Laplace transform, we obtain

$$\int_0^\infty e^{-pt}t^m \int_0^t f(v,l)dvdt = (-1)^m \frac{\partial^m F(x,p)}{\partial p^m}, \tag{3.6}$$

so

$$K_{(n,m)}\left[\int_0^x \int_0^t f(v,l)dvdl\right](s,p) = \int_0^\infty x^n e^{-sx}\left[\int_0^x (-1)^m \frac{\partial^m F(x,p)}{\partial p^m}\right]dldx.$$

Again making use of the property of the Laplace transform, we find that

$$K_{(n,m)}\left[\int_0^x \int_0^t f(v,l)dvdl\right](s,p) = (-1)^{n+m}\frac{\partial^{n+m}}{\partial s^n \partial p^m}\left[\frac{F(s,p)}{sp}\right].$$

□

4. Applications to a class of partial differential equations

In this section, we point out a possible application of the new integral operator for solving a type of second order partial differential equation with singularity called the *two dimensional Mboctara equation*. We will present the exact solutions for some examples.

Example 4.1. Consider the two dimensional Mboctara equation of the form

$$D_{xt}u(x,t) + \frac{1}{xt}u(x,t) = -\frac{2\exp[-t^2x^2]}{\sqrt{\pi}} + \frac{4x\exp[-t^2x^2]t^2x^2}{\sqrt{\pi}} + \frac{Erfc[tx]}{tx} \tag{4.1}$$

with initial condition $u(x,0) = 0$ and $u(0,t) = 0$.

Applying the Kanidem transform on both sides of the equation (4.1) we obtain

$$\begin{aligned}\frac{\partial^2}{\partial s \partial p}\left[spU(s,p) - pU(p,0) - sU(0,s)\right] + U(s,p) = \quad (4.2)\\ = K_{1,1}\left[-\frac{2\exp[-t^2x^2]}{\sqrt{\pi}} + \frac{4x\exp[-t^2x^2]t^2x^2}{\sqrt{\pi}} + \frac{Erfc[tx]}{tx}\right](s,p).\end{aligned}$$

$U(s,p)$ is the Laplace transform of $u(x,t)$. Using the initial conditions in Laplace space the above equation is reduced to

$$\frac{\partial^2}{\partial s \partial p}\left[spU(s,p)\right] + U(s,p) = K_{1,1}\left[-\frac{2\exp[-t^2x^2]}{\sqrt{\pi}} + \frac{4x\exp[-t^2x^2]t^2x^2}{\sqrt{\pi}} + \frac{Erfc[tx]}{tx}\right](s,p). \tag{4.3}$$

It is now possible to use some iteration method, for instance the homotopy decomposition method (the methodology can be found in [2, 3, 6]). Therefore applying the HDM scheme in the above equation, we assume that the solution is

$$U(s,p) = \sum_{n=0}^{\infty} \varepsilon^n U_n(s,p), \tag{4.4}$$

where ε is an embedding parameter. Using the iteration formula, we get

$$U_1(s,p) = \frac{1}{sp}\int_0^s\int_0^p K_{1,1}\left[-\frac{2\exp[-t^2x^2]}{\sqrt{\pi}} + \frac{4x\exp[-t^2x^2]t^2x^2}{\sqrt{\pi}} + \frac{Erfc[tx]}{tx}\right](\theta,\sigma)d\theta d\sigma, \tag{4.5}$$

and the remaining terms can be recovered by the recursive formula

$$U_n(s,p) = \frac{1}{sp}\int_0^s\int_0^p U_{n-1}(\theta,\sigma)d\theta d\sigma. \tag{4.6}$$

Then, by recovering the remaining terms and taking the embedding parameter to be equal to zero, we apply the inverse Laplace operator on both sides of equation (4.4) and using the linearity of the operator we obtain

$$u(x,t) = \mathcal{L}_{xt}^{-1}\left[\sum_{n=0}^{\infty} U_n(s,p)\right](x,t) = Erfc[xt], \tag{4.7}$$

where $Erfc$ is the complementary error function defined as

$$Erfc[x] = 1 - \frac{2}{\sqrt{\pi}}\int_0^x \exp[-t^2]dt. \tag{4.8}$$

It is very important to point out that $u(x,t) = Erfc[xt]$ is the exact solution of equation (4.1).

Example 4.2. Consider the two dimensional Mboctara equation of the form

$$D_{xt}u(x,t) + \frac{1}{xt}u(x,t) = \cos(xt) - \frac{\sin(xt)}{xt} - tx\sin(xt) \tag{4.9}$$

with initial condition $u(x,0) = 0$ and $u(0,t) = 0$, $u(x,t) = \sin(xt)$.

Now applying the Kanidem transform on both sides of the equation (4.1) we obtain

$$\frac{\partial^2}{\partial s \partial p}\left[spU(s,p) - pU(p,0) - sU(0,s)\right] + U(s,p) = K_{1,1}\left[\cos(xt) - \frac{\sin(xt)}{xt} - tx\sin(xt)\right](s,p). \tag{4.10}$$

$U(s,p)$ is the Laplace transform of $u(x,t)$. Using the initial conditions in Laplace space the above equation is reduced to the form

$$\frac{\partial^2}{\partial s \partial p}\left[spU(s,p)\right] + U(s,p) = K_{1,1}\left[\cos(xt) - \frac{\sin(xt)}{xt} - tx\sin(xt)\right](s,p). \tag{4.11}$$

It is now possible for us to use the homotopy decomposition method, as earlier, therefore we assume that the solution of the above equation is

$$U(s,p) = \sum_{n=0}^{\infty} \varepsilon^n U_n(s,p), \tag{4.12}$$

where ε is an embedding parameter. Using the iteration formula, we arrive at the following:

$$U_1(s,p) = \frac{1}{sp}\int_0^s \int_0^p K_{1,1}\left[\cos(xt) - \frac{\sin(xt)}{xt} - tx\sin(xt)\right](\theta,\sigma)d\theta d\sigma, \tag{4.13}$$

and the remaining terms can be recover by the recursive formula

$$U_n(s,p) = \frac{1}{sp}\int_0^s \int_0^p U_{n-1}(\theta,\sigma)d\theta d\sigma. \tag{4.14}$$

Then, by recovering the remaining terms and taking the embedding parameter to be equal to zero, we apply the inverse Laplace operator on both sides of equation (4.12), and using the linearity of the operator we obtain

$$u(x,t) = \mathcal{L}_{xt}^{-1}[U(s,p)](x,t) = \mathcal{L}_{xt}^{-1}\left[\sum_{n=0}^{\infty} U_n(s,p)\right](x,t) = \sin[xt]. \tag{4.15}$$

We note that the above solution is the exact solution of equation (4.9).

Example 4.3. Consider the two dimensional Mboctara equation of the form

$$D_{xt}u(x,t) - \frac{1}{x^m t^n}u(x,t) = \frac{2\exp[-t^2x^2]}{\sqrt{\pi}} - \frac{4x\exp[-t^2x^2]t^2x^2}{\sqrt{\pi}} - \frac{Erf[tx]}{t^n x^m}. \tag{4.16}$$

Applying the Kanidem transform on both sides of the equation (4.16) we obtain

$$\begin{aligned} (-1)^{n+m}\frac{\partial^{m+n}}{\partial s^n \partial p^m}\left[spU(s,p) - pU(p,0) - sU(0,s)\right] + U(s,p) = \\ = K_{1,1}\left[\frac{2\exp[-t^2x^2]}{\sqrt{\pi}} - \frac{4x\exp[-t^2x^2]t^2x^2}{\sqrt{\pi}} - \frac{Erfc[tx]}{t^n x^m}\right](s,p). \end{aligned} \tag{4.17}$$

$U(s,p)$ is the Laplace transform of $u(x,t)$. Using the initial conditions in Laplace space the above equation is reduced to

$$(-1)^{n+m}\frac{\partial^{m+n}}{\partial s^n \partial p^m}[spU(s,p)]+U(s,p)=K_{1,1}\left[\frac{2\exp[-t^2x^2]}{\sqrt{\pi}}-\frac{4x\exp[-t^2x^2]t^2x^2}{\sqrt{\pi}}-\frac{Erfc[tx]}{t^nx^m}\right](s,p). \quad (4.18)$$

Now we can use the homotopy decomposition method. Applying the HDM scheme in the above equation, we assume that its solution is

$$U(s,p)=\sum_{n=0}^{\infty}\varepsilon^n U_n(s,p), \quad (4.19)$$

where ε is an embedding parameter. Then we find

$$U_1(s,p)=\frac{1}{sp}\int_0^s\int_0^p K_{1,1}\left[\frac{2\exp[-t^2x^2]}{\sqrt{\pi}}-\frac{4x\exp[-t^2x^2]t^2x^2}{\sqrt{\pi}}-\frac{Erfc[tx]}{t^nx^m}\right](\theta,\sigma)d\theta d\sigma \quad (4.20)$$

and the remaining terms can be recovered by the recursive formula

$$U_n(s,p)=\frac{1}{sp\Gamma(m-1)\Gamma(n-1)}\int_0^s(s-\theta)^{n-1}\int_0^p(p-\sigma)^{m-1}U_{n-1}(\theta,\sigma)d\theta d\sigma. \quad (4.21)$$

By recuperating the remaining terms, and taking the fixed parameter to be equal to zero, we apply the inverse Laplace operator on both sides of equation (4.19) and using the linearity of the operator we obtain

$$u(x,t)=\mathcal{L}_{xt}^{-1}[U(s,p)](x,t)=\mathcal{L}_{xt}^{-1}\left[\sum_{n=0}^{\infty}U_n(s,p)\right](x,t)=Erfc[xt],$$

which is the exact solution of equation (4.16).

5. Numerical solutions

In this section, we present the numerical results of the exact solutions for the three examples. Figures 1, 2 and 3 show the graphical representation of the exact solution of the two dimensional Mboctara equations (4.1) (4.9) and (4.16), respectively.

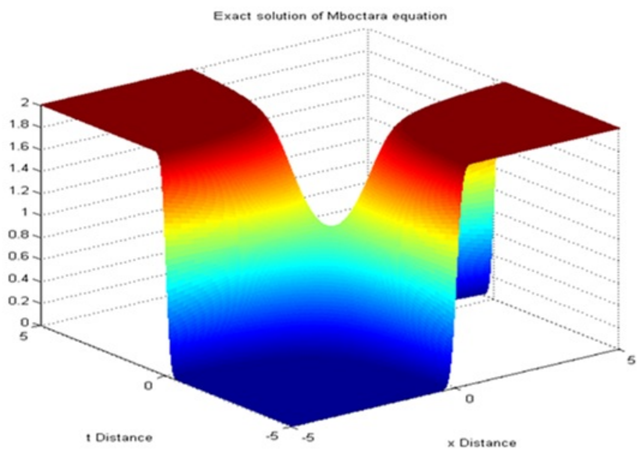

Figure 1: Exact solution of the two dimensional Mboctara equation (4.1) as function of time and space

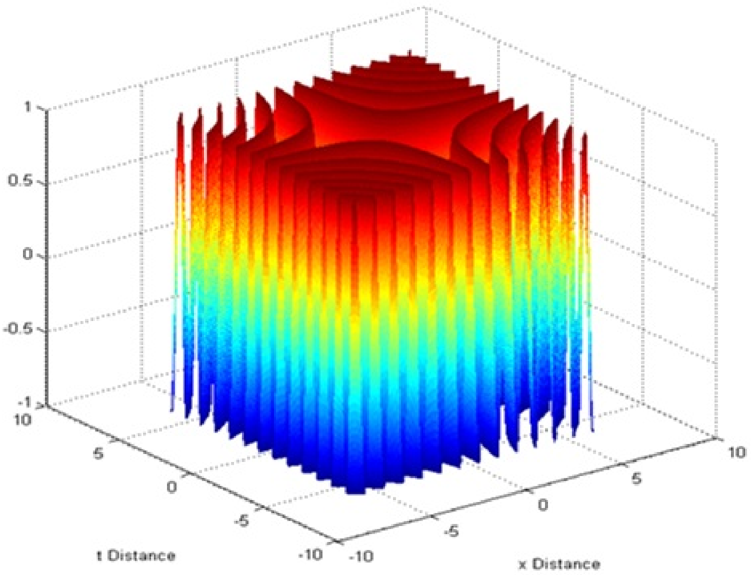

Figure 2: Exact solution of the two dimensional Mboctara equation (4.9) as function of time and space

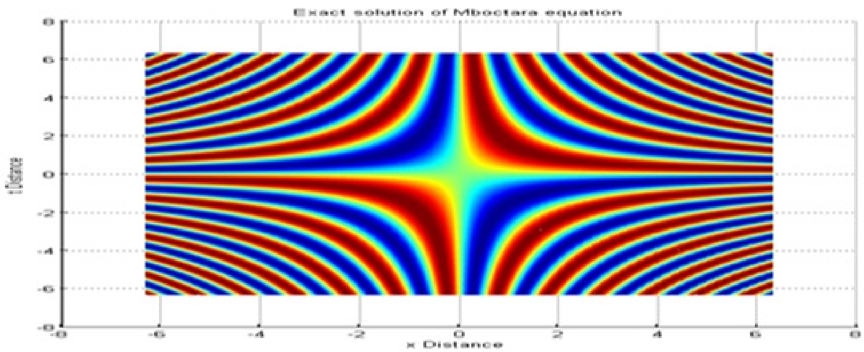

Figure 3: Exact solution of the two dimensional Mboctara equation (4.16) as function of time and space, top view

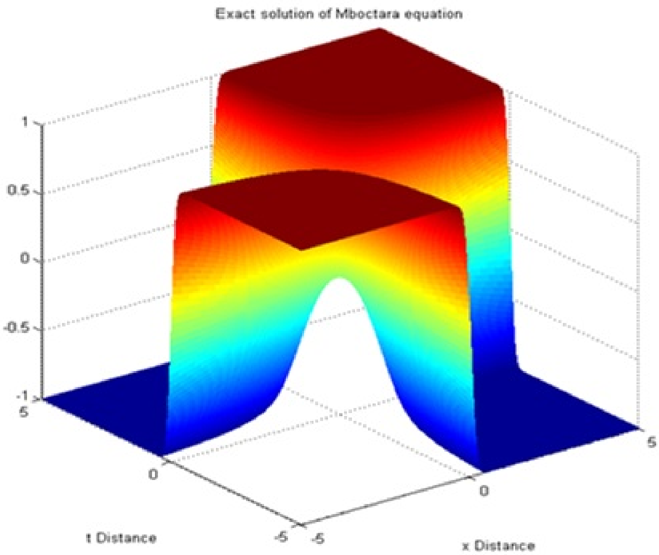

Figure 4: Exact solution of the two dimensional Mboctara equation (4.16) as function of time and space

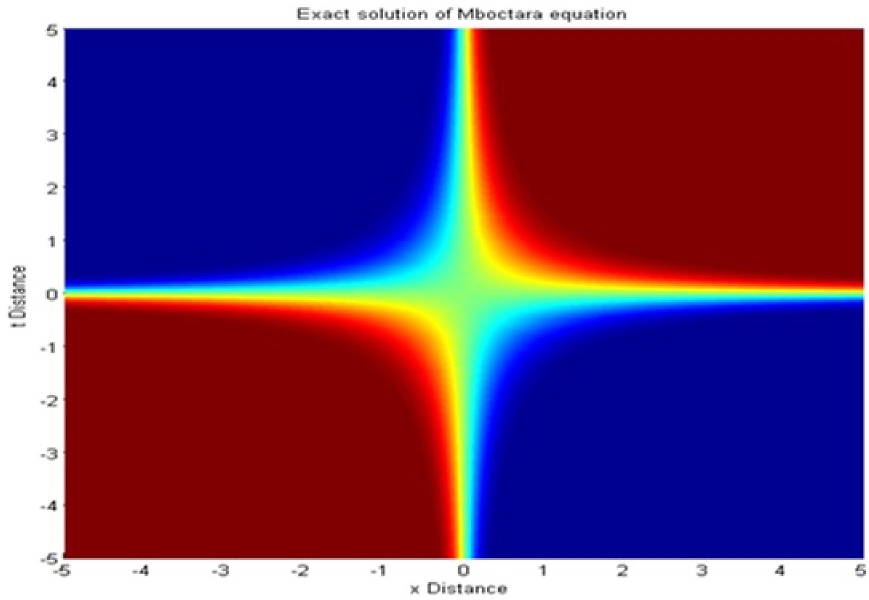

Figure 5: Exact solution of the two dimensional Mboctara equation (4.1) as function of time and space, top view

6. Conclusion

A new double integral transform was introduced in order to handle a particular class of second order partial differential equation with singularities. We presented some properties and theorems for this new operator transform, and provided some examples.

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